

# Pseudo-random graphs and bit probe schemes with one-sided error

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## Abstract

We study probabilistic bit-probe schemes for the membership problem. Given a set  $A$  of at most  $n$  elements from the universe of size  $m$  we organize such a structure that queries of type “ $x \in A$ ?” can be answered very quickly.

H. Buhrman, P.B. Miltersen, J. Radhakrishnan, and S. Venkatesh proposed a bit-probe scheme based on expanders. Their scheme needs space of  $O(n \log m)$  bits. The scheme has a randomized algorithm processing queries; it needs to read only one randomly chosen bit from the memory to answer a query. For every  $x$  the answer is correct with high probability (with two-sided errors).

In this paper we show that for the same problem there exists a bit-probe scheme with *one-sided* error that needs space of  $O(n \log^2 m + \text{poly}(\log m))$  bits. The difference with the model of Buhrman, Miltersen, Radhakrishnan, and Venkatesh is that we consider a bit-probe scheme with an auxiliary word. This means that in our scheme the memory is split into two parts of different size: the main storage of  $O(n \log^2 m)$  bits and a short word of  $\log^{O(1)} m$  bits that is pre-computed once for the stored set  $A$  and “cached”. To answer a query “ $x \in A$ ?” we allow to read the whole cached word and only one bit from the main storage. For some reasonable values of parameters (e.g., for  $\text{poly}(\log m) \ll n \ll m$ ) our space bound is better than what can be achieved by any scheme without cached data (the lower bound  $\Omega(\frac{n^2 \log m}{\log n})$  was proven in [15]).

We obtain a slightly weaker result (space of size  $n^{1+\delta} \text{poly}(\log m)$  bits and two bit probes for every query) for a scheme that is effectively encodable.

Our construction is based on the idea of naive derandomization, which is of independent interest. First we prove that a random combinatorial object (a graph) has the required properties, and then show that such a graph can be obtained as an outcome of a pseudo-random generator. Thus, a suitable graph can be specified by a short seed of a PRG, and we can put an appropriate value of the seed into the cache memory of the scheme.

# 1 Introduction

We investigate the static version of the membership problem. The aim is to represent a set  $A \subset \{1, \dots, m\}$  by some data structure so that queries “ $x \in A?$ ” can be easily replied. We are interested in cases when the number of elements in the set  $n = |A|$  is much less than size  $m$  of the universe (e.g.,  $n = \exp\{\text{poly}(\log \log m)\}$  or  $n = m^{0.01}$ ).

In practice, many different data structures are used to represent sets: simple arrays, different variants of height-balanced trees, hash tables, etc. The simplest solution is an array of  $m$  bits; to answer a query “ $x \in A?$ ” we need to read a single bit from the memory. However, the size of this data structure is excessive. In a more complicated data structure based on double hashing (Fredman, Komlós, and Szemerédi, [6]) a set is represented as a table of  $O(n)$  words of size  $\log m$  bits, and a query “ $x \in A?$ ” requires to read  $O(1)$  words from the memory. Another popular practical solution is Bloom’s filter [1]. This data structure requires only  $O(n)$  bits, whatever is the size of the universe; to answer a query we need to read  $O(1)$  bits from the memory. The drawback of this method is that we can get false answers to some queries. Only false positives answers are possible (for some  $x \notin A$  Bloom’s filter answers “yes”), but false negatives are not. For a “typical” set  $A$  the fraction of false answers is small.

An interesting approach was suggested by Harry Buhrman, Peter Bro Miltersen, Jaikumar Radhakrishnan, and Venkatesh Srinivasan [15]. They introduced some randomness into the query processing algorithm. That is, the data structure remains static (it is deterministically defined for each set  $A$ ), but when a query is processed we toss coins and read randomly chosen bit from the memory. In this model, we allow to return a wrong answer with some small probability. Notice the sharp difference with the Bloom’s filter: for *each*  $x$  we must correctly reply to the query “ $x \in A?$ ” with probability close to 1.

Buhrman, Miltersen, Radhakrishnan, and Venkatesh investigated both two-sided and one-sided errors. In this paper we will concentrate mostly on one-sided errors: if  $x \in A$ , then the answer must be always correct, and if  $x \notin A$ , then some small probability of error is allowed.

A trivial information-theoretic bound shows that the size of the structure representing a set  $A$  cannot be less than  $\log \binom{m}{n} = \Omega(n \log m)$  bits. Surprisingly, this bound can be achieved if we allow two-sided errors and use only single bit probe for each query. This result was proven in [15]. We refer to the scheme proposed there as the BMRV-scheme:

**Theorem 1 (two-sided BMRV-scheme, [15])** *For any  $\varepsilon > 0$  there is a scheme for storing subsets  $A$  of size at most  $n$  of a universe of size  $m$  using  $O(\frac{n}{\varepsilon^2} \log m)$  bits so that any membership query “Is  $x \in A?$ ” can be answered with error probability less than  $\varepsilon$  by a randomized algorithm which probes the memory at just one location determined by its coin tosses and the query element  $x$ .*

The bound achieved in this theorem is very close to the known lower bound. In fact, the trivial lower bound  $\log \binom{m}{n}$  can be improved: the less is probability of

an error, the greater must be the stored bit vector. For one-sided error schemes a stronger lower bound is known:

**Theorem 2 (lower bound, [15])** (a) For any  $\varepsilon > 0$  and  $\frac{n}{\varepsilon} < m^{1/3}$ , any  $\varepsilon$ -error randomized scheme which answers queries using one bitprobe must use space  $\Omega(\frac{n}{\varepsilon \log 1/\varepsilon} \log m)$ .

(b) Any scheme with one-sided error  $\varepsilon$  that answers queries using at most one bitprobe must use  $\Omega(\frac{n^2}{\varepsilon^2 \log(n/\varepsilon)} \log m)$  bits of storage.

The second part of the theorem above implies that we cannot achieve space of size  $O(n \log m)$  with a *one probe* scheme and one-sided error. However we can get very close if we allow  $O(1)$  probes (instead of a single probe):

**Theorem 3 (one-sided BMRV-scheme, [15])** Fix any  $\delta > 0$ . There exists a constant  $t$  such that the following holds: there is a one-sided  $\frac{1}{3}$ -error randomized scheme that uses space  $O(n^{1+\delta} \log m)$  and answers membership queries with at most  $t$  probes.

The constructions in [15] is not explicit: given the list of elements  $A$ , the corresponding scheme is constructed (with some brute force search) in time  $2^{\text{poly}(m)}$ . Moreover, each membership query requires exponential in  $m$  computations.

The crucial element of the constructions in Theorem 1 is an unbalanced expander graph. Existence of a graph with required parameters was proven in [15] probabilistically. We know that such a graph exists and we can find it by brute force search, but we do not know how to construct it explicitly. Since Bassalygo and Pinsker defined expanders [2, 3], many explicit (and poly-time computable) constructions of expander graphs were discovered, see a survey [19]. However, most of known constructions are based on spectral technique that is not suitable to get an expander of degree  $d$  with expansion parameter greater than  $d/2$ , see [12]. This is not enough for the construction used in the proof of Theorem 1 in [15]; we need there a graph with expansion parameter close to  $d$ .

There are only very few effective constructions of unbalanced graph with large expansion parameter. One of the known constructions was suggested by Capalbo et al in [16]; its parameters are close to optimal values if the size of the right part of the graph is *constant times* less than the size of the left part of the graph. However, in the BMRV-scheme we need a graph where the right part of the graph is much less than the left part. Some explicit construction suitable for BMRV-scheme was suggested in [18]. The best known explicit construction of a highly unbalanced expander graph was presented in [22]. This beautiful construction is based on the Parvaresh–Vardy code with an efficient list decoding. Thanks to the special structure of this expander, it enjoys some special property of effective decoding. Using this technique, the following variant of Theorem 1 can be proven:

**Theorem 4 ([22])** For any  $\delta > 0$  there exists a scheme for storing subset  $A$  of size at most  $n$  of a universe of size  $m$  using  $n^{1+\delta} \cdot \text{poly}(\log m)$  bits so that

any membership query can be answered with error probability less than  $\varepsilon$  by a randomized algorithm which probes the memory at one location determined by its coin tosses and the query element  $x$ .

Given the list of elements  $A$ , the corresponding storing scheme can be constructed in time  $\text{poly}(\log m, n)$ . When the storing scheme is constructed, a query for an element  $x$  can be calculated in time  $\text{poly}(\log m)$ .

In Theorems 1, 3, 4, a set  $A$  is encoded into a bit string, and when we want to know if  $x \in A$ , we just read from this string one randomly chosen bit (or  $O(1)$  bits in Theorem 3). The obtained information is enough to decide whether  $x$  is an element of the set. Let us notice that in all these computations we implicitly use more information. To make a query to the scheme and to process the retrieved bit, we need to know the parameters of the scheme: the size  $n$  of the set  $A$ , the size  $m$  of the universe, and the allowed error probability  $\varepsilon$ . This auxiliary information is very short (it takes only  $\log(m/\varepsilon)$  bits). It does not depend on the stored set  $A$ . We assume that it is somehow hardwired into the bitprobe scheme (we say that this information is *cached* in advance by the algorithms that processes queries).

In this paper we consider a more liberal model, where small *cached* information can depend on the set  $A$ . Technically, the data stored in our scheme consists of two parts of different size: a small cached string  $C$  of length  $\text{poly}(\log m)$ , and a long bit string  $B$  of length  $n \cdot \text{poly}(\log m)$ . Both these strings are prepared for a given set  $A$  of  $n$  elements (in the universe of size  $m$ ). When we need to answer a query “ $x \in A?$ ”, we use  $C$  to compute probabilistically a position in  $B$  and read there one bit. This is enough to answer whether  $x$  is an element of  $A$ , with a small one-sided error:

**Theorem 5** *Fix any constant  $\varepsilon > 0$ . There exists a one-sided  $\varepsilon$ -error randomized scheme that includes a string  $B$  of length  $O(n \log^2 m)$  and an auxiliary word  $C$  of length  $\text{poly}(\log m)$ . We can answer membership queries “ $x \in A?$ ” with one bit probe to  $B$ . For  $x \in A$  the answer is always correct; for each  $x \notin A$  probability of error is less than  $\varepsilon$ .*

*The position of the bit probed in  $B$  is computed from  $x$  and the auxiliary word  $C$  in time  $\text{poly}(\log m)$ .*

**Remark 1:** Schemes with ‘cached’ auxiliary information that depends on  $A$  (not only on its size  $n = |A|$  and the size  $m$  of the universe) makes sense only if this information is very small. If the size of the cached data is about  $\log \binom{m}{n}$  bits, then we can put there the list of all elements of  $A$ , so the problem becomes trivial. Since in our construction we need cached information of size  $\text{poly}(\log m)$  bits, the result is interesting when  $\text{poly}(\log m) \ll n \ll m$ , e.g., for  $n = \exp\{\text{poly}(\log \log n)\}$ . Notice that by Theorem 2 the space size  $O(n \log^2 m)$  with one-sided error cannot be achieved by any schemes without cached auxiliary information that depends on  $A$ .

**Remark 2:** The model of data structures with cached memory looks useful for practical applications. Indeed, most computer systems contain some hierarchy

of memory levels: CPU registers and several levels of processor caches, then random access memory, flash memory, magnetic disks, remote network-accessible drives, etc. Each next level of memory is cheaper but slower.

So, it is interesting to investigate the tradeoff between expensive and fast local memory and cheap and slow external memory. There is rich literature on algorithms with *external memory*, see, e.g, surveys [14, 20]. Thus, the idea of splitting the data structure into ‘cached’ and ‘remote’ parts is very natural. However, tradeoff between local and external memory is typically studied for *dynamic* data structures. The same time, it is not obvious that fast cache memory of negligible size can help to process queries in a *static* data structure. Since a small cache ‘knows’ virtually nothing about most objects in the database, so at first sight it seems to be useless. However, Theorem 5 shows that even a very small cache can be surprisingly efficient.

**Remark 3:** In the proof of Theorem 5 we derandomize a probabilistic proof of existence of some kind of expander graphs. In many works derandomization of probabilistic arguments involves highly sophisticated ad-hoc techniques. But we do derandomization in rather naive and straightforward way: take a value of a suitable pseudo-random number generator and check that with high probability (i.e., for most values of the seed) we obtain the required property. In fact, we observe that several types of generators fit our construction. Since the required property of a graph can be tested in  $AC^0$ , we can use the classic Nisan–Wigderson generator or (thanks to the recent result of Braverman [21]) any polylog-independent function. Also the required property of a pseudo-random graph can be tested by a machine with logarithmic space. Hence, we can use Nisan’s generator [10]. We stress that we do not need any unproven assumptions to construct all these generators.

Somewhat nonconventional part of our construction is that we consider a ‘local’ variant of the definition of expanders: we require that the usual expansion property holds not for all sets of vertices but only for one particular set  $A$ . This modification makes the property of the expander graph weaker, and this relaxation helps to derandomize the construction.

In Theorem 5 we construct a scheme such that decoding is effective: when the scheme is prepared, we can answer queries “ $x \in A?$ ” in time polynomial in  $\log m$ . However the encoding (preparing the bits string and the auxiliary word) runs in expected time  $\text{poly}(m)$ , which is much longer if  $m \gg n$ . Next theorem claims that the encoding time can be reduced if we slightly increase the space of the scheme:

**Theorem 6** *The scheme from theorem 5 can be made effectively encodable in the following sense. Fix any constants  $\varepsilon, \delta > 0$ . There exists randomized scheme that includes a bit string  $B$  of length  $n^{1+\delta} \text{poly}(\log m)$  and an auxiliary word  $C$  of length  $\text{poly}(\log m)$ . We can answer membership queries “ $x \in A?$ ” with two bits probe to  $B$ . For  $x \in A$  the answer is always correct; for  $x \notin A$  probability of error is less than  $\varepsilon$ .*

The position of the bit probed in  $A$  is computed by  $x$  and the auxiliary word  $C$  in time  $\text{poly}(\log(m))$ . Given  $A$ , the entire scheme (the string  $B$  and the word  $C$ ) can be computed probabilistically in average time  $\text{poly}(n, \log m)$ .

The rest of the paper is organized as follows. In Section 2 we remind the main ideas in the BMRV-scheme. We prove Theorem 5 in Section 3, and Theorem 6 in Section 4. In Section 5 we show that the proof of Theorem 2 (a) *mutatis mutandis* can be applied to the model with small cached memory. In Conclusion we discuss some open questions.

## 2 How BMRV-scheme works

Let us explain the main ideas of the proof of Theorem 1 in [15]. The construction is based on highly unbalanced *expanders*.

**Definition 1** A bipartite graph  $G = (L, R, E)$  (with left part  $L$ , right part  $R$  and set of edges  $E$ ) is called  $(m, s, d, k, \delta)$ -expander if  $L$  consists of  $m$  vertices,  $R$  consists of  $s$  vertices, degree of each vertex in  $L$  is equal to  $d$ , and for each subset of vertices  $A \subset L$  of size at most  $k$  the number of neighbors is at least  $(1 - \delta)d|A|$ .

We use a standard notation: for a vertex  $v$  we denote by  $\Gamma(v)$  the set of its neighbors; for a set of vertices  $A$  we denote by  $\Gamma(A)$  the set of neighbors of  $A$ , i.e.,  $\Gamma(A) = \cup_{v \in A} \Gamma(v)$ . So, the definition of expanders claims that for all small enough sets  $A$  of vertices in the left part of the graph,  $|\Gamma(A)| \geq (1 - \delta)d|A|$  (the maximal size of  $|\Gamma(A)|$  is obviously  $d|A|$ , since degree of all vertices on the left is equal to  $d$ ). The argument below is based on the following combinatorial property of an expander:

**Lemma 1 (see [16])** Let  $\varepsilon$  be a positive number, and  $G$  be an  $(m, s, d, k, \delta)$ -expander with  $\delta \leq \varepsilon/4$ . Then for every subset  $A \subset L$  such that  $|A| \leq k/2$ , the number of vertices  $x \in L \setminus A$  such that

$$|\Gamma(x) \cap \Gamma(A)| \geq \varepsilon d$$

is not greater than  $|A|/2$ .

Let  $G$  be a  $(m, s, d, k, \delta)$ -expander with  $\delta < \varepsilon/4$ . The storage scheme is defined as follows. We identify a set  $A \subset \{1, \dots, m\}$  of size  $n$  ( $n \leq k/2$ ) with a subset of vertices in the left part of the graph. We will represent it by some labeling (by ones and zeros) on the vertices of the right part of the graph. We do it in such a way that the vast majority (at least  $(1 - \varepsilon)d$ ) of neighbors of each vertex  $v$  from the left part of the graph correctly indicate whether  $v \in A$ . More precisely, if  $v \in A$  then at least  $(1 - \varepsilon)d$  of its neighbors in  $R$  are labeled by 1; if  $v \in L \setminus A$  then at least  $(1 - \varepsilon)d$  of its neighbors in  $R$  are labeled by 0. Thus, querying a random neighbor of  $v$  will return the right answer with probability  $> 1 - \varepsilon$ .

It remains to explain why such a labeling exists. In fact, it can be constructed by a simple greedy algorithm. First, we label all neighbors of  $A$  by 1, and the other vertices on the left by 0. This labeling classifies correctly all vertices in  $A$ . But it can misclassify some vertices outside  $A$ : some vertices in  $L \setminus A$  can have too many (more than  $\varepsilon d$ ) neighbors labeled by 1. Denote by  $B$  the set of all these “erroneous” vertices. We relabel all their neighbors, i.e., all vertices in  $\Gamma(B)$  to 0. This fixes the problem with vertices outside  $A$ , but it can create problems with some vertices in  $A$ . We take the set of all vertices in  $A$  that became erroneous (i.e., vertices in  $A$  that have at least  $\varepsilon d$  neighbors in  $\Gamma(B)$ ), and denote this set of vertices by  $A'$ . Then we relabel all  $\Gamma(A')$  to 1. This operation create new problems in some set of vertices  $B' \subset B$ , relabel  $\Gamma(B')$  to 0, etc. In this iterative procedure we get a sequence of sets

$$A \supset A' \supset A'' \supset \dots$$

whose neighbors are relabeled to 1 on steps 1, 3, 5, ... of the algorithm, and

$$B \supset B' \supset B'' \supset \dots$$

whose neighbors are relabeled to 0 on iterations 2, 4, 6, ... respectively. Lemma 1 guarantees that the number of the erroneous vertices on each iteration reduces by a factor of 2 ( $|B| \leq |A|/2$ ,  $|A'| \leq |B|/2$ , etc.). Hence, in  $\log m$  steps the procedure terminates.

To organize a storing scheme (and to estimate its size) for a set  $A$  of size  $n$  in the universe of size  $m$ , we should construct an  $(m, s, d, k = 2n, \delta = \varepsilon/4)$ -expander. Parameters  $m, k, \delta$  of the graph are determined directly by the parameters of the desired scheme (by the size of  $A$  and the universe and the allowed error probability  $\varepsilon$ ). We want to minimize the size of the left part of the graph  $s$ , which is the size of the stored data. Existence of expanders with good parameters can be proven by probabilistic arguments:

**Lemma 2 ([15])** *For all integers  $m, n$  and real  $\varepsilon > 0$  there exists an  $(m, s = O(\frac{n \log m}{\varepsilon^2}), d = \frac{\log m}{\varepsilon}, n, \varepsilon)$ -expander. Moreover, the vast majority of bipartite graphs with  $n$  vertices on the left,  $s = \frac{100n \log m}{\varepsilon^2}$  vertices on the right, and degree  $d = \frac{\log m}{\varepsilon}$  at all vertices on the left are expanders.*

Given the parameters  $m, n, \varepsilon$ , we can find an  $(m, O(\frac{n \log m}{\varepsilon^2}), \frac{\log m}{\varepsilon}, n, \varepsilon)$ -expander by brute force search. This can be done by a deterministic algorithm in time  $2^{\text{poly}(m/\varepsilon)}$ . Hence, to construct the bit-probe structure defined above we need exponential time. Moreover, when the structure is constructed and we want to answer a query “ $x \in A$ ?” we need to read only one bit from the stored bit string. But to select the position of this bit we need again to reconstruct the expander graph, which requires exponential computations. We can keep the structure of the computed graph in “cache” (compute the graph once, and then re-use it every time a new query should be answered). But then the size of this “cached data” (the size of the graph) becomes much greater than  $m$ , which makes the bit-probe scheme useless (it is cheaper to store  $A$  as a trivial  $m$ -bits array).

In [22] a nice and very powerful explicit construction of expanders was suggested:

**Theorem 7 ([22])** *Fix an  $\varepsilon > 0$  and  $\delta > 0$ . For all integers  $m, n$  there exists an explicit  $(m, s = n^{1+\delta} \cdot \text{poly}(\log m), d = \text{poly}(\log m), n, \varepsilon)$ -expander such that for an index of a vertex  $v$  from the left part (a binary representation of an integer between 1 and  $m$ ) and an index of an outgoing edge (a binary representation of an integer between 1 and  $d$ ), the corresponding neighbor on the right part of the graph (an integer between 1 and  $s$ ) can be computed in time polynomial in  $\log m$ .*

*Also, the following effective decoding algorithm exists. Given a set of vertices  $T$  from the right part of the graph, we can compute the list of vertices in the left part of the graph that have at least  $(4\varepsilon d)$  neighbors in  $T$ , i.e.,*

$$S = \{v : |\Gamma(v) \cap T| \geq 4\varepsilon d\},$$

*in time  $\text{poly}(|S|, n, \log m)$ .*

Theorem 4 is proven by plugging the expander from Theorem 7 in the general scheme explained above, see details in [22].

## 3 Proof of Theorem 5

### 3.1 Refinement of the property of $\varepsilon$ -reduction

The construction of bit-probe scheme for a set  $A$  of size  $n$  in the  $m$ -elements universe (with probability of an error bounded by some  $\varepsilon$ ) explained in the previous section involves an  $(m, s, d, k, \delta)$ -expander with  $s = O(\frac{n}{\varepsilon} \log m)$  and  $d = O(\frac{1}{\varepsilon} \log m)$ . Such a graph contains  $dm$  edges (degree of each vertex on the left is  $d$ ). The list of all its edges can be specified by a string of  $dm \log s$  bits: we sort all edges by their left ends, and specify for each edge its right end. Denote the size of the description of this graph by  $N = dm \log s$ .

In what follows we will assume that number  $s$  is a power of 2 (this will increase the parameters of the graph by only a factor at most 2). So, we may assume that every string of  $N(m, s, d) = dm \log s$  bits specifies a bipartite graph with  $m$  vertices on the left,  $s$  vertices on the right and degree  $d$  on the left. Lemma 2 claims that most of these bits string of length  $N$  describe an  $(m, s, d, k, \delta)$ -expander. By Lemma 1, if a graph is an expander with these parameters, then for  $\varepsilon = 4\delta$  and for every set  $A \subset L$  of size less than  $k/2$  the following *reduction property* holds:

**Combinatorial Property 1 ( $\varepsilon$ -reduction property)** *For every subset  $A \subset L$  such that  $|A| \leq k/2$ , the number of vertices  $x \in L \setminus A$  such that*

$$|\Gamma(x) \cap \Gamma(A)| \geq \varepsilon d$$

*is not greater than  $|A|/2$ .*



This property was the main ingredient of the BMRV-scheme. In our bit-probe scheme we will need another variant of Property 1:

**Combinatorial Property 2 (strong  $\varepsilon$ -reduction)** *Let  $G = (L, R, E)$  be a bipartite graph, and  $A \subset L$  be a subset of vertices from the left part. We say that the strong  $\varepsilon$ -reduction property holds for  $A$  in this graph if for all  $x \in L \setminus A$*

$$|\Gamma(x) \cap \Gamma(A)| \leq \varepsilon d.$$

**Lemma 3** *Fix an  $\varepsilon > 0$ . For all integers  $m, n$ , for every  $A \subset \{1, \dots, m\}$  of size  $n$  there exists a bi-partite graph  $G = (L, R, E)$  such that*

- $|L| = m$  (the size of the left part);
- $|R| = 2d^2n = O(n \log^2 m)$  (the size of the right part);
- degree of each vertex in the left part is  $d = \frac{2 \log m}{\varepsilon} = O(\log m)$ ;
- the property of strong  $\varepsilon$ -reduction holds for the set  $A$  (we identify it with a subset of vertices in left part of the graph).

*Moreover, the property of strong  $\varepsilon$ -reduction for  $A$  holds for the majority of graphs with the parameters specified above.*

The order of quantifiers is important here: we do not claim that in a random graph the strong  $\varepsilon$ -reduction property holds for all  $A$ ; we say only that for every  $A$  the strong  $\varepsilon$ -reduction is true in a random graph.

**Proof of lemma:** Let  $v$  be any vertex in  $L \setminus A$ . We estimate probability that at least  $\varepsilon d$  neighbors of  $x$  are at the same time neighbors of  $A$  (assuming that all edges are chosen at random independently). There are  $\binom{d}{\varepsilon d}$  choices of  $\varepsilon d$  vertices among all neighbors of  $v$ . Hence,

$$\text{Prob}[|\Gamma(v) \cap \Gamma(A)| \geq \varepsilon d] \leq \binom{d}{\varepsilon d} \cdot \left( \frac{|\Gamma(A)|}{|R|} \right)^{\varepsilon d} \leq d^{\varepsilon d} \cdot \left( \frac{dn}{2d^2n} \right)^{\varepsilon d} = \left( \frac{1}{2} \right)^{2 \log m}$$

This probability is less than  $1/m^2$  (for each vertex  $v$ ). So, the expected number of vertices  $v \in L$  such that  $|\Gamma(v) \cap \Gamma(A)| \geq \varepsilon d$ , is less than  $1/m < 1/2$ . Hence, the strong  $\varepsilon$ -reduction property holds for  $A$  for more than 50% of graphs.

### 3.2 Testing the property of strong $\varepsilon$ -reduction

Lemma 3 implies that a graph with the strong reduction property for  $A$  exists. Given  $A$ , we can find such a graph by brute force search. But we cannot use such a graph in our bit-probe scheme even if we do not care about computation complexity: the choice of the graph depends on  $A$ , and the size of the graph is too large to include it into the scheme. We need to find a suitable graph with a short description. We will do it using pseudo-random generators ('random' graphs will be parameterized by the seed of a generator).

Property 2 is a property of a graph and a set of vertices  $A$  in graph. We can interpret it as a property of an  $N$ -bits string (that determines a graph) and some  $A \subset \{1, \dots, m\}$ . Lemma 3 claims that for every  $A$ , for a randomly chosen graph (a randomly chosen  $N$ -bits string) with high probability the strong reduction property is true. We want to show that the same is true for a *pseudo-random* graph. At first, we observe that the strong reduction property can be tested by an  $AC^0$  circuit (a Boolean circuit of bounded depth, with polynomial number of gates *and, or* with unbounded fan-in, and *negations*).

Indeed, we need to check for each vertex  $v \in L \setminus A$  that the number of vertices in  $\Gamma(v) \cap \Gamma(A)$  is not large. For each vertex  $w$  in the right part of the graph we can compute by an  $AC^0$ -circuit whether  $w \in \Gamma(A)$ :

$$\bigvee_{u \in A} \bigvee_{i \leq d} [ \text{the } i\text{-th neighbor of } u \text{ is } w ]$$

(the condition in the square brackets is a statement on one edge in a graph, i.e., it is a conjunction of  $O(\log N)$  bits and negations of bits from the representation of this graph). So, for each  $v \in L \setminus A$  and for  $i = 1, \dots, d$  we can compute whether the  $i$ -th neighbor of  $v$  belongs to  $\Gamma(A)$ :

$$b_{v,i} = \bigvee_{w \in R} ([ \text{the } i\text{-th neighbor of } v \text{ is } w ] \ \& \ [ w \in \Gamma(A) ])$$

It remains to ‘count’ for each  $v \in L \setminus A$  the number of neighbors in  $\Gamma(A)$  and compute the thresholds

$$Th(b_{v,1}, \dots, b_{v,d}) = \begin{cases} 1, & \text{if } b_{v,1} + \dots + b_{v,d} \geq \varepsilon d, \\ 0, & \text{otherwise.} \end{cases}$$

In  $AC^0$  we cannot compute thresholds with linear number of inputs (e.g., the majority function is not in  $AC^0$ , see [8]). However, we need threshold functions with only logarithmic number of inputs. Such a function can be represented by a CNF of size  $2^{O(d)} = \text{poly}(N)$ .

Then we combine together these thresholds for all  $v \in L \setminus A$ , and get an  $AC^0$ -circuit that tests the property of strong  $\varepsilon$ -reduction.

### 3.3 Pseudo-random graphs

We need to generate a pseudo-random string of  $N$  bits that satisfies the strong  $\varepsilon$ -reduction property (for some fixed set  $A$ ). We know that (i) by Lemma 3, for a uniformly distributed random string this property is true with high probability, and (ii) this property can be checked in  $AC^0$ . It remains to choose a pseudo random generator that fools this particular  $AC^0$ -circuit. There exist several generators that fools such distinguishers. Below we mention three different solutions.

**Remark:** We can test by an  $AC^0$ -circuit Property 2 for every fixed set  $A$  but *not* for all sets  $A$  together.

*The first solution: the generator of Nisan–Wigderson.* The classic way to fool an  $AC^0$  circuit is the Nisan–Wigderson generator:

**Theorem 8 (Nisan–Wigderson generator, [11])** *For every constant  $c$  there exists an explicit family of functions*

$$G_m : \{0, 1\}^{\text{poly}(\log N)} \rightarrow \{0, 1\}^N$$

*such that for any family of circuits  $C_N$  (with  $N$  inputs) of polynomial in  $N$  size and depth  $c$ , the difference*

$$|\text{Prob}_y[C_N(y) = 1] - \text{Prob}_z[C_N(G_m(z)) = 1]|$$

*tends to zero (faster than  $1/\text{poly}(N)$ ).*

*The generator is effective: generator’s value  $G_m(x)$  can be computed from a given  $x$  in time  $\text{poly}(\log N)$ .*

From this theorem and Lemma 3 it follows that for each  $A \subset \{1, \dots, m\}$  of size at most  $n$ , for most values of the seed of the Nisan–Wigderson generator  $G_m$ , a pseudo-random graph  $G_m(x)$  satisfies the strong  $\varepsilon$ -expansion property for  $A$ .

*The second solution: polylog-independent strings.* M. Braverman proves that all polylog-independent functions fool  $AC^0$ -circuits:

**Theorem 9 (Braverman, [21])** *Let  $\mathcal{C}$  be a Boolean circuit of depth  $r$  and size  $M$ ,  $\varepsilon$  be a positive number, and*

$$D = \left( \log \frac{M}{\varepsilon} \right)^{\kappa r^2}$$

*(for some absolute constant  $\kappa$ ). Then  $\mathcal{C}$  cannot distinguish between the uniform distribution  $U$  and any  $D$ -independent distribution  $\mu$  on its inputs:*

$$|\text{Prob}_\mu[\mathcal{C} \text{ accepts}] - \text{Prob}_U[\mathcal{C} \text{ accepts}]| < \varepsilon$$

It follows that instead of the Nisan–Wigderson generator we can take any  $(\log^c n)$ -independent function (for large enough constant  $c$ ). The standard solution is a polynomial of degree  $r = \log^c n$  over a finite field of size about  $N$  (seeds of this ‘pseudo-random generator’ are coefficients of a polynomial). Alternatively, other constructions of polylog-independent functions can be used. E.g., the construction from [9, 13] provides a family of  $(\log^c n)$ -independent functions with very fast evaluation algorithm, and each function is specified by  $\text{poly}(\log n)$  bits (so, the size of the *seed* is again poly-logarithmic).

*The third solution: the generator of Nisan.* The property of strong  $\varepsilon$ -reduction can be tested by a Turing machine with logarithmic working space. Technically we need a machine with

- *advice tape:* read-only, two-way tape, where the list of elements of  $A$  is written;

- *input tape*: read-only tape with random (or pseudo-random) bits, with logarithmic number of passes (the machine is allowed to pass along the input on this tape only  $O(\log N)$  times);
- *index tape*: read-only, two-way tape with logarithmic additional information;
- *work tape* that is two-way and read-write; the zone of the working tape is restricted to  $O(\log N)$ .

We interpret the content of the input tape as a list of edges of a random (or pseudo-random) graph  $G = (L, R)$ . The content of the index tape is understood as an index of a vertex  $v \in L \setminus A$ . The machine reads the bits from the ‘input tape’ (understood as a list of edges of a random graph) and checks that the vast majority of neighbors of  $v$  does not belong to the set of neighbors of  $A$ . The machine needs to read the input tape  $2d = O(\log N)$  times (where  $d$  is degree of  $v$ ): on the first pass we find the index of the first neighbor of  $v$ ; on the second pass we check whether this neighbor of  $v$  is incident to any vertex of  $A$ ; then we find the second neighbor of  $v$ , check whether it is incident to any vertex of  $A$ , etc. The machine *accepts* the input if  $|\Gamma(v) \cap \Gamma(A)| < \varepsilon d$ .

We can use Nisan’s generator [10] to fool this machine. Indeed, this checking procedure fits the general framework of [17], where Nisan’s generator was used to derandomize several combinatorial constructions. The only nonconventional feature in our argument is that the input tape is not *read-once*: we allow to read the tape with random bits logarithmic number of times<sup>1</sup>. We still can apply Nisan’s technique for a machine that reads random bits several times. David, Papakonstantinou, and Sidiropoulos observed (see [24]) that a log-space machine with logarithmic (and even poly-logarithmic) number of passes on the input tape is fooled by Nisan’s generator with a seed of size  $\text{poly}(\log N)$ .

Now we are ready to prove Theorem 5. We fix an  $\varepsilon > 0$  and a set  $A \subset \{1, \dots, m\}$  of size  $m$ . Let  $G_m$  be one of the pseudo-random generators discussed above. For all these generators, for most values of the seed  $z$  the values  $G_m(z)$  encodes a graph such that the strong  $\varepsilon$ -reduction property holds for  $A$ . Let us fix one of such seeds. We label by 1 all vertices in  $\Gamma(A)$  and by 0 all other vertices in  $R$  in the graph encoded by the string  $G_m(z)$ .

The seed value  $z$  makes the ‘auxiliary word’  $C$ , and the specified above labeling of the right part of the graph makes the bit string  $B$ . To answer a query ‘ $x \in A$ ?’ we take a random neighbor of  $x$  in the graph and check its label. If the label is 1, we answer ‘ $x \in A$ ’; otherwise we answer  $x \notin A$ .

If  $x \in A$  then there are no errors, since all neighbors of  $A$  are labeled by 1. If  $x \notin A$  then probability of an error is bounded by  $\varepsilon$  because of the strong  $\varepsilon$ -reduction property. We can answer a query in polynomial (in  $\log m$ ) time since the generators under consideration are effectively computable.

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<sup>1</sup>The same argument can be presented in a more standard framework, with a read-once *input tape* and an *index tape* of poly-logarithmic size. However, we believe that the argument becomes more intuitive when we allow many passes on the *input tape*.

### 3.4 Complexity of encoding

The disadvantage of this construction is non-effective encoding procedure. We know that for most seeds  $z$  the corresponding graph  $G_m(z)$  enjoys the strong  $\varepsilon$ -reduction property. However, we need the brute force search over all vertices  $v \in L \setminus A$  (polynomial in  $m$  but not in  $\log m$ ) to check this property for any particular seed. Thus, we have a probabilistic encoding procedure that runs in *expected* time  $\text{poly}(m)$ : we choose random seeds until we find one suitable for the given  $A$ .

In the next section we explain how to make the encoding procedure more effective (in expected time  $\text{poly}(n, \log m)$ ) for the following price: we will need a slightly greater size of the data storage, and we will take 2 bit probes instead of one at each query.

## 4 Proof of theorem 6: effective encoding

To obtain a scheme with effective encoding and decoding we combine two constructions: the explicit expander from [22] and a pseudo-random graph from the previous section.

We fix an  $n$ -element set  $A$  in the universe  $\{1, \dots, m\}$ . Now we construct two bipartite graphs that share the same left part  $L = \{1, \dots, m\}$ . The first graph is the explicit  $(m, s = n^{1+\delta} \cdot \text{poly}(\log m), d = \text{poly}(\log m), n, \varepsilon)$ -expander  $G_1 = (L, R_1, E_1)$  from [22] with an effective decoding algorithm. We do the first two steps from the encoding procedure of the BMRV-scheme explained in Section 2. At first we label all vertices in  $\Gamma(A)$  by 1 and other vertices by 0. Denote the corresponding labeling (which is a  $n^{1+\delta} \cdot \text{poly}(\log m)$ -bits string) by  $C_1$ . Then we find the list of vertices outside  $A$  that have too many 1-labeled neighbors:

$$B := \{v \in L \setminus A : |\Gamma(v) \cap \Gamma(A)| \geq \varepsilon d\}.$$

We do not re-label neighbors of  $B$ , but we will use this set later (to find  $B$  effectively, we need the property of effective decoding of the graph).

Let  $v \in L$  be a vertex in the left part of the graph. There are three different cases:

- if  $v$  belongs to  $A$  then all neighbors of  $v$  are labeled by 1;
- if  $v$  does not belong to  $A \cup B$ , then a random neighbor of  $x$  with probability  $> (1 - \varepsilon)d$  is labeled by 0;
- if  $v$  belongs to  $B$ , we cannot say anything about the vast majority of its neighbors.

Thus, if we take a random neighbor of  $v$  and see label 0 in  $C_1$  then we can say that this point does not belong to  $A$ . If we see label 1 then more detailed investigation is needed. This investigation will involve the second part  $C_2$  of the scheme defined below.

Now our goal is to distinguish between  $A$  and  $B$ . To this end we take a pseudo-random graph  $G_2 = (L, R_2, E_2)$  specified by a value of a pseudo-random generator  $G_m(z)$  (any one from the previous section). We need a *restricted on  $B$  version of the strong  $\varepsilon$ -reduction property*:

*For every  $v \in B$ , at most  $\varepsilon d$  vertices in  $\Gamma(v)$  belong to  $\Gamma(A)$ .*

Set  $B$  is of size at most  $|A|/2$  (Lemma 1), and it can be effectively computed from  $A$  (effective decoding property of the graph). Hence, for a given  $z$  we can check the property above in time  $\text{poly}(n, \log m)$ . We know that for the majority of seeds  $z$  the graph  $G_m(z)$  satisfies the strong  $\varepsilon$ -reduction property, i.e., all vertices outside  $A$  have at most  $\varepsilon d$  neighbors in  $\Gamma(A)$ . We cannot effectively check this general property (we cannot check it for *all* vertices in the universe), but we can check its restricted version (i.e., only for vertices in  $B$ ).

Thus, in average time  $\text{poly}(n, \log m)$  we can probabilistically find some seed  $x$  such that the restricted (on  $B$ ) version of the strong  $\varepsilon$ -reduction property is true. In the corresponding graph we denote by 1 all vertices in  $\Gamma(A)$ , and by 0 all vertices of the right part of the graph outside  $\Gamma(A)$ . We denote this labeling (a  $O(m \log^2 n)$ -bits string) by  $C_2$  and take it as the second part of the data storage. The seed value  $x$  is taken as ‘cached’ memory.

The decoding procedure works as follows. Given  $x \in \{1, \dots, m\}$ , we take its random neighbor in both constructed graphs and look at their labels (bits from  $C_1$  and  $C_2$  respectively).

- if the first label is 0, we say that  $x \notin S$ ;
- if the first label is 1 and the second bit is 0 then we say that  $x \notin S$ .
- if both labels are 1 then we say that  $x \in S$ .

If  $x \notin A \cup B$  then by definition of  $B$  we get that the procedure above with probability  $> (1 - \varepsilon)$  returns the correct answer. If  $x \in A$  then by construction both labels are equal to 1. If  $x \in B$ , then we have no guarantee about labels in  $C_1$ ; but from the restricted strong reducibility property it follows that with probability  $> (1 - \varepsilon)$  the second label is 0. Thus, we have one-sided error probability bounded by  $\varepsilon$ .

## 5 A lower bound for schemes with cached memory

In [15] the lower bound  $\Omega(n^2 \log m)$  was proven for one-probe schemes with one-sided errors. This result cannot be applied to schemes with small ‘cached’ memory dependent on  $A$ . In fact, our scheme of size  $\Omega(n \log m)$  with a cache of size  $\text{poly}(\log m)$  bits (from Theorem 5) is below this bound for  $n \gg \log^{O(1)} m$ .

On the other hand, the proof of the lower bound  $\Omega(\frac{n}{\varepsilon \log(1/\varepsilon)} \log m)$  (theorem 2 in [15]) with minimal changes works for schemes with cached data if the size of the pre-computed and cached information is much less than  $n \log m$ :

**Theorem 10** *We consider randomized schemes that store sets of  $n$  elements from a universe of size  $m$ , with two-parts memory (the cached memory of size  $\text{poly}(\log m)$  and the main storage).*

*For all constant  $\varepsilon < 1$ , if  $\text{poly}(\log m) \ll n \ll \sqrt[3]{m}$ , then any such scheme with error probability  $\varepsilon$  (possibly with two-sided errors) that answers queries using cached memory of size  $\text{poly} \log(m)$  and one bit probe to the main storage, must use space  $\Omega(\frac{n}{\varepsilon \log(1/\varepsilon)} \log m)$ .*

**Proof:** We follow the arguments from theorem 2 in [15] (preserving the notation). Consider any randomized scheme with two-parts memory. Denote by  $C$  the cached memory (of size  $\text{poly}(\log m)$ ) and by  $B$  the main part of the memory of size  $s$  (the scheme answers queries with one bit probe to  $C$ ). Our aim is to prove a lower bound for  $s$ .

The proof is based on the bound for the size of cover-free families of sets proven by Dyachkov and Rykov [4]. Let us remind that a family of sets  $F$  is called  $r$ -cover-free if  $f_0 \not\subseteq f_1 \cup \dots \cup f_r$  for all distinct  $f_0, \dots, f_r \in F$ .

First we take a large enough  $\frac{1}{\varepsilon}$ -cover free family  $F$  of sets of size  $n$  from the universe  $\{1, \dots, m\}$ . The construction from [7, theorem 3.1] guarantees that there exists a family  $F$  such that

$$|F| \geq \frac{\binom{m}{\varepsilon n}}{\binom{n}{\varepsilon n}^2} = 2^{\varepsilon n \log \frac{m}{n^2} + O(\log m)}.$$

By assumption, each set  $f \in F$  can be represented in our scheme by some pair  $(B, C)$  (the main storage and the cached memory). Notice that

$$|F| = 2^{\varepsilon n \log \frac{m}{n^2} + O(\log m)} \gg 2^{|B|} = 2^{\text{poly}(\log m)}.$$

Hence, there exists some value  $C$  and some  $F' \subset F$  of size

$$|F'| \geq |F| / 2^{\text{poly}(\log m)} = 2^{\Omega(\varepsilon n \log m)}.$$

such that all sets from the family  $F'$  share in our scheme the same value  $C$  of the cached memory. Further we repeat word for word the proof of theorem 2 from [15] with substitute  $F'$  instead of  $F$ .

## 6 Conclusion

In this paper we constructed an effective probabilistic bit-probe scheme with one-sided error. The used space is close to the trivial information-theoretic lower bound  $\Omega(n \log m)$ . The scheme answers queries ‘ $x \in A?$ ’ with a small one-sided error and requires only poly-logarithmic (in the size of the universe) cached memory and *one bit (two bits)* in the version with effective encoding) from the main part of the memory.

For reasonable values of parameters (for  $n \gg \text{poly}(\log m)$ ) the size of our scheme  $O(n \log^2 m)$  with a cache of size  $\text{poly}(\log m)$  is below the lower bound

$\Omega(n^2 \log m)$  proven in [15] for one-probe schemes with one-sided errors without cached data dependent on  $A$ . The gap between our upper bounds and the trivial lower bound is a  $(\log m)$ -factor.

The following questions remain open: How to construct a bit-probe memory scheme with one-sided error and effective encoding and decoding that requires to read only *one* bit from the main part of memory to answer queries? What is the minimal size of the cached memory required for a bit-probe scheme with one-sided error, with space of size  $O(n \log m)$ ?

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## References

- [1] B. Bloom. Space-time trade-offs in hash coding with allowable errors. Communications of ACM, pages 13(7) July 1970, pp. 422–426.
- [2] M. S. Pinsker. On the complexity of a concentrator. In 7th International Telegrafic Conference, 1973, pp. 318/1–318/4.
- [3] L. A. Bassalygo and M. S. Pinsker. The complexity of an optimal non-blocking commutation scheme without reorganization. Problems of Information Transmission, 9 (1974) pp. 64–66.
- [4] A.G. Dyachkov and V.V. Rykov. Bounds on the length of disjunctive codes. Problems of information transmission, 18, No. 3 (1982), pp. 7–13.
- [5] A.K. Chandra, L. Stockmeyer, and U. Vishkin. Constant depth reducibility. SIAM Journal on Computing, 13 (1984) pp. 423–429.
- [6] M. L. Fredman, J. Komlós, and E. Szemerédi. Storing a sparse table with  $O(1)$  worst case access time. Journal of the Association for Computing Machinery, 31(3), 1984, pp. 538–544.
- [7] P. Erdős, P. Frankl, and Z. Füredi. Families of finite sets in which no set is covered by the union of  $r$  others. Israel Journal of Mathematics, 51, 1985, pp. 79–89.
- [8] J. Håstad. Almost optimal lower bounds for small depth circuits. Proc. of 18th ACM STOC, 1986, pp. 6–20.
- [9] A. Siegel. On universal classes of fast high performance hash functions, their time-space trade-off, and their applications. Proc. of 30th IEEE FOCS, 1989, pp. 20–25.
- [10] N. Nisan. Pseudorandom generators for space-bounded computation. Combinatorica, 12(4), 1992, pp. 449–461. Preliminary version: STOC 1990.



- [11] N. Nisan, A. Wigderson, Hardness vs Randomness. *J. Comput. Syst. Sci.* 49(2), 1994, pp. 149–167.
- [12] N. Kahale. Eigenvalues and expansion of regular graphs. *Journal of the ACM*, 42(5), 1995, pp. 1091–1106.
- [13] A. Siegel. On universal classes of extremely random constant time hash functions and their time-space tradeoff. Technical Report TR1995-684, Courant Institute, New York University, April 1995.
- [14] J.S. Vitter. External memory algorithms and data structures. *ACM Comput. Surv.* 33(2), 2001, pp. 209–271.
- [15] Harry Buhrman, Peter Bro Miltersen, Jaikumar Radhakrishnan, Venkatesh Srinivasan, Are bitvectors optimal? *Siam J. on Computing* 31(6), 2002, pp. 1723–1744. Preliminary version: in *Proc. of STOC 2000*, pp. 449–458.
- [16] M.R. Capalbo, O. Reingold, S.P. Vadhan, A. Wigderson. Randomness Conductors and Constant-Degree Lossless Expanders. In *Proc. of the 34th ACM STOC*, pp. 659–668.
- [17] D. Sivakumar. Algorithmic derandomization via complexity theory. In *Proc. ACM STOC 2002*, pp. 619–626.
- [18] Amnon Ta-Schma. Storing information with extractors. *Information Processing Letters* 83 (2002) pp. 267–274.
- [19] S. Hoory, N. Linial, A. Wigderson. Expander graphs and their applications. *Bulletin of the American Mathematical Society*, 43(4) 2006, pp. 439–561.
- [20] J.S. Vitter. *Algorithms and Data Structures for External Memory*. Series on Foundations and Trends in Theoretical Computer Science, now Publishers, Hanover, MA, 2008.
- [21] Mark Braverman. Poly-logarithmic Independence Fools  $AC^0$  Circuits. *IEEE Conference on Computational Complexity 2009*, pp. 3–8.
- [22] V. Guruswami, C. Umans and S. Vadhan. Unbalanced expanders and randomness extractors from Parvaresh–Vardy codes, *Journal of the ACM*, 56(4), 2009.
- [23] D. Musatov. Theorems about space-bounded Kolmogorov complexity obtained by “naive” derandomization. In *Proc. Computer Science in Russia*, 2011. Prelim. version: arXiv:1009.5108 (2010).
- [24] Matei David, Periklis A. Papakonstantinou, Anastasios Sidiropoulos. How strong is Nisan’s pseudorandom generator? 2010. Electronic preprint. [http://itcs.tsinghua.edu.cn/~papakons/pdfs/nisan\\_passes.pdf](http://itcs.tsinghua.edu.cn/~papakons/pdfs/nisan_passes.pdf)