Fixed point aperiodic tilings

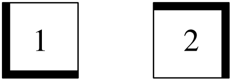
Bruno Durand, Andrei Romashchenko,
Alexander Shen
LIF Marseille,
CNRS & Univ. Aix–Marseille, France

AMS meeting, Jan. 2009

Tiles and tilings

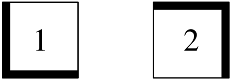
Tiles and tilings

Tile set:

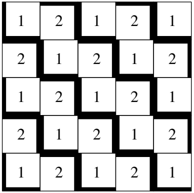


Tiles and tilings

Tile set:



Tiling:



Formal definitions

Formal definitions

C — a finite set of *colors*

Formal definitions

C — a finite set of *colors*

$\tau \subset C^4$ — a set of *tiles* (Wang tiles)

tile $t = \langle t.\text{left},\ t.\text{right},\ t.\text{up},\ t.\text{down} \rangle$

Formal definitions

C — a finite set of *colors*

$\tau \subset C^4$ — a set of *tiles* (Wang tiles)
tile $t = \langle t.\text{left},\ t.\text{right},\ t.\text{up},\ t.\text{down}\rangle$

configurations: mappings $\mathbb{Z}^2 \to \tau$

Formal definitions

C — a finite set of *colors*

$\tau \subset C^4$ — a set of *tiles* (Wang tiles)
tile $t = \langle t.\text{left},\ t.\text{right},\ t.\text{up},\ t.\text{down}\rangle$

configurations: mappings $\mathbb{Z}^2 \to \tau$

tilings: configuration that satisfy matching rules

$$C(i,j).\text{right} = C(i+1,j).\text{left}$$

$$C(i,j).\text{up} = C(i,j+1).\text{down}$$

Periodic tilings

Periodic tilings

A tiling C is *periodic* if it has some *period* T:

$$C(x + T) = C(x)$$

for all x.

Periodic tilings

A tiling C is *periodic* if it has some *period* T:

$$C(x + T) = C(x)$$

for all x.
Four possibilities for a tile set:

Periodic tilings

A tiling C is *periodic* if it has some *period* T:

$$C(x+T) = C(x)$$

for all x.

Four possibilities for a tile set:

- no tilings;

Periodic tilings

A tiling C is *periodic* if it has some *period* T:

$$C(x + T) = C(x)$$

for all x.

Four possibilities for a tile set:

- no tilings;
- only periodic tilings;

Periodic tilings

A tiling C is *periodic* if it has some *period* T:

$$C(x + T) = C(x)$$

for all x.

Four possibilities for a tile set:

- no tilings;
- only periodic tilings;
- both periodic and aperiodic tilings;

Periodic tilings

A tiling C is *periodic* if it has some *period* T:

$$C(x + T) = C(x)$$

for all x.

Four possibilities for a tile set:

- no tilings;
- only periodic tilings;
- both periodic and aperiodic tilings;
- only aperiodic tilings;

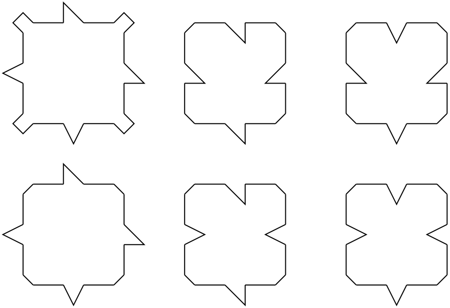
Aperiodic tile sets

Aperiodic tile sets

Theorem (Berger, 1966): there exists a tile set that has tilings but only aperiodic ones

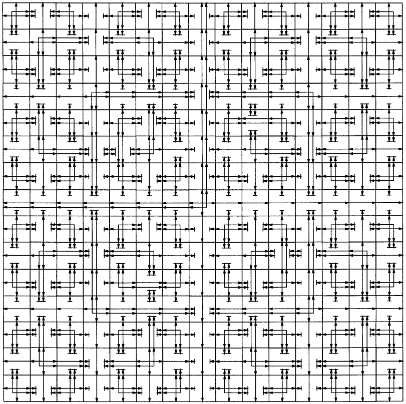
Aperiodic tile sets

Theorem (Berger, 1966): there exists a tile set that has tilings but only aperiodic ones

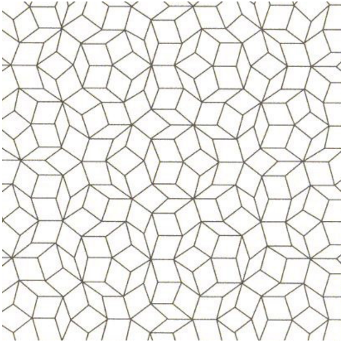


(Robinson tile set)

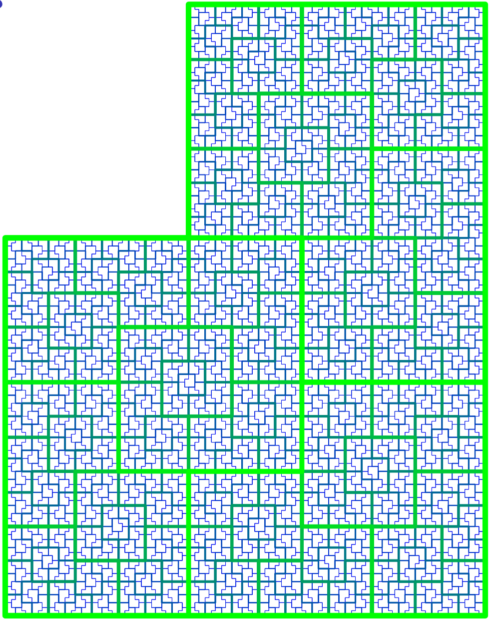
Tiling



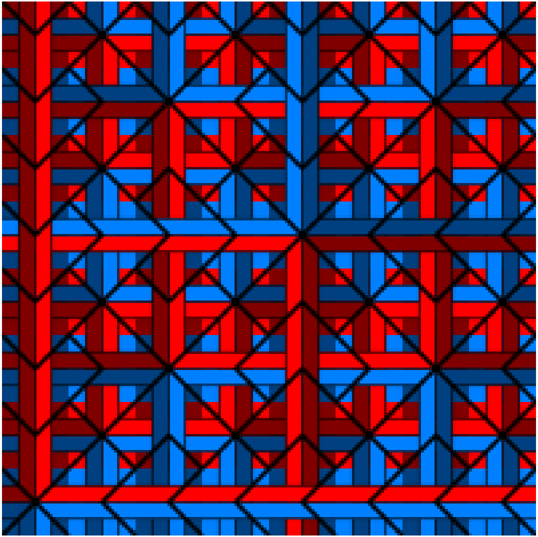
Penrose tiling



Ammann tiling



Ollinger tiling



Berger's theorem and theory of computation

Berger's theorem and theory of computation

- the question was asked by Hao Wang when he studied decision problems

Berger's theorem and theory of computation

- the question was asked by Hao Wang when he studied decision problems
- Berger's construction became an important tool to prove undecidability of many algorithmic problems

Berger's theorem and theory of computation

- the question was asked by Hao Wang when he studied decision problems
- Berger's construction became an important tool to prove undecidability of many algorithmic problems
- Aperiodic tiling can be constructed using self-referential argument widely used in logic and computation theory (Kleene's fixed point theorem)

History

History

- Hao Wang (1961) asked whether aperiodic tilings exist in connection with domino problem;

History

- Hao Wang (1961) asked whether aperiodic tilings exist in connection with domino problem;
- Alternative history: the self-referential aperiodic tile set should have been invented by von Neumann, inventor of self-reproducing automata (1952), but he died in 1957 and his work on cellular automata was published only in 1966

History

- Hao Wang (1961) asked whether aperiodic tilings exist in connection with domino problem;
- Alternative history: the self-referential aperiodic tile set should have been invented by von Neumann, inventor of self-reproducing automata (1952), but he died in 1957 and his work on cellular automata was published only in 1966
- Berger (1966) proved the existence of aperiodic tile sets and used this construction to prove the undecidability of the domino problem;

History

- Hao Wang (1961) asked whether aperiodic tilings exist in connection with domino problem;
- Alternative history: the self-referential aperiodic tile set should have been invented by von Neumann, inventor of self-reproducing automata (1952), but he died in 1957 and his work on cellular automata was published only in 1966
- Berger (1966) proved the existence of aperiodic tile sets and used this construction to prove the undecidability of the domino problem;
- Robinson tiling (1971)

History

- Hao Wang (1961) asked whether aperiodic tilings exist in connection with domino problem;
- Alternative history: the self-referential aperiodic tile set should have been invented by von Neumann, inventor of self-reproducing automata (1952), but he died in 1957 and his work on cellular automata was published only in 1966
- Berger (1966) proved the existence of aperiodic tile sets and used this construction to prove the undecidability of the domino problem;
- Robinson tiling (1971)
- Penrose tiling (1974)

History

- Hao Wang (1961) asked whether aperiodic tilings exist in connection with domino problem;
- Alternative history: the self-referential aperiodic tile set should have been invented by von Neumann, inventor of self-reproducing automata (1952), but he died in 1957 and his work on cellular automata was published only in 1966
- Berger (1966) proved the existence of aperiodic tile sets and used this construction to prove the undecidability of the domino problem;
- Robinson tiling (1971)
- Penrose tiling (1974)
- ...

History

- Hao Wang (1961) asked whether aperiodic tilings exist in connection with domino problem;
- Alternative history: the self-referential aperiodic tile set should have been invented by von Neumann, inventor of self-reproducing automata (1952), but he died in 1957 and his work on cellular automata was published only in 1966
- Berger (1966) proved the existence of aperiodic tile sets and used this construction to prove the undecidability of the domino problem;
- Robinson tiling (1971)
- Penrose tiling (1974)
- ...
- Ollinger tiling (2007)

Self-similar tile sets

Fix a integer *zoom factor* $M > 1$.

Self-similar tile sets

Fix a integer *zoom factor* $M > 1$.

Let τ be a tile set. A τ-*macro-tile* is a $M \times M$ square correctly tiled by τ-tiles.

Self-similar tile sets

Fix a integer *zoom factor* $M > 1$.

Let τ be a tile set. A τ-*macro-tile* is a $M \times M$ square correctly tiled by τ-tiles.

Let ρ be a set of τ-macro-tiles. We say that τ *implements* ρ if any τ-tiling can be uniquely split by a grid into ρ-macro-tiles

Self-similar tile sets

Fix a integer *zoom factor* $M > 1$.

Let τ be a tile set. A τ-*macro-tile* is a $M \times M$ square correctly tiled by τ-tiles.

Let ρ be a set of τ-macro-tiles. We say that τ *implements* ρ if any τ-tiling can be uniquely split by a grid into ρ-macro-tiles

Tile set τ is *self-similar* if it implements some set of macro-tiles ρ that is isomorphic to τ
(Isomorphism: 1-1-correspondence that preserves matching rules)

Berger's theorem and self-similar tile sets

Berger theorem follows from two statements:

Berger's theorem and self-similar tile sets

Berger theorem follows from two statements:

A. Any tiling by a self-similar tile set is aperiodic

Berger's theorem and self-similar tile sets

Berger theorem follows from two statements:

A. Any tiling by a self-similar tile set is aperiodic

B. There exists a self-similar tile set.

Proof of A

Let τ be a self-similar tile set with zoom factor M

Proof of A

Let τ be a self-similar tile set with zoom factor M
Let U be τ-tiling

Proof of A

Let τ be a self-similar tile set with zoom factor M
Let U be τ-tiling
Let T be a period of U

Proof of A

Let τ be a self-similar tile set with zoom factor M
Let U be τ-tiling
Let T be a period of U
U can be splitted into macro-tiles; T-shift preserves this splitting (uniqueness) and therefore T is a multiple of M

Proof of A

Let τ be a self-similar tile set with zoom factor M
Let U be τ-tiling
Let T be a period of U
U can be splitted into macro-tiles; T-shift preserves this splitting (uniqueness) and therefore T is a multiple of M
Zoom out: T/M is a period of a tiling by a tile set isomorphic to τ

Proof of A

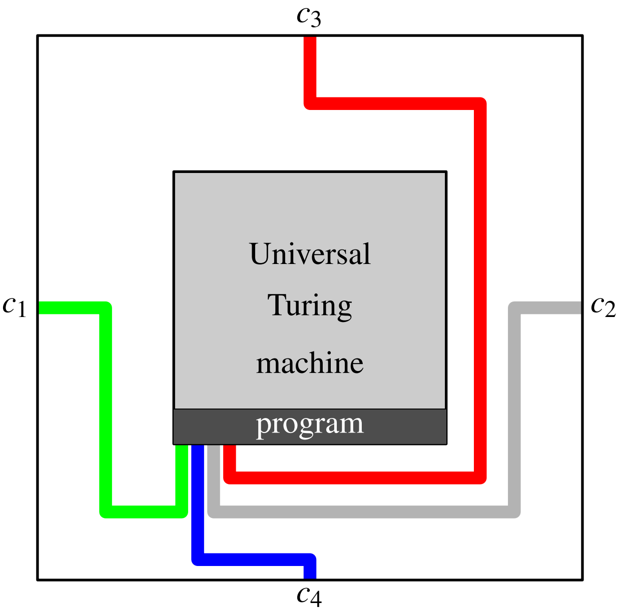
Let τ be a self-similar tile set with zoom factor M
Let U be τ-tiling
Let T be a period of U
U can be splitted into macro-tiles; T-shift preserves this splitting (uniqueness) and therefore T is a multiple of M
Zoom out: T/M is a period of a tiling by a tile set isomorphic to τ
T/M is a multiple of M etc.

Self-referential tile set

- For a given tile set σ we construct a tile set τ that implements σ
- This gives a mapping $\sigma \rightarrow \tau(\sigma)$
- It remains to find a fixed point:

$$\tau(\sigma) \text{ is isomorphic to } \sigma$$

The structure of a macro-tile that implements itself



Applications

Applications

- tile sets with variable zoom factor

Applications

- tile sets with variable zoom factor
- strongly aperiodic tile sets (each shift changes 99% positions)

Applications

- tile sets with variable zoom factor
- strongly aperiodic tile sets (each shift changes 99% positions)
- robust aperiodic tile sets (isolated or sparse holes can be patched)

Applications

- tile sets with variable zoom factor
- strongly aperiodic tile sets (each shift changes 99% positions)
- robust aperiodic tile sets (isolated or sparse holes can be patched)
- simple proof of the unidecidability of the domino problem

Applications

- tile sets with variable zoom factor
- strongly aperiodic tile sets (each shift changes 99% positions)
- robust aperiodic tile sets (isolated or sparse holes can be patched)
- simple proof of the unidecidability of the domino problem
- simple construction of a tile set that has only complex tilings

Applications

- tile sets with variable zoom factor
- strongly aperiodic tile sets (each shift changes 99% positions)
- robust aperiodic tile sets (isolated or sparse holes can be patched)
- simple proof of the unidecidability of the domino problem
- simple construction of a tile set that has only complex tilings
- tile set with any computable density