Improving the Space-Bounded Version of Muchnik's Conditional Complexity Theorem via "Naive" Derandomization

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St. Petersburg, June 14th, 2011

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- Why "naive"? Because we simply replace a random object by a pseudo-random one and it still does the job.

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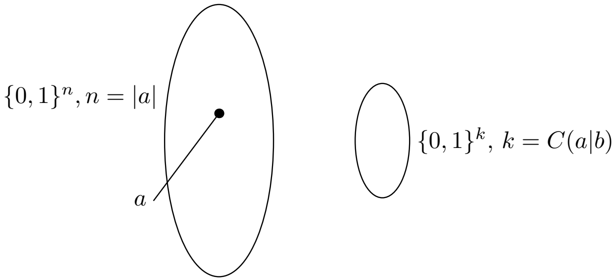
  - and $C^{\mathrm{poly}(n)}(p|a) = O(\log^3 n)$
- In current work we get rid of polylogarithmic terms and make them again logarithmic

Proof of Muchnik's theorem: fingerprints



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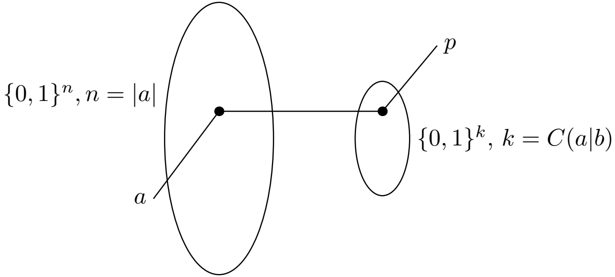

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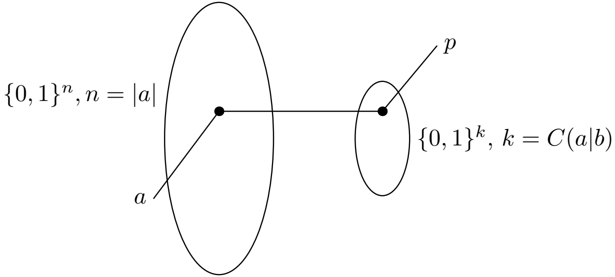

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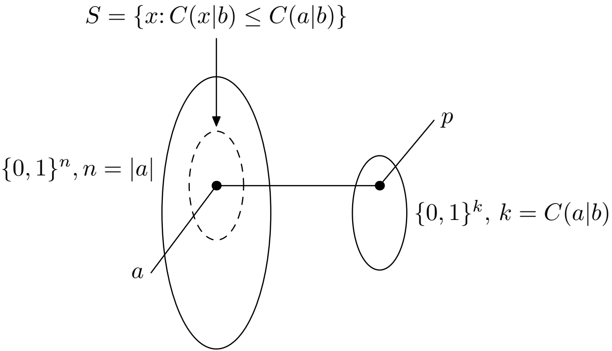

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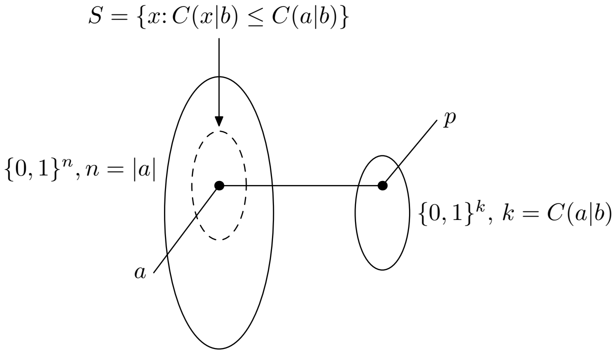

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For $C(a|p, b)$ to be small, there should be few preimages of p in S

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- Hence, the first graph in the enumeration has small complexity
- Hence, $C(p|a)$ and $C(a|p, b)$ are also small

What graph properties do we need?

Some graph properties leading to Muchnik's theorem:

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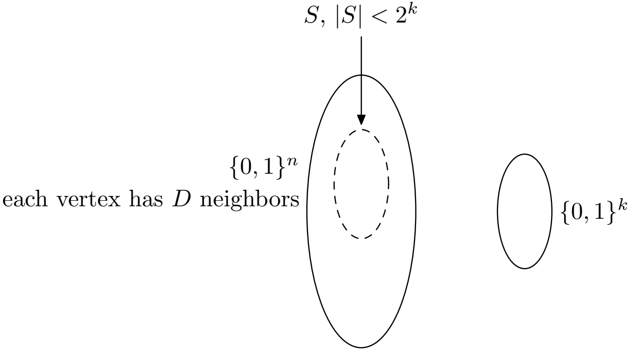
- (This paper) "Low-congesting"

The essential property: low-congesting graph



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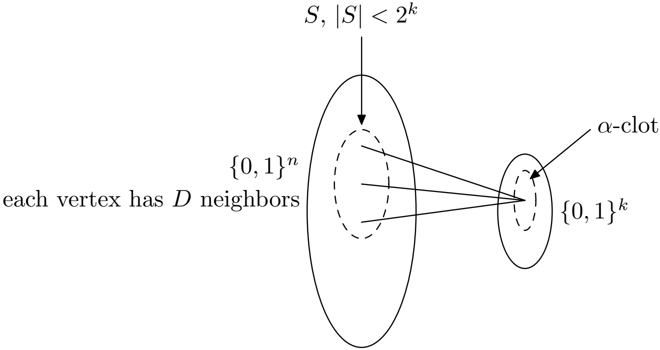

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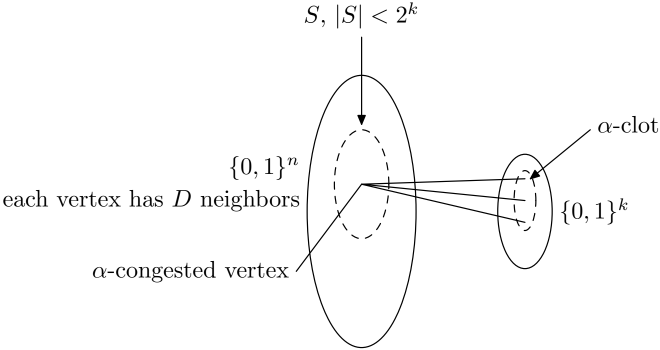

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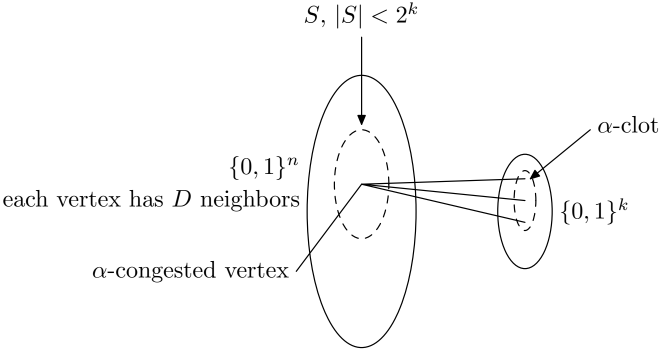

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- The set S is (α, β)-low-congested if it contains less than βK α-congested vertices.
- We call a set *relevant* if it has the form $\{x | C^s(x|b) < k\}$.
- We call a graph (α, β)-low-congesting if all relevant sets are (α, β)-low-congested.

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- **Central idea:** replace a random graph by a pseudorandom one
- To make this idea work, we need:
  - to prove that a pseudorandom graph is low-congesting with positive probability
  - to show that the seed this graph is generated from may be found in polynomial space

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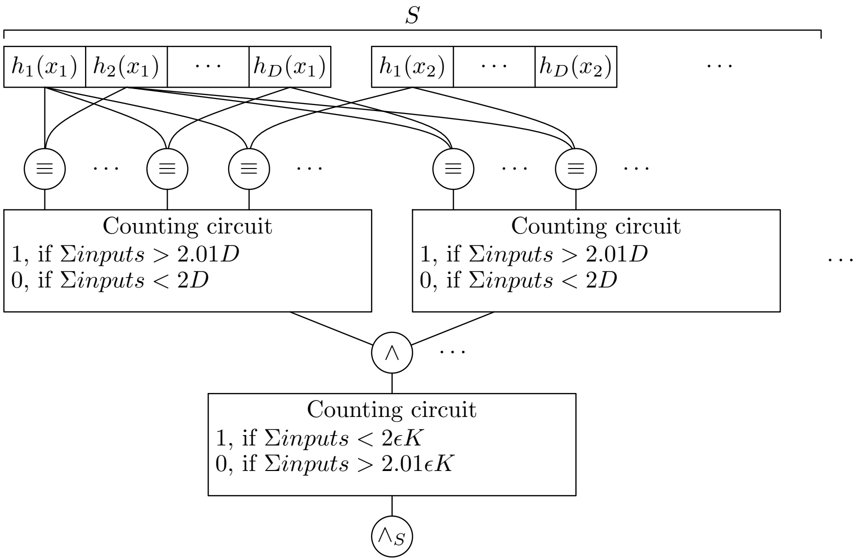
- This circuit accepts a random graph with sufficient probability, hence it does the same with a pseudorandom one.

The circuit


S
$h_1(x_1)$
$h_2(x_1)$
$\cdots$
$h_D(x_1)$
$h_1(x_2)$
$\cdots$
$h_D(x_2)$
$\cdots$
$\equiv$
Counting circuit
1, if $\Sigma inputs > 2.01D$
0, if $\Sigma inputs < 2D$
Counting circuit
1, if $\Sigma inputs > 2.01D$
0, if $\Sigma inputs < 2D$
$\wedge$
Counting circuit
1, if $\Sigma inputs < 2\epsilon K$
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$\wedge_S$

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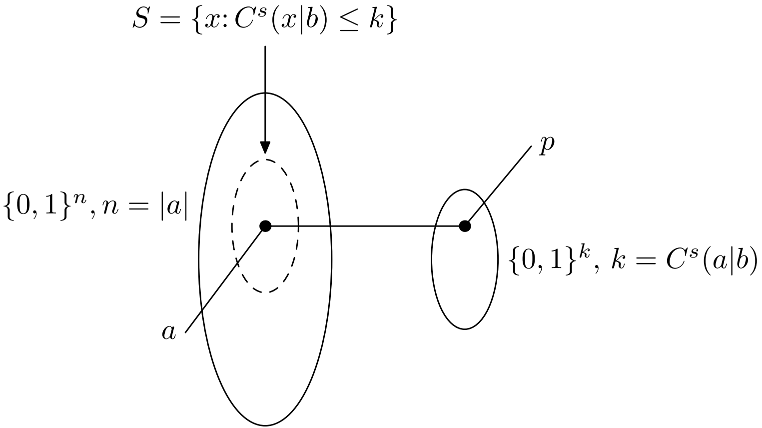

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- If a *is* congested then we repeat the whole construction replacing the relevant sets by the sets of congested vertices in relevant sets.
- There may be several iterations but since the upper bound on the size of a relevant set decreases exponentially there is at most linear number of steps, hence all polynomial bounds remain.

The final formulation

For any a and b of length n and for any s there exists p of length $C^s(a|b) + O(\log\log s + \log n)$ such that:

- $p(b) = a$;
- the computation of $p(b)$ performs in space $O(s) + \text{poly}(n)$
- and $C^{O(s)+\text{poly}(n)}(p|a) = O(\log\log s + \log n)$

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- ???????
- PROFIT

Thank you!

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