Twin-width and forbidden subdivisions

Amadeus Reinald
ENS de Lyon & IBS

joint work with
Édouard Bonnet[1], Eun Jung Kim,
Stéphan Thomassé, Rémi Watrigant

DIMAG Seminar
June 13th, 2022, IBS Daejeon

[1]Some figures are courtesy of Édouard

Graphs and trigraphs

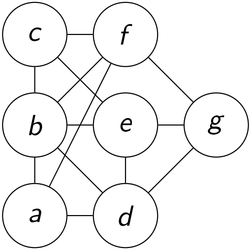


Fig. Graph: edges, non-edges

Graphs and trigraphs

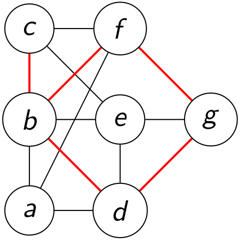
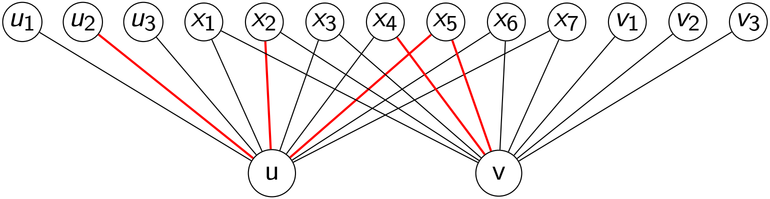


Fig. Trigraph: edges, non-edges, or **red** edges

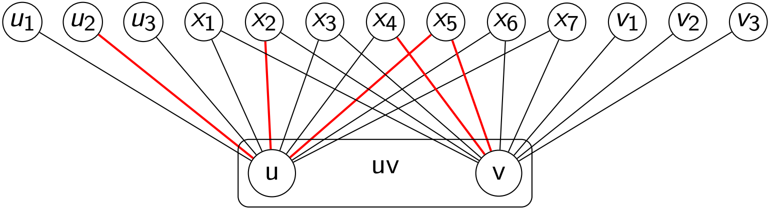
Contractions in trigraphs

- Idea behind twin-width: group near-twins.



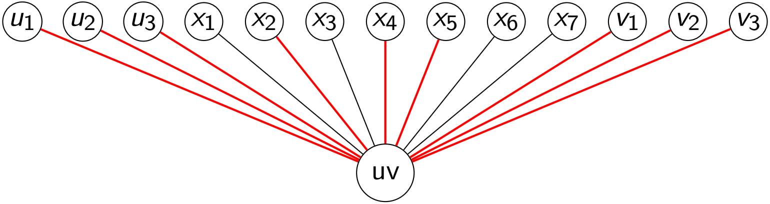
Contractions in trigraphs

- Idea behind twin-width: group near-twins.
- **Contraction** of u and v: record "twin" errors with red edges.



Contractions in trigraphs

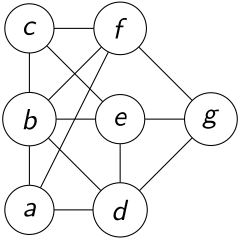
- Idea behind twin-width: group near-twins.
- **Contraction** of u and v: record "twin" errors with red edges.



edges to $N(u) \triangle N(v)$ turn red, for $N(u) \cap N(v)$ red is absorbing

Twin-width

Definition (Twin-width) *$tww(G)$: minimal d such that G admits a* ***contraction sequence*** *where all trigraphs have* ***maximum red degree*** *at most d.*

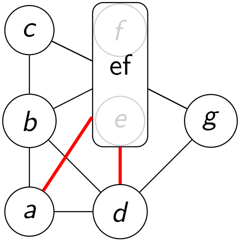


Maximum red degree $= 0$

overall maximum red degree = 0

Twin-width

Definition (Twin-width) *tww(G): minimal d such that G admits a* ***contraction sequence*** *where all trigraphs have* ***maximum red degree*** *at most d.*

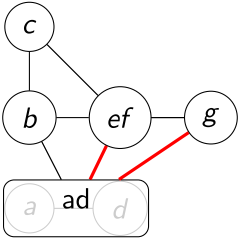


Maximum red degree = 2

overall maximum red degree = 2

Twin-width

Definition (Twin-width) *$tww(G)$: minimal d such that G admits a* ***contraction sequence*** *where all trigraphs have* ***maximum red degree*** *at most d.*

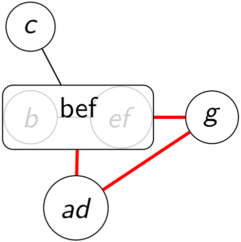


Maximum red degree $= 2$

overall maximum red degree = 2

Twin-width

Definition (Twin-width) *tww*(G): *minimal* d *such that* G *admits a* ***contraction sequence*** *where all trigraphs have* ***maximum red degree*** *at most* d.

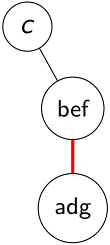


Maximum red degree $= 2$

overall maximum red degree = 2

Twin-width

Definition (Twin-width) *tww*(G): *minimal* d *such that* G *admits a* ***contraction sequence*** *where all trigraphs have* ***maximum red degree*** *at most* d.

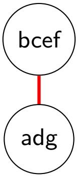


Maximum red degree $= 1$

overall maximum red degree = 2

Twin-width

Definition (Twin-width) *$tww(G)$: minimal d such that G admits a* ***contraction sequence*** *where all trigraphs have* ***maximum red degree*** *at most d.*



Maximum red degree $= 1$

overall maximum red degree = 2

Twin-width

Definition (Twin-width) *$tww(G)$: minimal d such that G admits a* ***contraction sequence*** *where all trigraphs have* ***maximum red degree*** *at most d.*

Maximum red degree $= 0$
overall maximum red degree $= 2$.

Twin-width

Definition (Twin-width) *$tww(G)$: minimal d such that G admits a* ***contraction sequence*** *where all trigraphs have* ***maximum red degree*** *at most d.*

Maximum red degree $= 0$

overall maximum red degree = 2 $\rightarrow$ $tww(G) \leqslant 2$.

Algorithmic Implications

Theorem (Bonnet, Kim, Thomassé, Watrigant '20)
For a graph G with tww d given a d-sequence, and any FO formula ϕ: Deciding $G \models \phi$ can be done in FPT time $f(|\phi|, d) \cdot n$.

Algorithmic Implications

Theorem (Bonnet, Kim, Thomassé, Watrigant '20)
For a graph G with tww d given a d-sequence, and any FO formula ϕ: Deciding $G \models \phi$ can be done in FPT time $f(|\phi|, d) \cdot n$.

Some **problems** expressible in FO:

Algorithmic Implications

Theorem (Bonnet, Kim, Thomassé, Watrigant '20)
For a graph G with tww d given a d-sequence, and any FO formula ϕ: Deciding $G \models \phi$ can be done in FPT time $f(|\phi|, d) \cdot n$.

Some **problems** expressible in FO:

- k-IS : $\exists x_1 ... \exists x_k \bigwedge_{1 \leqslant i \leqslant j \leqslant k} \neg(x_i = x_j \vee E(x_i, x_j))$,

Algorithmic Implications

Theorem (Bonnet, Kim, Thomassé, Watrigant '20)
For a graph G with tww d given a d-sequence, and any FO formula ϕ: Deciding $G \models \phi$ can be done in FPT time $f(|\phi|, d) \cdot n$.

Some **problems** expressible in FO:

- k-IS : $\exists x_1 \ldots \exists x_k \bigwedge_{1 \leqslant i \leqslant j \leqslant k} \neg(x_i = x_j \vee E(x_i, x_j))$,
- k-DS : $\exists x_1 \ldots \exists x_k \forall x \bigvee_{1 \leqslant i \leqslant k} (x = x_i) \vee \bigvee_{1 \leqslant i \leqslant k} E(x, x_i)$.

Graphs of bounded twin-width

Classes with bounded tww:

- Bounded treewidth, rank-width, queue/stack number,

Graphs of bounded twin-width

Classes with bounded tww:

- Bounded treewidth, rank-width, queue/stack number,
- Proper minor-closed, map graphs, posets of bounded width...

Graphs of bounded twin-width

Classes with bounded tww:

- Bounded treewidth, rank-width, queue/stack number,
- Proper minor-closed, map graphs, posets of bounded width...

In this talk: what structures/subgraphs are forced by high tww?

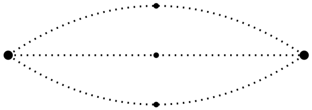
Graphs of bounded twin-width

Classes with bounded tww:

- Bounded treewidth, rank-width, queue/stack number,
- Proper minor-closed, map graphs, posets of bounded width...

In this talk: what structures/subgraphs are forced by high tww?

Theorem (B,K,R,T,W '21+)
Graphs of girth $\geqslant 5$ forbidding ***induced subdivisions*** *of theta have bounded twin-width.*



Parameters and graph minors

Question

- *High parameter $G \Rightarrow$ forced structure/subgraph H?*
- *Equivalently: G forbidding $H \Rightarrow$ low parameter?*

Parameters and graph minors

Question

- *High parameter $G \Rightarrow$ forced structure/subgraph H?*
- *Equivalently: G forbidding $H \Rightarrow$ low parameter?*

A restrictive approach: forbidding **minors**

Parameters and graph minors

Question

- *High parameter $G \Rightarrow$ forced structure/subgraph H?*
- *Equivalently: G forbidding $H \Rightarrow$ low parameter?*

A restrictive approach: forbidding **minors**

- minor: H obtained from G by vertex deletion, edge deletion, edge contraction.

Parameters and graph minors

Question

- *High parameter $G \Rightarrow$ forced structure/subgraph H?*
- *Equivalently: G forbidding $H \Rightarrow$ low parameter?*

A restrictive approach: forbidding **minors**

- minor: H obtained from G by vertex deletion, edge deletion, edge contraction.

Theorem (B,K,T,W '20)
For any H, H-minor free graphs have bounded twin-width.

Treewidth and Minors

Obstruction: $k \times k$-walls have treewidth $\Omega(k)$.

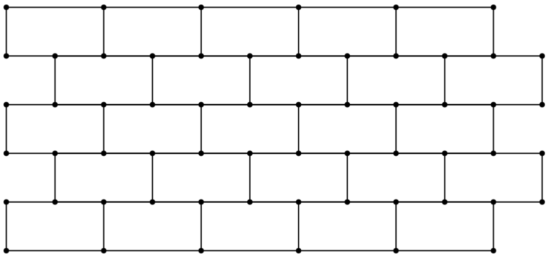


Fig. A 6×6-wall.

Treewidth and Minors

Obstruction: $k \times k$-walls have treewidth $\Omega(k)$.

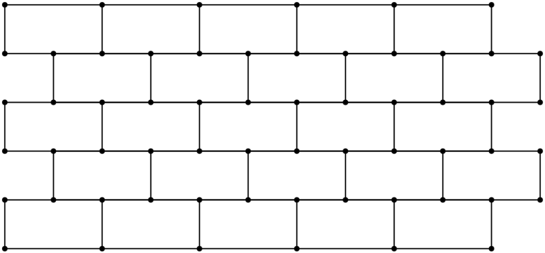


Fig. A 6×6-wall.

Theorem (Robertson, Seymour '86)
*Graphs forbidding a wall **minor** have bounded treewidth.*

Treewidth and induced subdivisions

Can we be less restrictive ?

Treewidth and induced subdivisions

Can we be less restrictive ?

- Forbid **induced subdivisions** of some H?

Treewidth and induced subdivisions

Can we be less restrictive ?

- Forbid **induced subdivisions** of some H?
- K_t breaks induced subgraphs → sparse classes

Treewidth and induced subdivisions

Can we be less restrictive ?

- Forbid **induced subdivisions** of some H?
- K_t breaks induced subgraphs → sparse classes
- Sparse: K_t-free, $K_{t,t}$-free, high girth...

Treewidth and induced subdivisions

Can we be less restrictive ?

- Forbid **induced subdivisions** of some H?
- K_t breaks induced subgraphs $\rightarrow$ sparse classes
- Sparse: K_t-free, $K_{t,t}$-free, high girth...

Question

Sparse + forbidden subdivision of subwall $\Rightarrow$ bounded tw?

Treewidth and induced subdivisions

Can we be less restrictive ?

- Forbid **induced subdivisions** of some H?
- K_t breaks induced subgraphs → sparse classes
- Sparse: K_t-free, $K_{t,t}$-free, high girth...

Question

Sparse + forbidden subdivision of subwall ⇒ bounded tw?

A **theta** is a subdivision of $K_{2,3}$.

Treewidth and induced subdivisions

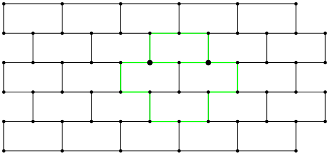
Can we be less restrictive ?

- Forbid **induced subdivisions** of some H?
- K_t breaks induced subgraphs → sparse classes
- Sparse: K_t-free, $K_{t,t}$-free, high girth...

Question

Sparse + forbidden subdivision of subwall ⇒ bounded tw?

A **theta** is a subdivision of $K_{2,3}$.

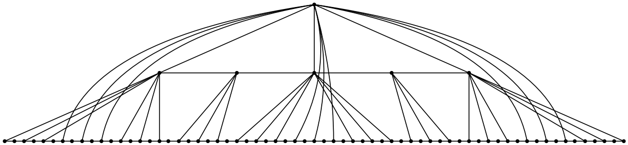


Treewidth and induced thetas

Theorem (Sintiari, Trotignon '19)
There exist theta-free graphs of arbitrarily large girth and treewidth.

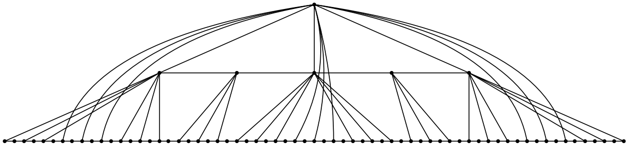
Treewidth and induced thetas

Theorem (Sintiari, Trotignon '19)
There exist theta-free graphs of arbitrarily large girth and treewidth.



Treewidth and induced thetas

Theorem (Sintiari, Trotignon '19)
There exist theta-free graphs of arbitrarily large girth and treewidth.



Theorem (Abrishami, Chudnovsky, Hajebi, Spirkl '21)
There exists c s.t. for any (Theta, triangle)-free G, $tw(G) \leqslant c \log(|V(G)|)$.

Degeneracy and induced subdivisions

For degeneracy:

Theorem

Any $K_{t,t}$-free graph forbidding an induced subdivision of H is $f(t, H)$-degenerate.

Degeneracy and induced subdivisions

For degeneracy:

Theorem

Any $K_{t,t}$-free graph forbidding an induced subdivision of H is $f(t,H)$-degenerate.

What about twin-width?

Twin-width and induced subgraphs

What graphs can we hope to forbid?

Twin-width and induced subgraphs

What graphs can we hope to forbid?

- $o(\log(n))$-subdivisions of K_n have unbounded tww.

Twin-width and induced subgraphs

What graphs can we hope to forbid?

- $o(\log(n))$-subdivisions of K_n have unbounded tww.
- Subcubic graphs have unbounded tww

Twin-width and induced subgraphs

What graphs can we hope to forbid?

- $o(\log(n))$-subdivisions of K_n have unbounded tww.
- Subcubic graphs have unbounded tww

Optimistic conjecture: forbidding any subcubic graph + sparse $\Rightarrow$ bounded twin-width?

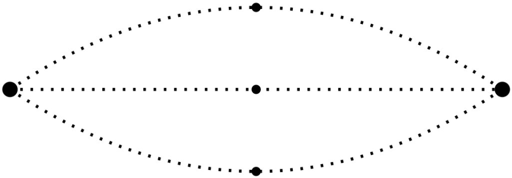
Twin-width and induced subgraphs

What graphs can we hope to forbid?

- $o(\log(n))$-subdivisions of K_n have unbounded tww.
- Subcubic graphs have unbounded tww

Optimistic conjecture: forbidding any subcubic graph + sparse $\Rightarrow$ bounded twin-width?

In this talk: forbid **theta-free, girth** $\geqslant 5$



Twin-width and mixed minors

Theorem (Bonnet, Kim, Thomassé, Watrigant '20)
If $\exists \sigma$ s.t. $Adj_\sigma(G)$ is t-mixed free, then $tww(G) = 2^{2^{O(t)}}$.

Twin-width and mixed minors

Theorem (Bonnet, Kim, Thomassé, Watrigant '20)
If $\exists\sigma$ s.t. $Adj_\sigma(G)$ is t-mixed free, then $tww(G) = 2^{2^{O(t)}}$.

$$\left[\begin{array}{cc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

Fig. A 3 mixed-minor and a 4-grid minor

Twin-width and mixed minors

Theorem (Bonnet, Kim, Thomassé, Watrigant '20)
If $\exists\sigma$ s.t. $Adj_\sigma(G)$ is t-mixed free, then $tww(G) = 2^{2^{O(t)}}$.

$$\left[\begin{array}{cc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]
\qquad
\left[\begin{array}{cc|cc|cc|cc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

Fig. A 3 mixed-minor and a 4-grid minor

In particular: σ **forbidding t-grid minor $\Rightarrow$ bounded tww.**

Twin-width and mixed minors

Theorem (Bonnet, Kim, Thomassé, Watrigant '20)
If $\exists\sigma$ s.t. $Adj_\sigma(G)$ is t-mixed free, then $tww(G) = 2^{2^{O(t)}}$.

$$\left[\begin{array}{cc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]
\qquad
\left[\begin{array}{cc|cc|cc|cc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
\hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
\hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

Fig. A 3 mixed-minor and a 4-grid minor

In particular: σ **forbidding** t**-grid minor** $\Rightarrow$ **bounded tww.**
Example: DFS on trees.

Roadmap

- Find a grid-free **order** $\Rightarrow$ bounded twin-width,

$$\left[\begin{array}{cc|cc|cc|cc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
\hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
\hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

Roadmap

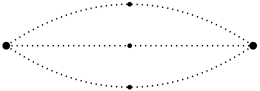
- Find a grid-free **order** $\Rightarrow$ bounded twin-width,
- Build an order according to structural properties of the class,

$$\left[\begin{array}{cc|cc|cc|cc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
\hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
\hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

Roadmap

- Find a grid-free **order** ⇒ bounded twin-width,
- Build an order according to structural properties of the class,
- **Large grid minors ⇒ witness induced thetas.**

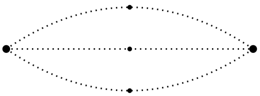
$$\left[\begin{array}{cc|cc|cc|cc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$



Roadmap

- Find a grid-free **order** $\Rightarrow$ bounded twin-width,
- Build an order according to structural properties of the class,
- **Large grid minors $\Rightarrow$ witness induced thetas.**

$$\left[\begin{array}{cc|cc|cc|cc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

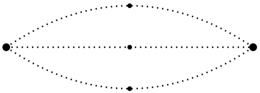


Candidates: BFS, DFS...

Roadmap

- Find a grid-free **order** ⇒ bounded twin-width,
- Build an order according to structural properties of the class,
- **Large grid minors ⇒ witness induced thetas.**

$$\left[\begin{array}{cc|cc|cc|cc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
\hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
\hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$



Candidates: BFS, DFS...
Little information on **connected** subgraphs, hard finding thetas.

Connected BFS Decomposition

Idea: BFS exploring **connected parts** instead of vertices.

Connected BFS Decomposition

Idea: BFS exploring **connected parts** instead of vertices.
$Y \subseteq V$ is a **minimal connected neighbourhood** of $X \subseteq V$ if:

Connected BFS Decomposition

Idea: BFS exploring **connected parts** instead of vertices.

$Y \subseteq V$ is a **minimal connected neighbourhood** of $X \subseteq V$ if:

- $N(X) \subseteq Y$,

Connected BFS Decomposition

Idea: BFS exploring **connected parts** instead of vertices.

$Y \subseteq V$ is a **minimal connected neighbourhood** of $X \subseteq V$ if:

- $N(X) \subseteq Y$,
- $y, y' \in Y$ in the same CC of $G \backslash X$ are in the same CC of $G[Y]$.

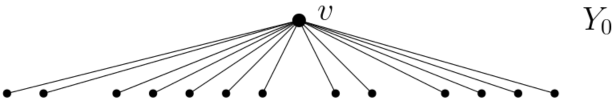
Connected BFS Decomposition

Idea: BFS exploring **connected parts** instead of vertices.

$Y \subseteq V$ is a **minimal connected neighbourhood** of $X \subseteq V$ if:

- $N(X) \subseteq Y$,
- $y, y' \in Y$ in the same CC of $G \backslash X$ are in the same CC of $G[Y]$.

Connected BFS: $Y_0 = \{v\}$,



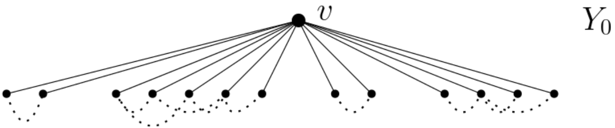
Connected BFS Decomposition

Idea: BFS exploring **connected parts** instead of vertices.

$Y \subseteq V$ is a **minimal connected neighbourhood** of $X \subseteq V$ if:

- $N(X) \subseteq Y$,
- $y, y' \in Y$ in the same CC of $G \backslash X$ are in the same CC of $G[Y]$.

Connected BFS: $Y_0 = \{v\}$,



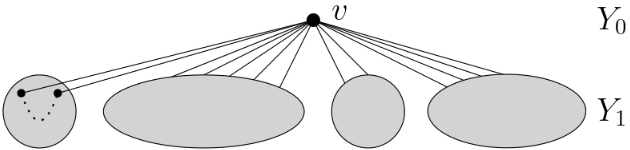
Connected BFS Decomposition

Idea: BFS exploring **connected parts** instead of vertices.

$Y \subseteq V$ is a **minimal connected neighbourhood** of $X \subseteq V$ if:

- $N(X) \subseteq Y$,
- $y, y' \in Y$ in the same CC of $G \backslash X$ are in the same CC of $G[Y]$.

Connected BFS: $Y_0 = \{v\}$, $Y_{i+1} := mcn(Y_0 \cup \ldots \cup Y_i)$



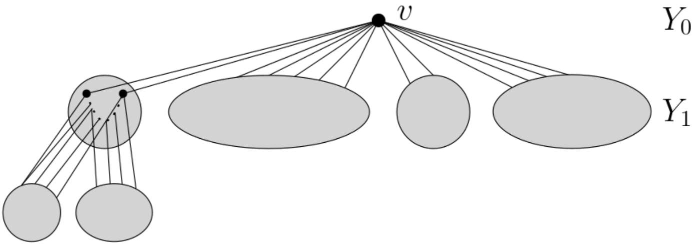
Connected BFS Decomposition

Idea: BFS exploring **connected parts** instead of vertices.

$Y \subseteq V$ is a **minimal connected neighbourhood** of $X \subseteq V$ if:

- $N(X) \subseteq Y$,
- $y, y' \in Y$ in the same CC of $G \backslash X$ are in the same CC of $G[Y]$.

Connected BFS: $Y_0 = \{v\}$, $Y_{i+1} := mcn(Y_0 \cup \ldots \cup Y_i)$



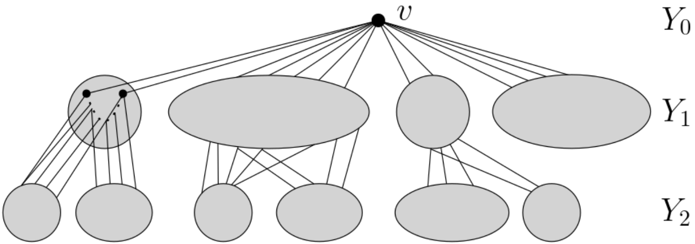
Connected BFS Decomposition

Idea: BFS exploring **connected parts** instead of vertices.

$Y \subseteq V$ is a **minimal connected neighbourhood** of $X \subseteq V$ if:

- $N(X) \subseteq Y$,
- $y, y' \in Y$ in the same CC of $G \backslash X$ are in the same CC of $G[Y]$.

Connected BFS: $Y_0 = \{v\}$, $Y_{i+1} := mcn(Y_0 \cup \ldots \cup Y_i)$



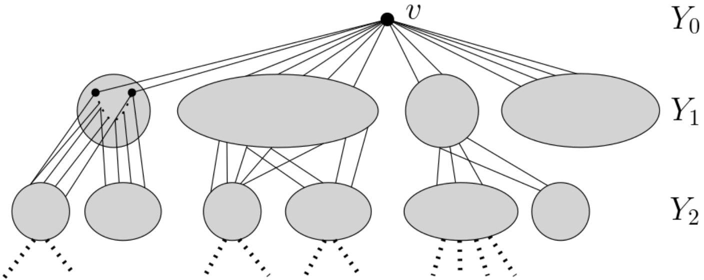
Connected BFS Decomposition

Idea: BFS exploring **connected parts** instead of vertices.

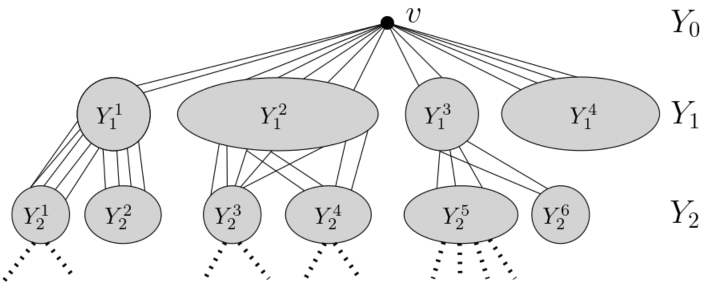
$Y \subseteq V$ is a **minimal connected neighbourhood** of $X \subseteq V$ if:

- $N(X) \subseteq Y$,
- $y, y' \in Y$ in the same CC of $G \backslash X$ are in the same CC of $G[Y]$.

Connected BFS: $Y_0 = \{v\}$, $Y_{i+1} := mcn(Y_0 \cup \ldots \cup Y_i)$

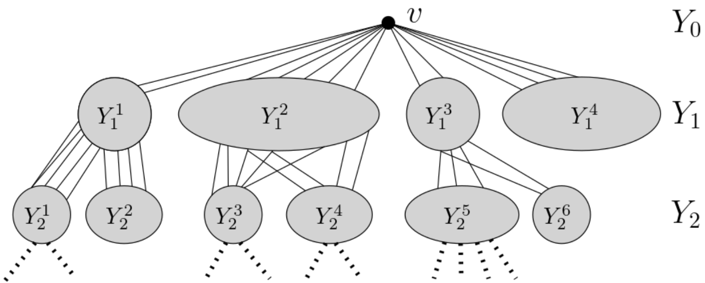


Global structure



Let Y_i^j be the j-th connected component of $G[Y_i]$ ordered lexicographically.

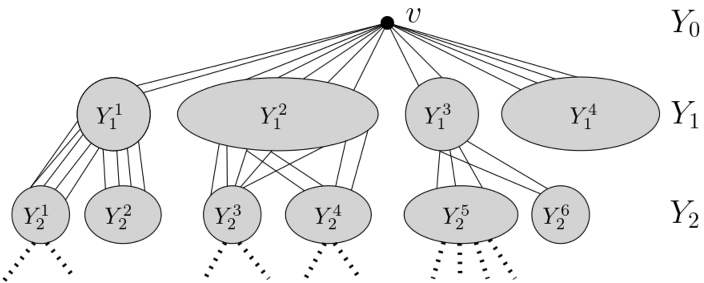
Global structure



Let Y_i^j be the j-th connected component of $G[Y_i]$ ordered lexicographically.

- Y_i is only adjacent to Y_{i-1} and Y_{i+1},

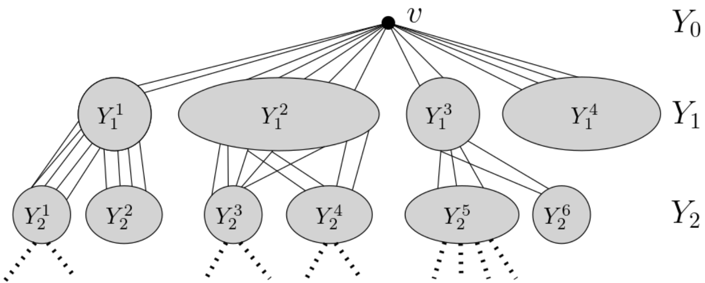
Global structure



Let Y_i^j be the j-th connected component of $G[Y_i]$ ordered lexicographically.

- Y_i is only adjacent to Y_{i-1} and Y_{i+1},
- Y_i has an unique antecedent Y_{i-1}^j (otherwise, better Y_{i-1}^j).

Global structure



Let Y_i^j be the j-th connected component of $G[Y_i]$ ordered lexicographically.

- Y_i is only adjacent to Y_{i-1} and Y_{i+1},
- Y_i has an unique antecedent Y_{i-1}^j (otherwise, better Y_{i-1}^j).

Global relation between components: tree = "simple" / tww.

Component structure

Does complexity lie **inside** components Y_i^j?

Component structure

Does complexity lie **inside** components Y_i^j? **No**

Lemma *For any CBFS of G, each Y_i^j is a tree.*

Component structure

Does complexity lie **inside** components Y_i^j? **No**

Lemma *For any CBFS of G, each Y_i^j is a tree. Moreover, each $v \in Y_i^j$ has an unique antecedent $v^{-1} \in Y_{i-1}$.*

Component structure

Does complexity lie **inside** components Y_i^j? **No**

Lemma *For any CBFS of G, each Y_i^j is a tree. Moreover, each $v \in Y_i^j$ has an unique antecedent $v^{-1} \in Y_{i-1}$.*

$\hookrightarrow$ Complexity must lie in adjacencies **between** components...

Component structure

Does complexity lie **inside** components Y_i^j? **No**

Lemma *For any CBFS of G, each Y_i^j is a tree. Moreover, each $v \in Y_i^j$ has an unique antecedent $v^{-1} \in Y_{i-1}$.*

$\hookrightarrow$ Complexity must lie in adjacencies **between** components...

1. Describe the **structure** between layers,

Component structure

Does complexity lie **inside** components Y_i^j? **No**

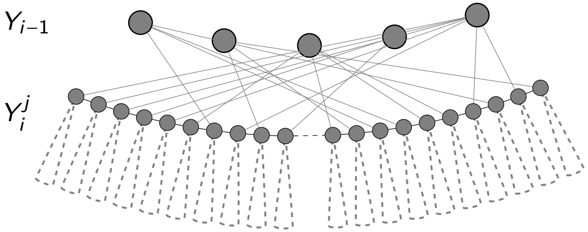
Lemma *For any CBFS of G, each Y_i^j is a tree. Moreover, each $v \in Y_i^j$ has an unique antecedent $v^{-1} \in Y_{i-1}$.*

$\hookrightarrow$ Complexity must lie in adjacencies **between** components...

1. Describe the **structure** between layers,
2. Use it to guide our **ordering** choice.

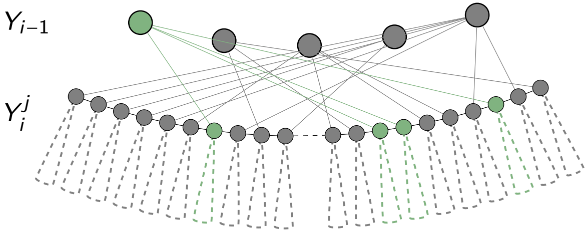
Component-Antecedent Structure

How do successors of **different** vertices of Y_{i-1} relate in Y_i?



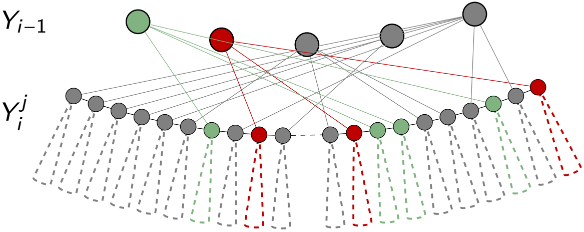
Component-Antecedent Structure

How do successors of **different** vertices of Y_{i-1} relate in Y_i?



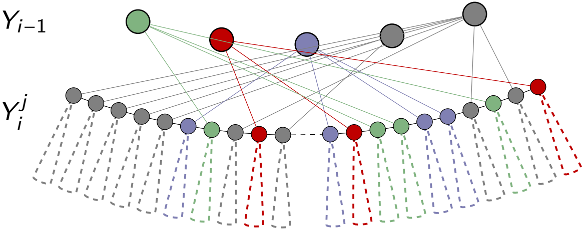
Component-Antecedent Structure

How do successors of **different** vertices of Y_{i-1} relate in Y_i?



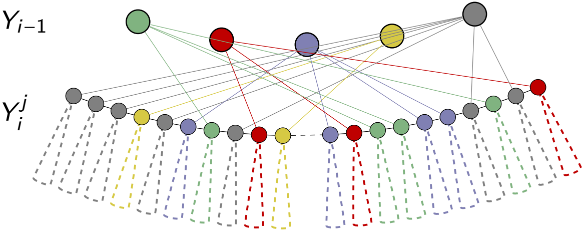
Component-Antecedent Structure

How do successors of **different** vertices of Y_{i-1} relate in Y_i?



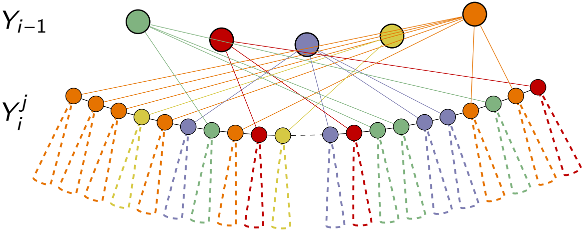
Component-Antecedent Structure

How do successors of **different** vertices of Y_{i-1} relate in Y_i?



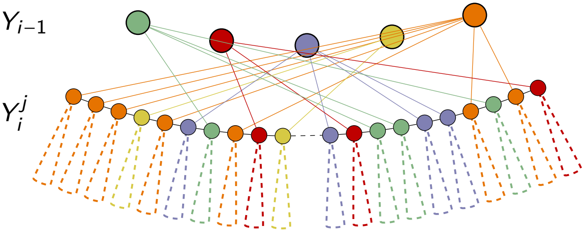
Component-Antecedent Structure

How do successors of **different** vertices of Y_{i-1} relate in Y_i?



Component-Antecedent Structure

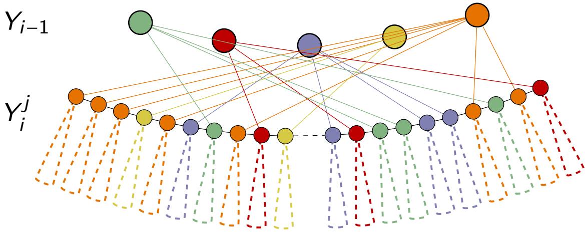
How do successors of **different** vertices of Y_{i-1} relate in Y_i?



Lemma *For any Y_i^j, there is a* ***principal path*** *P_i^j s.t. any successor $v \in Y_i^j$ belongs to a v^{-1}-private branch of P_i^j.*

Component-Antecedent Structure

How do successors of **different** vertices of Y_{i-1} relate in Y_i?



Lemma *For any Y_i^j, there is a **principal path** P_i^j s.t. any successor $v \in Y_i^j$ belongs to a v^{-1}-private branch of P_i^j.*

Natural order: follow P_i^j exhausting private branches along the way.

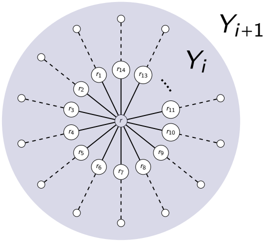
Component-Successors Structure

Locally: for any vertex r in component Y_i^j:

Component-Successors Structure

Locally: for any vertex r in component Y_i^j:

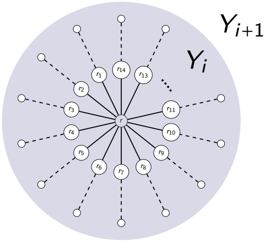
- Y_i^j = tree → r-**branches are simple**,



Component-Successors Structure

Locally: for any vertex r in component Y_i^j:

- Y_i^j = tree → r-**branches are simple**,
- Complexity = r-branches intertwining **using** Y_{i+1},

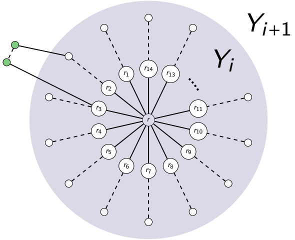


Component-Successors Structure

Locally: for any vertex r in component Y_i^j:

- Y_i^j = tree → r-**branches are simple**,
- Complexity = r-branches intertwining **using** Y_{i+1},

Definition *Consecutivity: shortest path between successors of two different r-branches.*



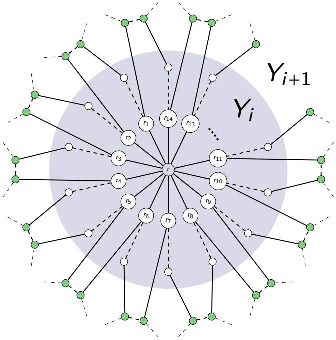
Component-Successors Structure

Locally: for any vertex r in component Y_i^j:

- Y_i^j = tree → r-**branches are simple**,
- Complexity = r-branches intertwining **using** Y_{i+1},

Definition *Consecutivity: shortest path between successors of two different r-branches.*

Lemma *All but two r-branches admit at most two consecutivities.*



Constructing the total order

- **Globally:** components Y_i^j related as a tree

Constructing the total order

- **Globally:** components Y_i^j related as a tree
- $\rightarrow$ follow the **CBFS order** lexicographically.

Constructing the total order

- **Globally:** components Y_i^j related as a tree
- $\rightarrow$ follow the **CBFS order** lexicographically.

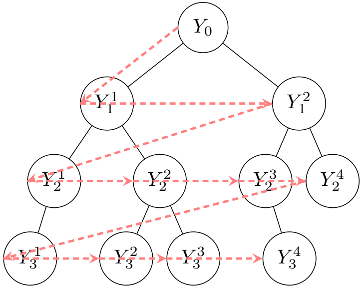


Fig. Inter-component **BFS**

Constructing the total order

- **Globally:** components Y_i^j related as a tree
- $\rightarrow$ follow the **CBFS order** lexicographically.

- **Locally:** branches in Y_i^j: little intertwining on Y_{i+1}

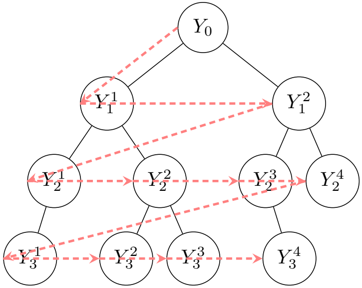


Fig. Inter-component **BFS**

Constructing the total order

- **Globally:** components Y_i^j related as a tree
- → follow the **CBFS order** lexicographically.

- **Locally:** branches in Y_i^j: little intertwining on Y_{i+1}
- → **DFS**: order branches / consecutivity cycles

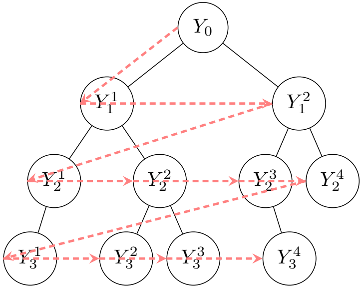


Fig. Inter-component **BFS**

Constructing the total order

- **Globally:** components Y_i^j related as a tree
- $\rightarrow$ follow the **CBFS order** lexicographically.

- **Locally:** branches in Y_i^j: little intertwining on Y_{i+1}
- $\rightarrow$ **DFS**: order branches / consecutivity cycles

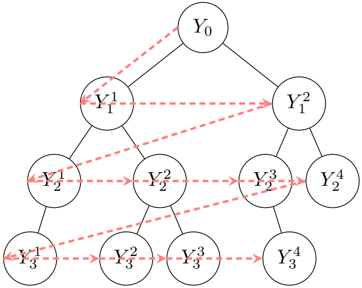


Fig. Inter-component **BFS**

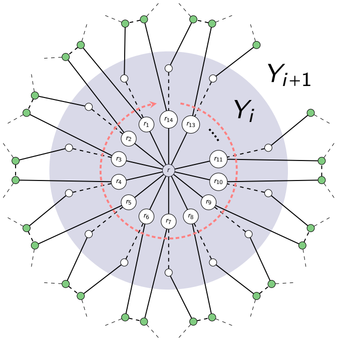


Fig. Intra-component DFS

Bounding the twin-width

Graphs in the class → corresponding **ordered** matrices $M_<$:

Bounding the twin-width

Graphs in the class → corresponding **ordered** matrices $M_<$:

Roadmap:

Bounding the twin-width

Graphs in the class → corresponding **ordered** matrices $M_<$:

Roadmap:

1. Assume existence of arbitrarily large minors,

Bounding the twin-width

Graphs in the class → corresponding **ordered** matrices $M_<$:

Roadmap:

1. Assume existence of arbitrarily large minors,
2. Use global order to localize them,

Bounding the twin-width

Graphs in the class → corresponding **ordered** matrices $M_<$:

Roadmap:

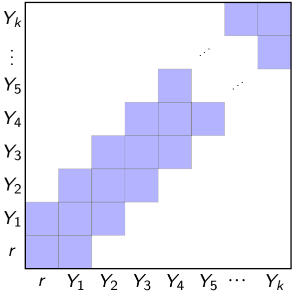
1. Assume existence of arbitrarily large minors,
2. Use global order to localize them,
3. Use local order to yield a contradiction.

Grid minors among successive layers

- **Globally:** We have ordered layers $Y_0 < Y_1 < \ldots < Y_k$.
- Where can a large grid minor lie?

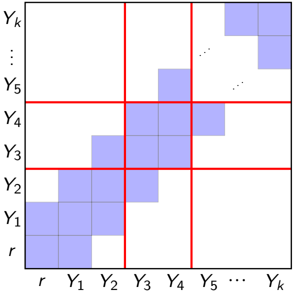
Grid minors among successive layers

- **Globally:** We have ordered layers $Y_0 < Y_1 < \dots < Y_k$.
- Where can a large grid minor lie?



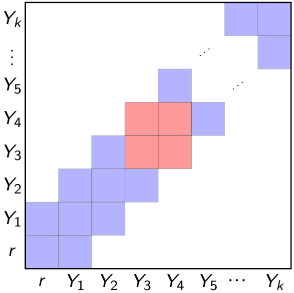
Grid minors among successive layers

- **Globally:** We have ordered layers $Y_0 < Y_1 < ... < Y_k$.
- Where can a large grid minor lie?



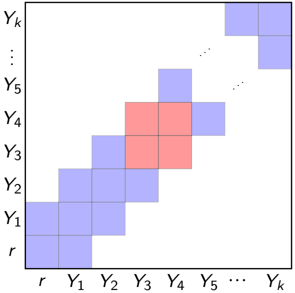
Grid minors among successive layers

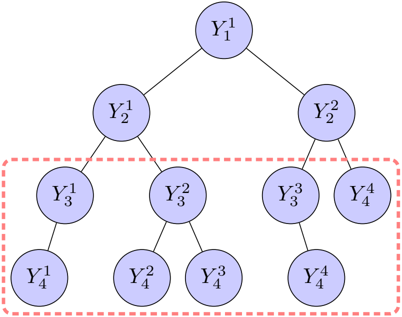
- **Globally:** We have ordered layers $Y_0 < Y_1 < \ldots < Y_k$.
- Where can a large grid minor lie?



Grid minors among successive layers

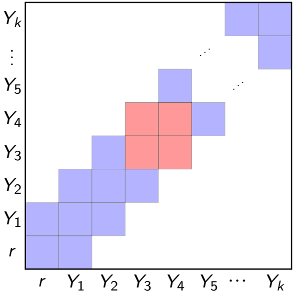
- **Globally:** We have ordered layers $Y_0 < Y_1 < \ldots < Y_k$.
- Where can a large grid minor lie?

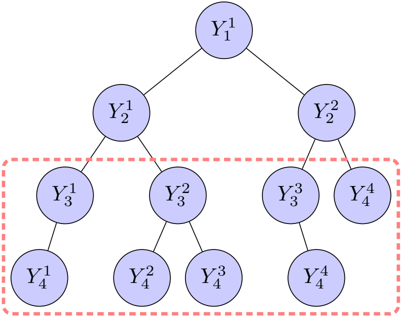




Grid minors among successive layers

- **Globally:** We have ordered layers $Y_0 < Y_1 < \ldots < Y_k$.
- Where can a large grid minor lie?

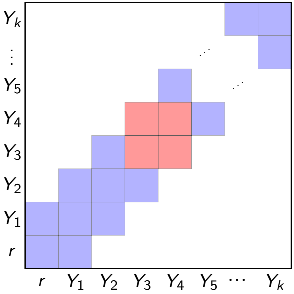




Lemma *If matrices $M_<$ have arbitrarily large grid minors, some submatrices indexed by $(Y_i \cup Y_{i+1})^2$ also do.*

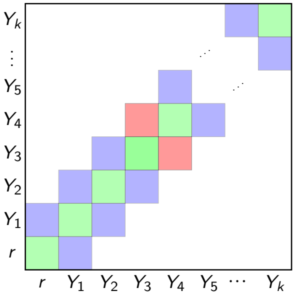
Grid minors between successive layers

Locally: Y_i are forests → bounded tww (DFS) →



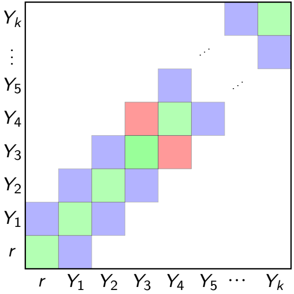
Grid minors between successive layers

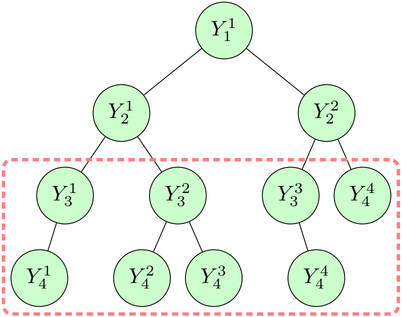
Locally: Y_i are forests → bounded tww (DFS) → no large grid minors



Grid minors between successive layers

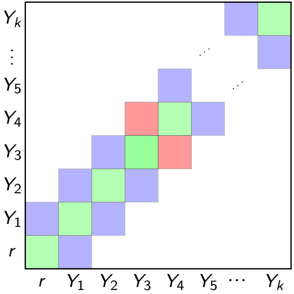
Locally: Y_i are forests → bounded tww (DFS) → no large grid minors

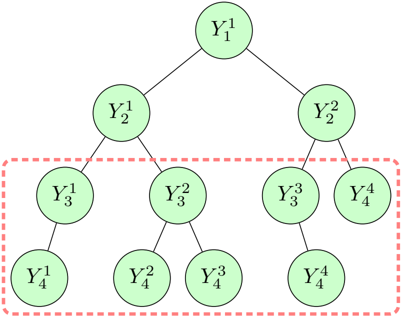




Grid minors between successive layers

Locally: Y_i are forests → bounded tww (DFS) → no large grid minors





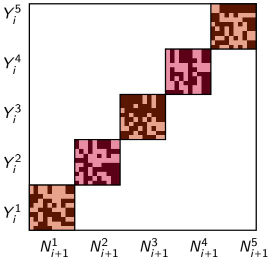
Lemma *If matrices $M_<$ have arbitrarily large grid minors, some submatrices indexed by $Y_i \times Y_{i+1}$ also do.*

Grid Minors from One Component

Grid minor on $Y_i \times Y_{i+1}$ + lexicographical CBFS:

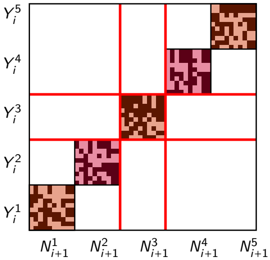
Grid Minors from One Component

Grid minor on $Y_i \times Y_{i+1}$ + lexicographical CBFS:



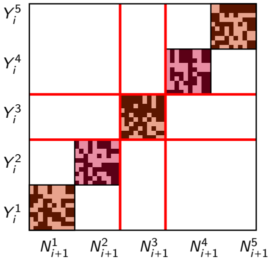
Grid Minors from One Component

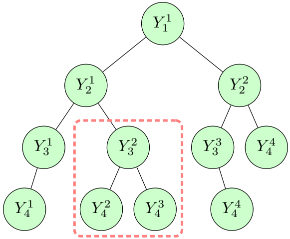
Grid minor on $Y_i \times Y_{i+1}$ + lexicographical CBFS:



Grid Minors from One Component

Grid minor on $Y_i \times Y_{i+1}$ + lexicographical CBFS:





Lemma *If $M_<$ admit arbitrarily large grid minors, some submatrices indexed by Y_i^j and its successors do too.*

Excluding Grid Minors

Assume large grid minor on Y_i^j and successors:

R_t	1	1	1	1	1	1
$\vdots$				$\vdots$		1
R_3	1	1	1	1	$\cdots$	1
R_2	1	1	1	1		1
R_1	1	1	1	1		1
R_1	1	1	1	1		1
	C_1	C_1	C_2	C_3	$\cdots$	C_t

Excluding Grid Minors

Assume large grid minor on Y_i^j and successors:

- Row sets $\leftrightarrow$ subforests of Y_i^j,

R_t	1	1	1	1	1	1
$\vdots$				$\vdots$		1
R_3	1	1	1	1	$\cdots$	1
R_2	1	1	1	1		1
R_1	1	1	1	1		1
R_1	1	1	1	1		1
	C_1	C_1	C_2	C_3	$\cdots$	C_t

Excluding Grid Minors

Assume large grid minor on Y_i^j and successors:

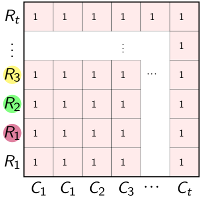
- Row sets $\leftrightarrow$ subforests of Y_i^j,
- Column sets $\leftrightarrow$ subpaths of Y_{i+1},

R_t	1	1	1	1	1	1
$\vdots$				$\vdots$		1
R_3	1	1	1	1	$\cdots$	1
R_2	1	1	1	1		1
R_1	1	1	1	1		1
R_1	1	1	1	1		1
	C_1	C_1	C_2	C_3	$\cdots$	C_t

Excluding Grid Minors

Assume large grid minor on Y_i^j and successors:

- Row sets ↔ subforests of Y_i^j,
- Column sets ↔ subpaths of Y_{i+1},
- Non-zero entries → **consecutivities**.



Excluding Grid Minors

Assume large grid minor on Y_i^j and successors:

- Row sets $\leftrightarrow$ subforests of Y_i^j,
- Column sets $\leftrightarrow$ subpaths of Y_{i+1},
- Non-zero entries $\rightarrow$ **consecutivities**.

R_t	1	1	1	1	1	1
$\vdots$				$\vdots$		1
R_3	1	1	1	1	$\cdots$	1
R_2	1	1	1	1		1
R_1	1	1	1	1		1
R_1	1	1	1	1		1
	C_1	C_1	C_2	C_3	$\cdots$	C_t

Proof scheme:

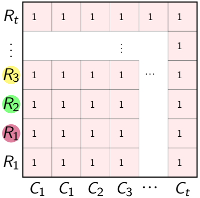
Excluding Grid Minors

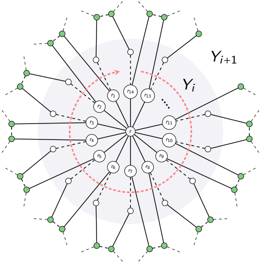
Assume large grid minor on Y_i^j and successors:

- Row sets $\leftrightarrow$ subforests of Y_i^j,
- Column sets $\leftrightarrow$ subpaths of Y_{i+1},
- Non-zero entries $\rightarrow$ **consecutivities**.

Proof scheme:

- Concentrate rows around $r \in Y_i^j$,





Excluding Grid Minors

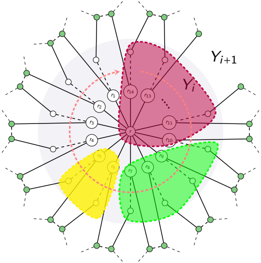
Assume large grid minor on Y_i^j and successors:

- Row sets $\leftrightarrow$ subforests of Y_i^j,
- Column sets $\leftrightarrow$ subpaths of Y_{i+1},
- Non-zero entries $\rightarrow$ **consecutivities**.

Proof scheme:

- Concentrate rows around $r \in Y_i^j$,

R_t	1	1	1	1	1	1
$\vdots$				$\vdots$		1
R_3	1	1	1	1	$\cdots$	1
R_2	1	1	1	1		1
R_1	1	1	1	1		1
R_1	1	1	1	1		1
	C_1	C_1	C_2	C_3	$\cdots$	C_t



Excluding Grid Minors

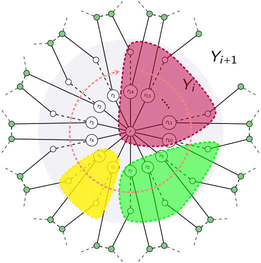
Assume large grid minor on Y_i^j and successors:

- Row sets $\leftrightarrow$ subforests of Y_i^j,
- Column sets $\leftrightarrow$ subpaths of Y_{i+1},
- Non-zero entries $\rightarrow$ **consecutivities**.

Proof scheme:

- Concentrate rows around $r \in Y_i^j$,
- Intertwining around r **violates** our order

R_t	1	1	1	1	1	1
$\vdots$				$\vdots$		1
R_3	1	1	1	1	$\cdots$	1
R_2	1	1	1	1		1
R_1	1	1	1	1		1
R_1	1	1	1	1		1
	C_1	C_1	C_2	C_3	$\cdots$	C_t



Excluding Grid Minors

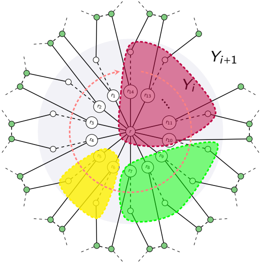
Assume large grid minor on Y_i^j and successors:

- Row sets ↔ subforests of Y_i^j,
- Column sets ↔ subpaths of Y_{i+1},
- Non-zero entries → **consecutivities**.

R_t	1	1	1	1	1	1
$\vdots$				$\vdots$		1
R_3	1	1	1	1	$\cdots$	1
R_2	1	1	1	1		1
R_1	1	1	1	1		1
R_1	1	1	1	1		1
	C_1	C_1	C_2	C_3	$\cdots$	C_t

Proof scheme:

- Concentrate rows around $r \in Y_i^j$,
- Intertwining around r **violates** our order



Theorem (B,K,R,T,W '21)
Theta-free graphs of girth at least 5 have bounded twin-width.

Further paths

- Do sparse θ-free graphs have **bounded queue/stack number**? Possibly through the same order.

Further paths

- Do sparse θ-free graphs have **bounded queue/stack number**? Possibly through the same order.
- Extend the approach to classes forbidding any subcubic subgraph of the wall.

Further paths

- Do sparse θ-free graphs have **bounded queue/stack number**? Possibly through the same order.
- Extend the approach to classes forbidding any subcubic subgraph of the wall.

Thank you!

Bibliography

N. L. D. Sintiari and N. Trotignon. "(Theta, triangle)-free and (even hole, K_4)-free graphs. Part 1 : Layered wheels". In: *CoRR* abs/1906.10998 (2019). arXiv: 1906.10998. URL: http://arxiv.org/abs/1906.10998

T. Abrishami, M. Chudnovsky, S. Hajebi, and S. Spirkl. "Induced subgraphs and tree decompositions III. Three-path-configurations and logarithmic treewidth". In: (2021). arXiv: 2109.01310 [math.CO]

É. Bonnet, E. J. Kim, S. Thomassé, and R. Watrigant. "Twin-width I: tractable FO model checking". In: (2020). arXiv: 2004.14789 [cs.DS]