# Logical operations and Kolmogorov complexity

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**Abstract**

Conditional Kolmogorov complexity $K(x|y)$ can be understood as the complexity of the problem "$Y \to X$", where $X$ is the problem "construct $x$" and $Y$ is the problem "construct $y$". Other logical operations ($\wedge, \vee, \leftrightarrow$) can be interpreted in a similar way, extending Kolmogorov interpretation of intuitionistic logic and Kleene realizability. This leads to interesting problems in algorithmic information theory. Some of these questions are discussed.

Keywords: intuitionistic logic, realizability, Kolmogorov complexity.

Kolmogorov complexity $K(x)$ of a binary string $x$ is defined as minimal length of a program that generates this string. This definition can be extended to sets of strings. Let $A$ be a (finite or infinite) set of strings. We define the complexity $K(A)$ as the length of a shortest program that generates some string $x \in A$. Informally, we consider $A$ as a problem "Generate any element of $A$"; $K(A)$ is complexity of this problem. Evidently, $K(A) = \min\{K(x) \mid x \in A\}$, so this generalization gives nothing really new.

However, it can be combined with the definition of logical operations on sets of strings that goes back to Kolmogorov's paper [3] and Kleene's notion of realizability [2, p. 501]. Let $A$ and $B$ be two sets of strings. We define sets $A \wedge B$, $A \vee B$ and $A \to B$ as follows:

- $A \wedge B = \{\langle a, b\rangle \mid a \in A, b \in B\}$
- $A \vee B = \{\langle 0, a\rangle \mid a \in A\} \cup \{\langle 1, b\rangle \mid b \in B\}$
- $A \to B = \{p \mid [p](x) \in B \text{ for all } x \in A\}$

Here $\langle \cdot, \cdot \rangle$ is computable encoding of pair of strings; $[p](x)$ stands for the output of $p$ (considered as a program) when applied to input $x$. Note that a program $p$ in $A \to B$ may give arbitrary results (or no results) when applied to $x \notin A$.

**Example 1.** Let $x$ and $y$ be two strings. Consider the set $x \to y$ (to simplify notation we identify a string $s$ and the singleton $\{s\}$). This set contains all programs that map $x$ to $y$. It is easy to see that $K(x \to y) = K(y|x) + O(1)$ where $K(y|x)$ denotes conditional complexity of the string $y$ when $x$ is known.

Note also that $K(x \wedge y)$ is the complexity of the pair $(x, y)$ and $K(x \vee y) = \min(K(x), K(y))$.

**Example 2.** Now consider the set $x \leftrightarrow y$ defined as $(x \to y) \wedge (y \to x)$. By definition, $K(x \leftrightarrow y)$ is the complexity of pair of programs transforming $x$ to $y$ and vice versa. It turns out (as proved in [1]) that $K(x \leftrightarrow y) = \max(K(x|y), K(y|x)) + O(\log(K(x|y) + K(y|x)))$.

**Example 3.** Let $x$ and $y$ be two strings. Consider the set $(x \to y) \to y$. Its elements are programs that map every program in $(x \to y)$ (i.e., transforming $x$ to $y$) to $y$. Let us prove that $K((x \to y) \to y) = \min(K(x), K(y)) + O(\log(|x| + |y|))$. (Here $|s|$ stands for the length of $s$.)

It is easy to see that $K((x \to y) \to y) \le K(y) + O(1)$. Indeed, any program that prints $y$ can be considered as a program that maps every element of $(x \to y)$ to $y$.

It is also easy to see that $K((x \to y) \to y) \le K(x) + O(1)$. Indeed, for a given string $x$ consider the program $p_x$ that maps any program $s$ to $[s](x)$. If $s$ belongs to $x \to y$, then $[p_x](s) = [s](x) = y$. Therefore, $p_x \in ((x \to y) \to y)$. On the other hand, $K(p_x) \le K(x) + O(1)$.

Therefore, $K((x \to y) \to y) \le \min(K(x), K(y)) + O(1)$. It remains to prove that $\min(K(x), K(y)) \le K((x \to y) \to y) + O(\log n)$ if $x$ and $y$ are strings of length at most $n$.

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Let $s$ be a program in $(x \to y) \to y$. Let $S$ be the set of all strings of length at most $n$. For any function $\tau\colon S \to S$ we fix some program $l_\tau$ that computes this function; therefore, we have $|S|^{|S|}$ different programs $l_\tau$. A pair $(u, v) \in S \times S$ is called *$s$-coherent* if $[s](l_\tau) = v$ for any function $\tau$ such that $\tau(u) = v$. (Note that we do not require that $[s](p) = v$ for any program $p \in (u \to v)$.)

By definition the pair $(x, y)$ is $s$-coherent. Other coherent pairs may exist. (For example, if $s(p) = y$ for any $p$, then any pair $(x', y)$ is $s$-coherent. If $s(p) = p(x)$ for any $p$, then any pair $(x, y')$ will be $s$-coherent.) However, either all coherent pairs have $x$ as the first component or all coherent pairs have $y$ as the second component. To prove that, it is enough to prove the following statement: if $(x', y')$ and $(x'', y'')$ are $s$-coherent pairs then either $x' = x''$ or $y' = y''$. If it is not the case and $x' \neq x''$, $y \neq y''$, consider a function $\tau$ such that $\tau(x') = y'$ and $\tau(x'') = y''$. Then we have $[s](l_\tau) = y'$ and $[s](l_\tau) = y''$ at the same time, which is impossible, since $y' \neq y$.

If all $s$-coherent pairs have $x$ as the first component, then $K(x) \leq K(s) + O(\log n)$, because we can find $x$ when $s$ and $n$ are given (searching for a $s$-coherent pair and taking its first component). Similarly, if all $s$-coherent pairs have $y$ as the second component, then $K(y) \leq K(s) + O(\log n)$. Therefore, $\min(K(x), K(y)) \leq K(s) + O(\log n)$. (End of the proof.)

Similar argument can be used to prove that $\min(K(x), K(z)) \leq K((x \to y) \to z) + O(\log n)$ for any strings $x, y, z$ having length at most $n$. Indeed, let $t$ be a program in $(x \to y) \to z$. A triple $(x, y, z)$ is called $t$-coherent if $t(l_\tau) = z$ for any $\tau$ such that $\tau(x) = y$. If two triples $(x', y', z')$ and $(x'', y'', z'')$ are $t$-coherent then either $x' = x''$ or $z' = z''$. Therefore, either $x$ or $z$ can be reconstructed from $t$.

In particular, for $x = z$ we get $K((x \to y) \to x) = K(x) + O(\log n)$.

**Example 4.** For any three strings $x, y, z$ having length at most $n$ we have $K((x \vee y) \to z) = \max(K(z|x), K(z|y)) + O(\log n)$. This was recently proved by Andrei Muchnik with a very nice argument. One direction is easy: $K(z|x) = K(x \to z) \leq K((x \vee y) \to z)$; for the same reason $K(z|y) \leq K((x \vee y) \to z)$. To prove the reverse inequality, Muchnik assumes that $K(z|x) \leq k$ and $K(z|y) \leq k$ and proves that one can find a "fingerprint" $f$ of $z$ having length $k$ such that $z$ can be reconstructed from $f$ and $x$ and also from $f$ and $y$. The proof uses expander graphs and will be published elsewhere.

**Question.** Let $A(p, q, \ldots)$ be a propositional formula with connectivities $\wedge, \vee, \to$. For any strings $x, y, \ldots$ consider the set $A(x, y \ldots)$ defined in a natural way. The question is whether $K(A(x, y, \ldots))$ is determined by complexities $K(x), K(y) \ldots$ and conditional complexities of their combinations up to $O(\log n)$-term if $x, y, \ldots$ are strings of length at most $n$. Examples 1–4 support this conjecture.

**Remark 1.** The goal of Kolmogorov and Kleene was to provide an interpretation of the intuitionistic logic. Following this idea, we can prove the following statement: if $A(p, q, \ldots)$ is provable in the intuitionistic propositional calculus (IPC), then $K(A(x, y, \ldots)) = O(1)$ for any strings $x, y, \ldots$. Indeed, there exists a string $s$ that belongs to $A(x, y, \ldots)$ for all $x, y, \ldots$ (induction by the length of the proof in IPC). See also [4] where a slightly different approach using Scott domains is used.

We do not know whether the reverse implication is true (i.e., whether $K(A(x, y, \ldots)) = O(1)$ implies that $A$ is provable in IPC) for any propositional formula $A$ containing $\wedge$, $\vee$ and $\to$. It is easy to see, however, that if $K(A(x, y, \ldots)) = O(1)$ then $A$ is provable in classical propositional calculus.

**Remark 2.** It is easy to see that

$$K(B) \leq K(A) + K(A \to B) + O(\log K(A \to B)))$$

Indeed, one can combine (self-delimiting encoding of) a program from $A \to B$ and (encoding of) any element of $A$ to get an encoding of some element of $B$.

We can combine this remark with the previous one: if $A(p, q, \ldots) \to B(p, q, \ldots)$ is provable in IPC, then $K(B(x, y, \ldots)) \leq K(A(x, y, \ldots)) + O(1)$ for all strings $x, y, \ldots$ For example, the formula $(x \vee y) \to ((x \to y) \to y))$ is provable in IPC, therefore $K((x \to y) \to y)) \leq K(x \vee y) + O(1) = \min(K(x), K(y)) + O(1)$ (as we mentioned in example 3 above).

**Example 5**. The Pierce law $((x \to y) \to x) \to x$ (being provable in classical logic) is not provable in IPC. However, Pierce law has low complexity: $K(((x \to y) \to x) \to x) = O(\log n)$ for any strings $x, y$ of length at most $n$. Indeed, let $s$ belong to $(x \to y) \to x$. This time call a pair $(u, v) \in S \times S$ *$s$-coherent* if $[s](l_\tau) = u$ for any $\tau$ such that $\tau(u) = v$. If $(u, v)$ is $s$-coherent then $u = x$, since otherwise there exists $\tau$ with $\tau(x) = y$, $\tau(u) = v$ and we have $x = [s](l_\tau) = u$. Given $s$ find an $s$-coherent pair and output its first component. This instruction describes a program from $((x \to y) \to x) \to x$ of complexity $\log n$ (note that we need to know $n$).

Recalling Remark 2, we see again that $K(x) \leq K((x \to y) \to x) + O(\log n)$ (cf. Example 3, last sentence).

However, as the next example shows, the inequality for complexities may be valid even in the case when the corresponding implication has large complexity.

**Example 6.** $K(((x \to y) \to y) \to (x \vee y)) = \min(K(x|y), K(y|x)) + O(\log(|x| + |y|))$.

(Recall that $K((x \to y) \to y) = K(x \vee y) + O(\log n)$, as we have seen in Example 3, so one may expect that $K(((x \to y) \to y) \to (x \vee y)) = O(\log n)$. But this is not the case.)

The inequality $K(((x \to y) \to y) \to (x \vee y)) \leq K(y|x) + O(1)$ is straightforward since given a program $p$ with $[p](x) = y$ and a program $s$ in $(x \to y) \to y$ we can find $y$ (because $s[p] = y$).

Let us prove that $K(((x \to y) \to y) \to (x \vee y)) \leq K(x|y) + O(\log n)$, where $n = \max\{|x|, |y|\}$. It suffices to prove that given the triple $\langle n$, a program $p$ with $[p](y) = x$, a program $s$ in the set $(x \to y) \to y\rangle$ we are able to find either $x$ or $y$. Recall the notion of $s$-coherent pair (see Example 3). Given $n$, $p$ and $s$ find an $s$-coherent pair $(u, v)$. Then continue to enumerate other $s$-coherent pairs and run in parallel $p$ on input $v$. We stop if we either find another $s$-coherent pair $(u', v')$, or we find out that $[p](v) = u$. In the former case we know either $x$, or $y$: if $v' \neq v$ then $x = u$ and if $u' \neq u$ then $y = v$ (recall that either the first component of all $s$-coherent pairs is equal to $x$, or the second component of all $s$-coherent pairs is equal to $y$). In the latter case (when $[p](v) = u$) we know that $x = u$. Indeed, if $x \neq u$ then $y = v$ hence $u = [p](v) = [p](y) = x$. Note that computation terminates (if there are no other $s$-coherent pairs except $(u, v)$ then $(u, v) = (x, y)$ hence $[p](v) = u$).

Let us prove that $K(((x \to y) \to y) \to (x \vee y)) \geq \min\{K(y|x), K(x|y)\} - O(1)$. Assume that a program $p$ is in the set $((x \to y) \to y) \to (x \vee y)$. We wish to prove that either given $p$ we can find a program $r_1$ with $[r_1](x) = y$, or given $p$ we can find a program $r_2$ with $[r_2](y) = x$.

We know that $p[s] = \langle 0, x\rangle$ or $p[s] = \langle 1, y\rangle$ for any $s$ in $(x \to y) \to y$. Using a standard trick from recursion theory, we choose a pair $A, B$ of enumerable inseparable subsets of $\mathbb{N}$. For any $i \in \mathbb{N}$ and any strings $u, v$ consider the following program $q_i(u, v)$ in $(u \to v) \to v$: given a program $s$ run it on input $u$ and enumerate in parallel $A$ and $B$; if it turns out that $[s](u) = v$ then output $v$ and stop, if it turns out that $i \in A$ then output $v$ and stop, if it turns out that $i \in B$ and $[s](u)$ is defined then output $[s](u)$ and stop (note that it does not matter which option to choose if several of them happen simultaneously; note also that output is undefined if $[s](u)$ is undefined and $i \notin B$).

As for any $i$ the program $q_i(x, y)$ is in $(x \to y) \to y$, the program $p$ applied to $q_i(x, y)$ outputs either $\langle 0, x\rangle$ or $\langle 1, y\rangle$. And either there is $i \in A$ such that $[p](q_i(x, y)) = \langle 0, x\rangle$, or there is $i \in B$ such that $[p](q_i(x, y)) = \langle 1, y\rangle$ (otherwise the decidable set $\{i \mid [p](q_i(x, y)) = \langle 1, y\rangle\}$ separates $A$ and $B$). In the former case given $p$ and $y$ we are able to find $x$: find $i \in A$ and $u$ such that $[p](q_i(u, y)) = \langle 0, u\rangle$ and output $u$; note that $q_i(u, y)$ is in $(x \to y) \to y$ hence $u = x$. In the latter case given $p$ and $x$ we are able to find $y$: find $i \in B$ and $v$ such that $[p](q_i(x, v)) = \langle 1, v\rangle$ and output $v$; note that $q_i(x, v)$ is in $(x \to y) \to y$ hence $v = y$. (End of proof.)

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