Probabilistic Proofs, Kolmogorov Compleixity and Laszlo Lovasz Local Lemma

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- If a random variable has an expectation greater than c, it is sometimes greater than c

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- Similar argument: every 3-CNF has an assignment that satisfies at least $7/8$ of all clauses

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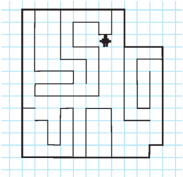

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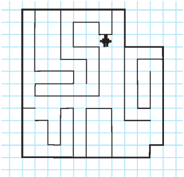

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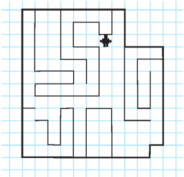

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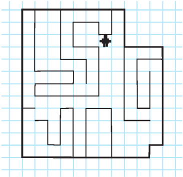

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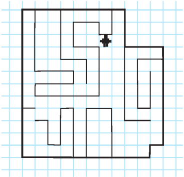

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- Theorem: for every board size there exists a sequence that guarantees that the robot visits all cells (independent of the maze and initial position).

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- for this k a random program of length kN with positive probability works for all mazes and initial positions

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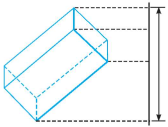

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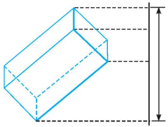

- (Expected value of the projection to a random line is proportional to the sum of dimensions)

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- Probability argument: take a random $n \times n$ matrix. For a given position of a minor the probability to see an uniform minor there is $2^{-k^2} \times 2$. There are at most n^{2k} positions for a $k \times k$ minor. So if $n^{2k}2^{-k^2+1} < 1$, a matrix without uniform minors exists. Taking logarithms, we get $2k\log n - k^2 + 1 < 0$ which is guaranteed if $k > 2\log n + 1$.

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- Complexity version: let us prove that a incompressible matrix has no uniform minors. In other words, a matrix that has uniform minor is compressible. Indeed, in can be described by specifying the position of that minor ($2k$ indices in $1 \dots n$ range, i.e., $2k \log n$ bits), the bit in the minor (1 bit) and the remaining $n^2 - k^2$ bits in the matrix, so if $2k \log n + 1 + n^2 - k^2 < n^2$, the matrix is compressible

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- This is not a rigorous proof since complexity is defined up to a constant.

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- Sometimes an open problem (e.g., the existence of Boolean functions that require circuits of exponential size)
- Sometime explicit constructions exist but are rather complicated and do no achieve the best possible parameters (expanders, codes)

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- Laslo Lovasz Local Lemma deals with partial independence
- Each node in a rectangular grid may have one of 10 colors; each edge prohibits one of 100 color combinations (different for different edges). LLLL guarantees that there exists a coloring that satisfies all restrictions.

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Let $A_1, \ldots, A_n$ are events indexed by vertices of an (undirected) graph. Let $N(i)$ be the set of all neighbors of i (not including i). Assume that for every i the event A_i is independent with the tuple of all events A_j with $j \notin N(i)$. Assume that for every i an upper bound $\varepsilon_i < 1$ for $\Pr[A_i]$ is chosen and, moreover,

$$\Pr[A_i] \leq \varepsilon_i \prod_{j \in N(i)} (1 - \varepsilon_j).$$

Then

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- If there is no information about dependence (complete graph), and $\sum \Pr[A_i] < 1/4$, one can let $\varepsilon_i = 2\Pr[A_i]$: the product of $(1 - \varepsilon_i)$ is at least $1 - \sum \varepsilon_i > 1/2$.

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- In our example events correspond to edges; neighbors are edges that share a vertex (6 of them). Choosing the same ε for every edge, we need

$$1/100 < \varepsilon(1 - \varepsilon)^6$$

If $\varepsilon = 1/6$, the rhs is about $1/6e \gg 1/100$.

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- using induction (less events in the condition)

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- (Kolmogorov complexity version) There exists a sequence ω such that any substring x is ω has complexity at least $\alpha|x| - O(1)$.

Statements are equivalent: there is at most $2^{\alpha n}$ sequences of length n and complexity $< \alpha n$; on the other hand, if X is a set of forbidden strings and there is at most $2^{\alpha n}$ forbidden strings of length n, they are all simple (have compexity $\alpha n + o(n)$) relative to X. (Non-relativized version can be also used.)

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- such a block exists since we can take a block that is random relative to the prefix of the sequence; each group of s consequtive blocks increases complexity at least by βsM and therefore has complexity at least βsM. For non-aligned blocks we discard some part of them (using the difference between α and β to compensate for the losses).

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- Corollary: if A has small complexity with respect to every its element, then $K(\omega(A))$ is large.

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- In other terms, we have Boolean variables (bits of ω) and clauses: e.g., $w_5 \vee \neg w_7 \vee w_{11}$ says that $\omega(\{5, 7, 11\}) \neq 010$.
- Rumyantsev: for every $\alpha < 1$ there exists a sequence ω such that for every finite A the complexity $K(A, \omega(A)|t)$ exceeds $\alpha\#A - O(1)$ for some $t \in A$. [Proof: use LLLL]
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- and a two-dimensional sequence such that every rectangle has a high complexity (close to its area)

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- but a very simple argument exists for special cases (as explained by Fortnow using complexity)

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- ...and it can be found with high probability by a polynomial probabilistic algorithm

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- The correctness of *Fix* assuming it terminates is trivial (induction: if recursive calls are correct, the calling procedure is correct)
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- Indeed, each clause is violated only for one combination of bits, so resampling can be "undone" and random bits can be extracted
- So if we can describe the sequence of resampled clauses using less than m bits per clause, we get a contradiction (N is fixed and for the large number of steps we get a contradiction with incompressibility)

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- so we need an additional bit (up/down) for recursive call (down step) and one bit to describe exits (up steps)
- so instead of $\log t$ per resampling we get $(\log t + 1) + 1$ – still less than m by assumption ($\log t < m - 3$)