IP = PSPACE: Simplified Proof

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Abstract. Lund et al. [1] have proved that PH is contained in IP. Shamir [2] improved this technique and proved that PSPACE = IP. In this note, a slightly simplified version of Shamir's proof is presented, using degree reductions instead of simple QBFs.

Categories and Subject Descriptors: F. 1. 2 [**Computation by Abstract Devices**]: Modes of computation—*Alternation and nondeterminism*; *probabilistic computation*; F.1.3 [**Computation by Abstract Devices**]: Complexity classes—*relation among complexity classes*; F.4.1 [**Mathematical Logic and Formal Languages**]; Mathematical Logic—*proof theory*

General Terms: Theory

Additional Key Words and Phrases: Interactive proofs, PSPACE

1. *Introduction*

It is well known that IP is contained in PSPACE. So, for equality, it is enough to show that some PSPACE-complete language has an IP-protocol. We use the language of true Quantified Boolean Formulas (QBF), that is, formulas $Q_1 x_1 \cdots Q_n x_n B(x_1 \cdots x_n)$, where $B(x_1 \cdots x_n)$ is a Boolean formula (without quantifiers) and $Q_1 \cdots Q_n \in \{\forall, \exists\}$.

Each Boolean formula $B(x_1 \cdots x_n)$ corresponds to a polynomial $b(x_1 \cdots x_n)$ where $\alpha \wedge \beta$ is replaced by $\alpha \cdot \beta$, $\neg \alpha$ by $1 - \alpha$ and $\alpha \vee \beta$ by $\alpha * \beta = \alpha + \beta - \alpha \cdot \beta (= 1 - (1 - \alpha)(1 - \beta))$. Its value coincides with the value of B on boolean arguments ($0 =$ False, $1 =$ True).

Let $P(x, \dots)$ be a polynomial. Define three polynomials

$$(AxP)(\cdots) = P(0, \dots) \cdot P(1, \dots),$$
$$(ExP)(\cdots) = P(0, \dots) * P(1, \dots),$$
$$(RxP)(x, \dots) = P \bmod (x^2 - x)$$
$$\text{(i.e., all } x^n \text{ with } n > 1 \text{ are replaced by } x\text{).}$$

The polynomial RxP has the same variables as P; in AxP and ExP, variable x is absent. Note that P and RxP coincide on Boolean arguments.

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Let $S(x_1 \cdots x_k)$ be a polynomial over some finite field F. Assume that we know an IP-protocol α allowing P to persuade V that $S(u_1 \cdots u_k) = v$ with probability 1 for any input $u_1 \cdots u_k, v \in F$ when it is true and with probability less than ϵ when it is false. Let U be a polynomial obtained from S by one of the operations (Ax, Ex, Rx). Let the degree of S with respect to x be less than some constant d (known to V). We construct a protocol β allowing P to persuade V that $U(c_1 \cdots c_l) = e$ with probability 1 for any input $c_1 \cdots c_l, e$ when it is true and with probability $< \epsilon + d/\#F$ when it is false. (Here, $\#F$ denotes the cardinality of F.) This protocol uses α as a procedure *called only once*.

2. *A Construction of IP-protocol* β

Case A. $U(y_1 \cdots y_l) = AxS(x, y_1 \cdots y_l)$.

P wants to persuade V that $U(c_1 \cdots c_l) = e$. P sends V the coefficients of a polynomial $s(x) = S(x, c_1 \cdots c_l)$. If $degree(s) > d$ or $s(0)s(1) \neq e$, V rejects. Otherwise, V sends P a random element $r \in F$. Now (using protocol α), P must persuade V that $S(r, c_1 \cdots c_l) = s(r)$.

Case E. $U(y_1 \cdots y_l) = ExS(x, y_1 \cdots y_l)$.

Replace $s(0)s(1)$ by $s(0) * s(1)$.

Case R. $U(x, y_1 \cdots y_l) = RxS(x, y_1 \cdots y_l)$.

P wants to persuade V that $U(f, c_1 \cdots c_l) = e$. P sends V the coefficients of a polynomial $s(x) = S(x, c_1 \cdots c_l)$. If $degree(s) > d$ or $s(0) + (s(1) - s(0))f \neq e$, V rejects (note that $s(0) + (s(1) - s(0))f$ is the value of $s(x) \bmod (x^2 - x)$ at f). Otherwise, V sends P a random element $r \in F$. Now (using protocol α), P must persuade V that $S(r, c_1 \cdots c_l) = s(r)$.

P can fool V either during α (probability less than ϵ) or if different polynomials $s(x)$ and $S(x, c_1 \cdots c_l)$ coincide at the random point r (probability not greater than $d/\#F$).

Let $\phi = Q_1 x_1 \cdots Q_n x_n B(x_1 \cdots x_n)$ be a *QBF*; $Q_1 \cdots Q_n \in \{\forall, \exists\}$. Consider a polynomial $b(x_1 \cdots x_n)$ corresponding to $B(x_1 \cdots x_n)$ and apply (sequentially) operations

$$
\begin{aligned}
&Rx_1, Rx_2, \ldots, Rx_n,\\
&q_n x_n,\\
&Rx_1, Rx_2, \ldots, Rx_{n-1},\\
&q_{n-1} x_{n-1},\\
&\vdots\\
&Rx_1, Rx_2,\\
&q_1 x_1,
\end{aligned}
$$

where $q_i = A$ or E if $Q_i = \forall$ or $\exists$, respectively. After these operations, we get a constant equal to 0 or 1, depending on the truth value of ϕ. P can persuade V that this constant is 1 using the reduction steps described. Ultimately, the equality $b(u_1 \cdots u_n) = v$ must be checked for some $u_1 \cdots u_n, v$; V can do this alone because the formula B is known. The probability of error does not exceed

$$\frac{(\text{number of operations A, E, R}) \cdot (\text{maximal degree})}{(\#F)}.$$

If the length of QBF was l, then number of operations is $O(l^2)$ and maximal degree is $O(l)$ (degree of t does not exceed l, R-operations reduce it to 1 and later all degrees are not greater than 2). If $\#F$ is about l^4, the probability of error tends to 0 when $l \to \infty$. So we can use $F = Z/pZ$ where p is a prime of logarithmic length (p can be chosen by P or V because primality testing is trivial for numbers of this size). It is easy to see that Verifier is weak in the sense of Shamir [2].

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