Ergodic-type characterizations of algorithmic randomness

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Abstract. A theorem of Kučera states that given a Martin-Löf random infinite binary sequence ω and an effectively open set A of measure less than 1, some tail of ω is not in A. We show that this result can be seen as an effective version of Birkhoff's ergodic theorem (in a special case). We prove several results in the same spirit and generalize them via an effective ergodic theorem for bijective ergodic maps.

1 Introduction

The classical setting for the ergodic theorem is as follows. Let X be a space with a probability measure μ on it, and let $T\colon X \to X$ be a measure-preserving transformation. Let f be a real-valued integrable function on X. Birkhoff's ergodic theorem (see for example [Shi96]) says that the time-average

$$\frac{f(x) + f(T(x)) + f(T(T(x))) + \ldots + f(T^{(n-1)}(x))}{n}$$

has a limit (as $n \to \infty$) for all x except for some null set, and this limit (the "time average") equals the space average $\int f(x)\,d\mu(x)$ if the transformation T is ergodic (i.e., has no non-trivial invariant subsets).

The classical example is the left shift on Cantor space Ω (the set of infinite binary sequences, denoted also by $2^{\mathbb{N}}$ or 2^{ω}): $\sigma(\omega_0\omega_1\omega_2\ldots) = \omega_1\omega_2\ldots$ It preserves Lebesgue measure (a.k.a. uniform measure) μ on Ω and is ergodic. Therefore, the time and space averages coincide for almost every starting point x. For a special case where f is an indicator function of some (measurable) set A, we conclude that almost surely (for all x outside some null set) the fraction of terms in the sequence $x, \sigma(x), \sigma(\sigma(x)), \ldots$ that are inside A, converges to the measure of A.

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Assuming that A has positive measure, we conclude that almost surely at least one element of this sequence belongs to A. Switching to complements: if A has measure less than 1, then (almost surely) some elements of this sequence are outside A. Kučera [Kuč85] proved an effective version of this statement:

Theorem 1. *If A is an effectively open set of measure less than 1, then for every Martin-Löf random sequence ω at least one of ω, $\sigma(\omega)$, $\sigma(\sigma(\omega))$,... does not belong to A.*

Recalling the definition of Martin-Löf randomness (a sequence is random if it is outside any effectively null set) we can reformulate Kučera's theorem as follows:

> *Let A be an effectively open set of measure less than 1. Consider the set A^* of all sequences ω such that every tail $\sigma^{(n)}(\omega)$ belongs to A. Then A^* is an effectively null set.*

Before presenting the proof, let us mention an interpretation of this result. Recall that the universal Martin-Löf test is a computable sequence $U_1, U_2, \ldots$ of effectively open sets such that $\mu(U_i) \leq 1/2^i$ and the intersection $\cap_i U_i$ is the maximal effectively null set, i.e., the set of all non-random sequences. Kučera's theorem shows that randomness can be (in a paradoxical way) characterized by U_1 alone: a sequence is non-random if and only if all its tails belong to U_1. (In one direction it is Kučera's theorem, in the other direction we need to note that a tail of a non-random sequence is non-random.)

Proof (of Kučera's theorem). We start with the following observation: it is enough to show that for every interval I, we can uniformly construct an effectively open set $J \subset I$ that contains $I \cap A^*$ and such that $\mu(J) \leq r\mu(I)$ for some fixed $r < 1$ (here we call an *interval* any set of type $x\Omega$, where x is some finite string, which is the set of infinite binary sequences that start with x). Then we represent the effectively open set A of measure $r < 1$ as a union of disjoint intervals $I_1, I_2, \ldots$, construct the sets J_i for every I_i and note that the union A_1 of all J_i is an effectively open set that contains A^* and has measure r^2 or less. Splitting A_1 into disjoint intervals and repeating this argument, we get a set A_2 of measure at most r^3, etc. In this way we get a effectively open cover for A^* of arbitrarily small measure, so A^* is an effectively null set.

It remains to show how to find J given I. The interval I consists of all sequences that start with some fixed prefix x, i.e., $I = x\Omega$. Since sequences in A^* have all their tails in A, the intersection $I \cap A^*$ is contained in xA, and the latter set has measure $r\mu(I)$ (where $r = \mu(A)$). □

Note that this proof also shows the following: suppose A is an effectively open set of measure less than 1, and A can be written as a disjoint union of intervals $A = x_1\Omega \cup x_2\Omega \cup \ldots$. Let ω be an infinite sequence that can be written as $\omega = w_1 w_2 w_3 \ldots$ where for all i, $w_i = x_j$ for some j. Then ω is not random. (If A contains all non-random sequences, the reverse implication is also true, and we get yet another criterion of randomness.)

Effective versions of the ergodic theorem in a general setting have been studied in several papers (see for example [V'y97,V'y98,GHR09,HR09a]). In this paper, we present characterizations of randomness that resemble Kučera's and which (to the best of our knowledge) cannot be directly derived from previous papers.

2 Effective Kolmogorov 0-1-law

Trying to find characterizations of randomness similar to Kučera's theorem, one may look at Kolmogorov's 0-1-law. It says that any measurable subset A of the Cantor space that is stable under finite changes of bits (i.e. if $\omega \in A$ and ω' is equal to ω up to a finite change of bits, then $\omega' \in A$) has measure 0 or 1. It can be reformulated as follows: let A be a (measurable) set of measure less than 1. Consider the set A^* defined as follows: $\omega \in A^*$ if and only if all sequences that are obtained from ω by changing finitely many terms, belong to A. Then A^* has measure zero (indeed, A^* is stable and cannot have measure 1). Note also that we may assume without loss of generality that A is open (replacing it by an open cover of measure less than 1).

A natural effective version of Kolmogorov's 0-1-law can then be formulated as follows.

Theorem 2. *Let A be an effectively open set of measure $r < 1$. Consider the set A^* of all sequences that belong to A and remain in A after changing finitely many terms. Then A^* is an effectively null set.*

(As we have seen, the last two sentences can be replaced by the following claim: *any Martin-Löf random sequence can be moved outside A by changing finitely many terms.*)

Proof. To prove this effective version of the 0-1-law, consider any interval I. As before, we want to find an effectively open set $U \subset I$ that contains $A^* \cap I$ and has measure at most $r\mu(I)$. Let x be a prefix that defines I, i.e., $I = x\Omega$. For every string y of the same length as x, consider the set $A_y = \{\omega \mid y\omega \in A\}$. It is easy to see that the average measure of A_y (over all y of a given length) equals $\mu(A) = r$. Therefore, the set $B = \bigcap_y A_y$ (which is effectively open as an intersection of an effectively defined finite family of open sets) has measure at most r. Now take $U = xB$. Let us show that U is as wanted. First U is an effectively open set, contained in I, and of measure $r\mu(I)$. Also, it contains every element of $A^* \cap I$. Indeed, if $\alpha \in A^* \cap I$, x is a prefix of α, so one can write $\alpha = x\beta$. Since $\alpha \in A^*$, any finite variation of α is in A, so for all y of the same length as x, $y\beta \in A$. Therefore, β is in all A_y, and therefore is in B. Since $\alpha = x\beta$, it follows that α is in $xB = U$. □

3 Adding prefixes

We have considered left shifts (deletion of prefixes) and finite changes. Another natural question is about *adding* finite prefixes. It turns out that a similar result can be proven in this case (although the proof becomes a bit more difficult).

Theorem 3. *Let A be an effectively open set of measure $r < 1$. Let A^* be the set of all sequences ω such that $x\omega \in A$ for every binary string x. Then A^* is an effectively null set.*

(Reformulation: *for every Martin-Löf random sequence ω there exists a string x such that $x\omega \notin A$.*)

Proof. To prove this statement, consider again some interval $I = x\Omega$. We want to cover $A^* \cap I$ by an effectively open set of measure $r\mu(I)$. (In fact, we get a cover of measure $s\mu(I)$ for some constant $s \in (r, 1)$, but this is enough.) Consider some string z. We know that the density of A^* in I does not exceed the density of A in $zI = zx\Omega$. Indeed, $x\omega \in A^*$ implies $zx\omega \in A$ by definition of A^*.

Moreover, for any finite number of strings $z_1, \ldots, z_k$ the set A^* is contained in the intersection of sets $\{\omega \mid z_i\omega \in A\}$, and the density of A^* in I is bounded by the minimal (over i) density of A in $z_iI = z_ix\Omega$.

Now let us choose $z_1, \ldots, z_k$ in such a way that the intervals $z_ix\Omega$ are disjoint and cover Ω except for a set of small measure. This is possible for the same reason as in a classic argument that explains why the Cantor set in $[0, 1]$ has zero measure. We start, say, with $z_1 = \Lambda$ and get the first interval $x\Omega$. The rest of Ω can be represented as a union of disjoint intervals, and inside each interval $u\Omega$ we select a subinterval $ux\Omega$ thus multiplying the size of the remaining set by $(1 - 2^{-|x|})$. Since this procedure can be iterated indefinitely, we can make the rest as small as needed.

Then we note that the density of A in the union of disjoint intervals (and this density is close to r if the union covers Ω almost entirely) is greater than or equal to the density of A in one of the intervals, so the intersection (an effectively open set) has density at most s for some constant $s \in (r, 1)$, as we have claimed. (We need to use the intersection and not only one of the sets since our construction should be effective even when we do not know for which interval the density is minimal.) □

4 Bidirectional sequences and shifts

Recall the initial discussion in terms of ergodic theory. In this setting it is more natural to consider bi-infinite binary sequences, i.e., mappings of type $\mathbb{Z} \to \mathbb{B} = \{0, 1\}$; the uniform Bernoulli measure μ can be naturally defined on this space, too. On this space the transformation T corresponding to the shift to the left is reversible: any sequence can be shifted left or right.

The result of Theorem 1 remains true in this setting.

Theorem 4. *Let A be an effectively open set of measure $r < 1$. The set A^* of all sequences that remain in A after any arbitrary shift (any distance in any direction) is an effectively null set.*

To prove this statement, consider any $s \in (r, 1)$. As usual, it is enough to find (effectively) for every interval I_x an effectively open subset of I_x that contains $A^* \cap I_x$ and has measure at most $s\mu(I_x)$. Here x is a finite partial function from $\mathbb{Z}$ to $\mathbb{B}$ and I_x is the set of all its extensions. (One may assume that x is contiguous, since every other interval is a finite union of disjoint contiguous intervals, but this is not important for us.) Then we may iterate this construction, replacing each interval of an effectively open set by an open set inside this interval, and so on until the total measure (s^k, where k is the number of iterations) becomes smaller than any given $\varepsilon > 0$.

Assume that some I_x is given. Note that A^* is covered by every shift of A, so any intersection of I_x with a finite collection of shifted versions of A (i.e. sets of type $T^n(A)$ for $n \in \mathbb{Z}$) is a cover for $I_x \cap A^*$. It remains to show that the intersection of properly chosen shifts of A has density at most s inside I_x. To estimate the measure of the intersection, it is enough to consider the minimum of measures, and the minimum can be estimated by estimating the average measure.

More formally, we first note that by reversibility of the shift and the invariance of the measure, we have

$$\mu\big(I_x \cap T^{-n}(A)\big) = \mu\big(A \cap T^n(I_x)\big)$$

for all n. Then we prove the following lemma:

Lemma 1. *Let $J_1, \ldots, J_k$ be independent intervals of the same measure d corresponding to disjoint functions $x_1, \ldots, x_k$ of the same length. Then the average of the numbers*

$$\mu(A \cap J_1), \ldots, \mu(A \cap J_k)$$

does not exceed sd if k is large enough. Moreover such a k can be found effectively.

Proof (of Lemma 1). The average equals

$$\frac{1}{k} \sum_i \mathsf{E}(\chi_A \cdot \chi_i)$$

where χ_A is the indicator function of A and χ_i is the indicator function of J_i. Rewrite this as

$$\mathsf{E}\left(\chi_A \cdot \frac{1}{k} \sum_i \chi_i\right),$$

and note that

$$\frac{1}{k} \sum_i \chi_i$$

is the frequency of successes in k independent trials with individual probability d. (Since the functions x_i are disjoint, the corresponding intervals J_i are independent events.) This frequency (as a function on the bi-infinite Cantor space) is close to d everywhere except for a set of small measure (by the central limit theorem; in fact Chebyshev's inequality is enough). The discrepancy and the measure of this exceptional set can be made as small as needed using a large k, and the difference is then covered by the gap between r and s. This ends the proof of the lemma.

Now, given an interval I_x, we cover $I_x \cap A^*$ as follows. First, we take a integer N larger than the size of the interval I_x. The intervals

$$T^N(I_x), T^{2N}(I_x), T^{3N}(I_x), \ldots$$

are independent and have the same measure as I_x, so we can apply the above lemma and effectively find a k such that the average of

$$\mu(A \cap T^N(I_x)), \ldots, \mu(A \cap T^{kN}(I_x))$$

does not exceed $s\mu(I_x)$. This means that for some $i \leq k$ one has

$$\mu(I_x \cap T^{-iN}(A)) = \mu(A \cap T^{iN}(I_x)) \leq s\mu(I_x)$$

Therefore, $I_x \cap \bigcap_{i \leq k} T^{-iN}(A)$ is an effectively open cover of A^* of measure at most $s\mu(I_x)$. □

The statement can be strengthened: we can replace all shifts by any infinite enumerable family of shifts.

Theorem 5. *Let A be an effectively open set* (*of bi-infinite sequences*) *of measure $\alpha < 1$. Let S be an computably enumerable infinite set of integers. Then the set*

$$A^* = \{\omega \mid \omega \textit{ remains in } A \textit{ after shift by } s, \textit{ for every } s \in S\}$$

is an effectively null set.

(Reformulation: *let A be an effectively open set of measure less than* 1*; let S be an infinite computably enumerable set of integers; let α be a Martin-Löf random bi-infinite sequences. Then there exists $s \in S$ such that the s-shift of ω is not in A.*)

Proof. The proof remains the same: indeed, having infinitely many shifts, we can choose as many disjoint shifts of a given interval as we want. □

Our last argument is more complicated than the previous ones (that do not refer to the central limit theorem): previously we were able to use disjoint intervals instead of independent ones. In fact the results about shifts in unidirectional sequences (both) are corollaries of the last statement. Indeed, let A be an effectively open set of right-infinite sequences of measure less than 1. Let ω be a right-infinite Martin-Löf random sequence. Then it is a part of a bi-infinite random sequence $\bar{\omega}$ (one may use, e.g., van Lambalgen's theorem [vL87] on the random pairs, see last section for a precise statement). So there is a right shift that moves $\bar{\omega}$ outside $\bar{A}$, and also a left shift with the same property (here by $\bar{A}$ we denote the set of bi-infinite sequences whose right halves belong to A).

5 A generalization: computable ergodic transformations

First let us recall the notion of a computable transformation of Cantor space Ω. Consider a Turing machine with a read-only input tape and write-only output tape (where head prints a bit and moves to the next blank position). Such a machine determines a computable mapping of Ω into the space of all finite and infinite binary sequences. Restricting this mapping to the inputs where the output sequence is infinite, we get a (partial) computable mapping from Ω into Ω.

Theorem 6. *Let μ be a computable measure on Ω. Let $T : \Omega \to \Omega$ be a computable almost every defined measure-preserving ergodic transformation of Ω. Let A be an effectively open subset of Ω of measure less than 1. Let A^* be the set of points $x \in X$ such that $T^n(x) \in A$ for all $n \geq 0$. Then, A^* is an effectively null set.*

Proof. Let r be a real number such that $\mu(A) < r < 1$. As before, given an interval I, we want to effectively find an n such that $I \cap \bigcap_{i \leq n} T^{-i}(A)$ has measure at most $r\mu(I)$. This gives us an effectively open cover of $A^* \cap I$ having measure at most $r\mu(I)$; iterating this process, we conclude that A^* is an effectively null set.

(A technical clarification is needed here. If we consider T only on inputs where output sequence is infinite, the set $T^{-1}(A)$ (and in general $T^{-i}(A)$) is no more open in Ω. But since T is almost everywhere defined, we may extend T to the space of finite and infinite sequence in a natural way and get an effectively open cover of the same measure.)

To estimate $\mu(I \cap \bigcap_i T^{-i}(A))$, we note that it does not exceed the minimal value of $\mu(I \cap T^{-i}(A))$, which in its turn does not exceed the average (over $i \leq n$) of $\mu(I \cap T^{-i}(A))$. This average,

$$\tfrac{1}{n+1}\left[\mu(I \cap A) + \mu(I \cap T^{-1}(A)) + \ldots + \mu(I \cap T^{-n}(A))\right]$$

can be rewritten as

$$\tfrac{1}{n+1}\left[\mu(T^{-n}(I) \cap T^{-n}(A)) + \mu(T^{-(n-1)}(I) \cap T^{-n}(A)) + \ldots + \mu(I \cap T^{-n}(A))\right]$$

since T is measure preserving. The latter expression is the scalar product of the characteristic function of $T^{-n}(A)$ and the average $(\chi_0 + \ldots + \chi_n)/(n+1)$, where χ_i is the characteristic function of $T^{-i}(I)$.

This average pointwise converges to $\mu(I)$ due to ergodic theorem and therefore converges in L_2. This means that the scalar product converges to $\mu(A)\mu(I)$ and therefore does not exceed $r\mu(I)$ for large n.

It remains to find effectively a value of n when L_2-distance between the average and the constant $\mu(I)$ is small. Note that for all i the set $T^{-i}(I)$ is an effectively open set of measure $\mu(I)$ (recall that T is measure preserving). And $\mu(I)$ is computable. Therefore, for any i and $\varepsilon > 0$, one can uniformly approximate $T^{-i}(I)$ by a subset U which is a finite union of intervals such that $\mu(T^{-i}(I) \setminus U) < \varepsilon$. This means that the L_2-distance between the average and constant function $\mu(I)$ can be computed effectively, and we can wait until we find a term with any precision needed. □

Remark 1. Theorem 6 can be easily extended to other natural probability spaces (e.g., to the space of bi-infinite sequences). It is possible to further extend Theorem 6 to the general setting of computable probability spaces (see [Gác] and [HR09b]).

Now we get the previous theorems as corollaries: the effective ergodic theorem for the bidirectional shift (Theorem 4) immediately follows as the bidirectional shift is clearly computable, measure-preserving and ergodic. Moreover, we have already seen that from this theorem one can derive both Theorem 1 (Kučera's theorem for deletion of finite prefixes) and Theorem 3 (addition of finite prefixes).

It turns out that even Theorem 2 (finite change of bits) can also be proven in this setting. Indeed, let us consider the map F defined on Ω by:

$$F(1^n 0\omega) = 0^n 1\omega \quad \text{for all } n, \quad \text{and} \quad F(11111\ldots) = 00000\ldots$$

(F adds 1 to the sequence in the dyadic sense). It is clear that F is computable and measure-preserving. That it is ergodic comes from Kolmogorov's 0-1 law, together with the observation that any two binary sequences ω, ω' that agree on all but finitely many bits are in the same orbit: $\omega' = F^n(\omega)$ for some $n \in \mathbb{Z}$. The reverse is also true except for the case when sequences have finitely many zeros or finitely many ones. This cannot happen for a random sequence, so this exceptional case does not prevent us to derive Theorem 2 from Theorem 5.

Remark 2. Theorem 5 asserts that given a random ω, and a c.e. open set U, there exists an n such that $T^n(\omega) \notin U$ (T being the bidirectional shift), and that moreover n can be taken in a computable enumerable set fixed in advance. This of course still holds for the unidirectional shift on Ω (by the above discussion), but this does not hold for all ergodic maps. Indeed, this fact follows from the so-called *strong mixing property* of the shift, which not all ergodic maps have (e.g. a rotation of the circle by an irrational angle is an ergodic map but does not have this property).

6 An application

The celebrated van Lambalgen theorem [vL87] asserts that in the probability space Ω^2 (pairs of binary sequences with independent uniformly distributed components) a pair (ω_0, ω_1) is random if and only if ω_0 is random and ω_1 is ω_0-random (random relative to the oracle ω_0). This can be easily generalized to k-tuples: an element $(\omega_0, \omega_1, \ldots, \omega_{k-1})$ of Ω^k is random if and only if ω_0 is random and ω_i is $(\omega_0, \ldots, \omega_{i-1})$-random for all $i = 1, 2 \ldots, k-1$. Can we generalize this statement to infinite sequences? Not completely: there exists an infinite sequence $(\omega_i)_{i\in\mathbb{N}}$ such that ω_0 is random and ω_i is $(\omega_0, \ldots, \omega_{i-1})$-random for all $i \geq 1$ and nevertheless $(\omega_i)_{i\in\mathbb{N}}$ is non-random as an element of $\Omega^{\mathbb{N}}$. To construct such an example, take a random sequence in $\Omega^{\mathbb{N}}$ and then replace the first i bits of ω_i by zeros.

Informally, in this example all ω_i are random, but their "randomness deficiency" increases with i, so the entire sequence (ω_i) is not random (in $\Omega^{\mathbb{N}}$).

K. Miyabe [Miy] has shown recently that one can overcome this difficulty allowing finitely many bit changes in each ω_i (number of changed bits may depend on i):

Theorem 7 (Miyabe). *Let $(\omega_i)_{i\in\mathbb{N}}$ be a sequence of elements of Ω such that ω_0 is random and ω_i is $(\omega_0,\dots,\omega_{i-1})$-random for all $i \geq 1$. Then there exists a sequence $(\omega'_i)_{i\in\mathbb{N}}$ such that*

- *For every i the sequence ω'_i is equal to ω_i except for a finite number of places.*
- *The sequence $(\omega'_i)_{i\in\mathbb{N}}$ is a random element of $\Omega^{\mathbb{N}}$.*

Informally, this result can be explained as follows: as we have seen (Theorem 2), a change in finitely many places can decrease the randomness deficiency (starting from any non-random sequence, we get a sequence that is not covered by a first set of a Martin-Löf test) and therefore can prevent "accumulation" of randomness deficiency.

This informal explanation can be formalized and works not only for finite changes but also for adding/removing prefixes. In fact, the results of this paper allow us to get a simple proof of the following generalization of Miyabe's result (Miyabe's original proof used a different approach, namely martingale characterizations of randomness). We restrict ourselves to the uniform measure, but the same argument works for arbitrary computable measures.

Theorem 8. *Let $(\omega_i)_{i\in\mathbb{N}}$ be a sequence of elements of Ω such that ω_0 is random and ω_i is $(\omega_0,\dots,\omega_{i-1})$-random for all $i \geq 1$. Let $T : \Omega \to \Omega$ be a computable bijective ergodic map. Then, there exists a sequence $(\omega'_i)_{i\in\mathbb{N}}$ such that*

- *For every i, the sequence ω'_i is an element of the orbit of ω_i (i.e. $\omega'_i = T^{n_i}(\omega_i)$ for some integer n_i).*
- *The sequence $(\omega'_i)_{i\in\mathbb{N}}$ is a random element of $\Omega^{\mathbb{N}}$.*

Proof. Let U be the first level of a universal Martin-Löf test on $\Omega^{\mathbb{N}}$, with $\mu(U) \leq 1/2$. We will ensure that the sequence $(\omega'_i)_{i\in\mathbb{N}}$ is outside U, and this guarantees its randomness.

Consider the set V_0 consisting of those $\alpha_0 \in \Omega$ such that the section

$$U_{\alpha_0} = \{(\alpha_1, \alpha_2, \dots) \mid (\alpha_0, \alpha_1, \alpha_2, \dots) \in U\}$$

has measure greater than 2/3. The measure of V_0 is less than 1, otherwise we would have $\mu(U) > 1/2$. It is easy to see that V_0 is an effectively open subset of Ω. Since ω_0 is random, by Theorem **??** there exists an integer n_0 such that $\omega'_0 = T^{n_0}(\omega_0)$ is outside V_0. This ω'_0 will be the first element of the sequence we are looking for.

Now we repeat the same procedure for $U_{\omega'_0}$ instead of U. Note that it is an open set of measure at most 2/3, and, moreover, an effectively open set with respect to oracle ω'_0. Since ω_0 and ω'_0 differ by a computable transformation, the set $U_{\omega'_0}$ is effectively open with oracle ω_0. We repeat the same argument (where 1/2 and 2/3 are replaced by 2/3 and 3/4 respectively) and conclude that there exists an integer n_1 such that the sequence $\omega'_1 = T^{n_1}(\omega_1)$ has the following property: the set

$$U_{\omega'_0\omega'_1} = \{(\alpha_2, \alpha_3, \ldots) \mid (\omega'_0, \omega'_1, \alpha_2, \alpha_3, \ldots) \in U\}$$

has measure at most 3/4. (Note that we need to use ω_0-randomness of ω_1, since we apply Theorem **??** to an ω_0-effectively open set.)

At the next step we get n_2 and $\omega'_2 = T^{(n_2)}\omega_2$ such that

$$U_{\omega'_0\omega'_1\omega'_2} = \{(\alpha_3, \alpha_4, \ldots) \mid (\omega'_0, \omega'_1, \omega'_2, \alpha_3, \alpha_4, \ldots) \in U\}$$

has measure at most 4/5, etc.

Is it possible that the resulting sequence $(\omega'_0, \omega'_1, \omega'_2, \ldots)$ is covered by U? Since U is open, it would be then covered by some interval in U. This interval may refer only to finitely many coordinates, so for some m all sequences

$$(\omega'_0, \omega'_1, \ldots, \omega'_{m-1}, \alpha_m, \alpha_{m+1}, \ldots)$$

would belong to U (for every $\alpha_m, \alpha_{m+1}, \ldots$). However, this is impossible, because our construction ensures that the measure of the set of all $(\alpha_m, \alpha_{m+1}, \ldots)$ with this property is less than 1. □

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References

[Gác] Peter Gács. Lecture notes on descriptional complexity and randomness. Manuscript, available at http://www.cs.bu.edu/fac/gacs/recent-publ.html.

[GHR09] Péter Gács, Mathieu Hoyrup, and Cristóbal Rojas. Randomness on computable probability spaces - a dynamical point of view. In *Symposium on Theoretical Aspects of Computer Science (STACS 2009)*, volume 09001 of *Dagstuhl Seminar Proceedings*, pages 469–480. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, Germany Internationales Begegnungs- und Forschungszentrum fuer Informatik (IBFI), Schloss Dagstuhl, Germany, 2009.

[HR09a] Mathieu Hoyrup and Cristóbal Rojas. Applications of effective probability theory to Martin-löf randomness. In *International Colloquium on Automata, Languages and Programming (ICALP 2009)*, volume 5555 of *Lecture Notes in Computer Science*, pages 549–561. Springer, 2009.

[HR09b] Mathieu Hoyrup and Cristóbal Rojas. Computability of probability measures and Martin-Löf randomness over metric spaces. *Information and Computation*, 207(7):2207–2222, 2009.

[Kuč85] Antonin Kučera. Measure, Π^0_1 classes, and complete extensions of PA. *Lecture Notes in Mathematics*, 1141:245–259, 1985.

[Miy] Kenshi Miyabe. An extension of van Lambalgen's theorem to infinitely many relative 1-random reals. To appear in the Notre Dame Journal of Formal Logic.

[Shi96] Albert Shiryaev. *Probability*. Springer, 2nd edition, 1996.

[vL87] Michiel van Lambalgen. *Random sequences*. PhD dissertation, University of Amsterdam, Amsterdam, 1987.

[V'y97] Vladimir V'yugin. Effective convergence in probability and an ergodic theorem for individual random sequences. *SIAM Theory of Probability and Its Applications*, 42(1):39–50, 1997.

[V'y98] Vladimir V'yugin. Ergodic theorems for individual random sequences. *Theoretical Computer Science*, 207(2):343–361, 1998.