Kolmogorov complexity as a language

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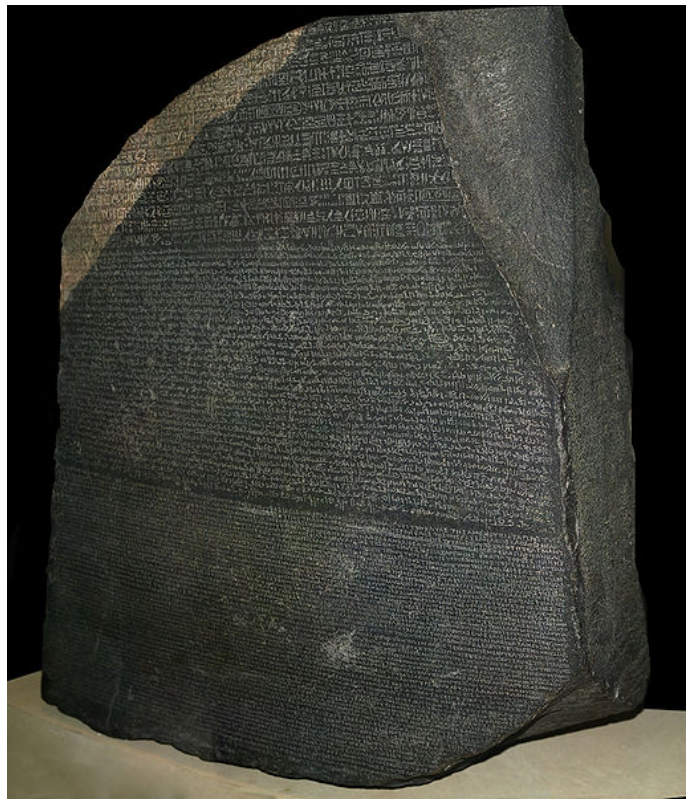

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There is also a priori probability (in two versions: discrete, on strings; continuous, on prefixes)

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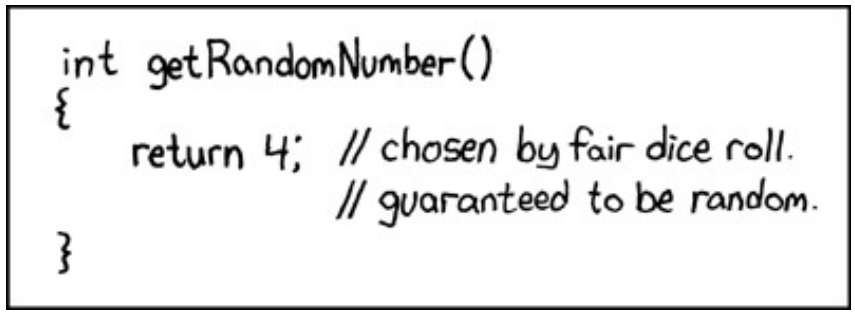

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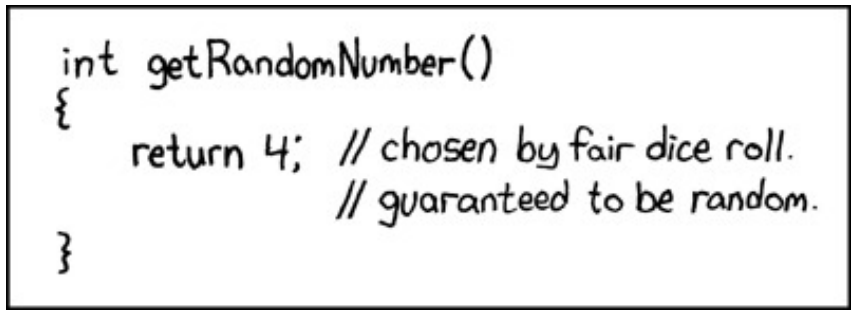

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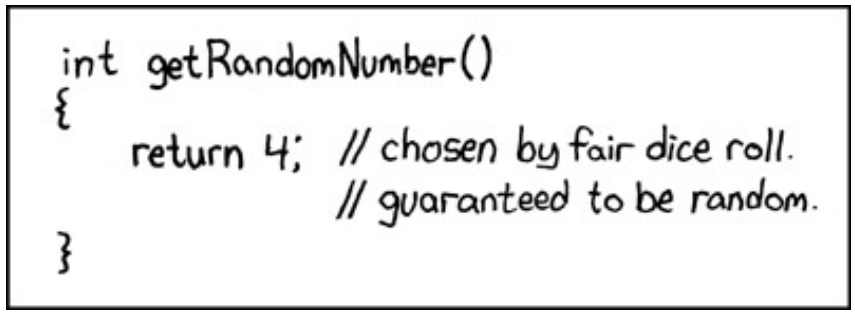

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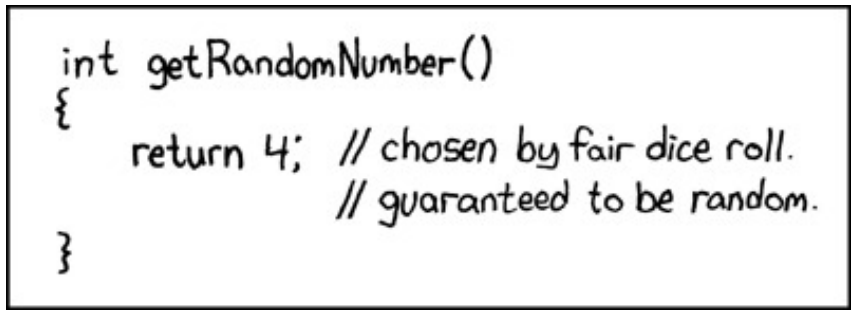

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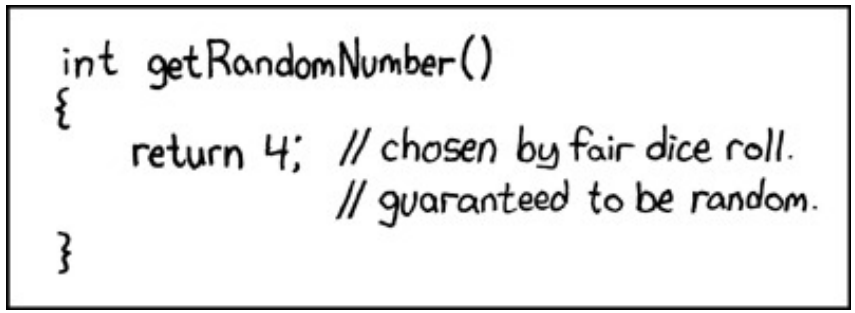

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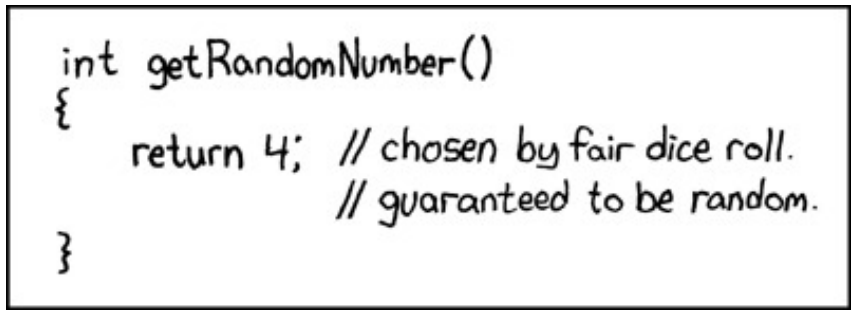

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- algorithmic $\Rightarrow$ classical: Martin-Löf random sequences form a set of measure 1.

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- but are related under some assumptions

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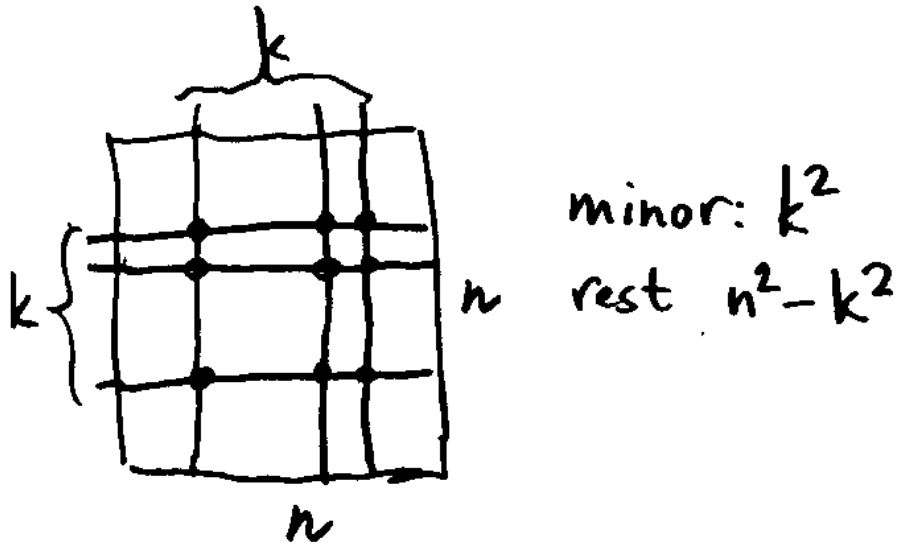

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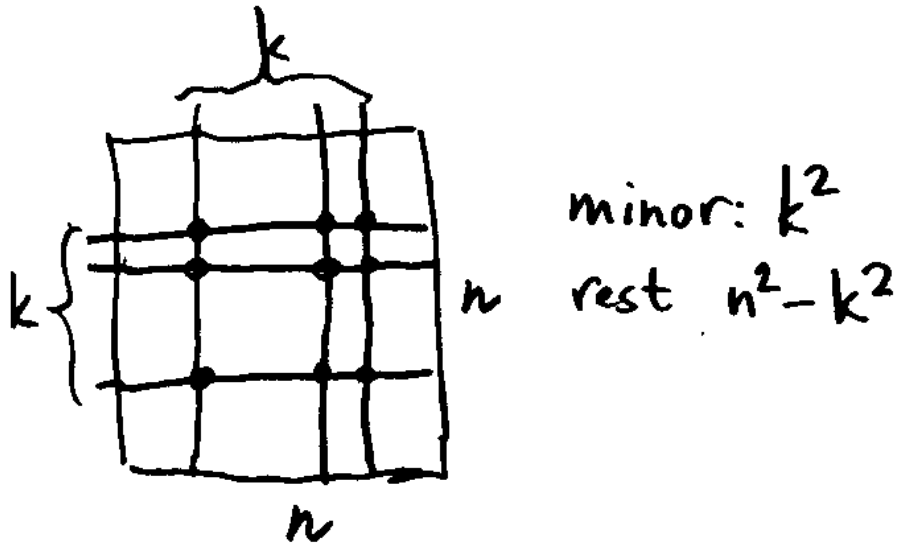

- minor is uniform if it is all-0 or all-1.
- claim: there is a $n \times n$ bit matrix without $k \times k$ uniform minors for $k = 3 \log n$.

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- $2k \log n + 1 + (n^2 - k^2) = 6 \log^2 n + 1 + (n^2 - 9 \log^2 n) < n^2$.

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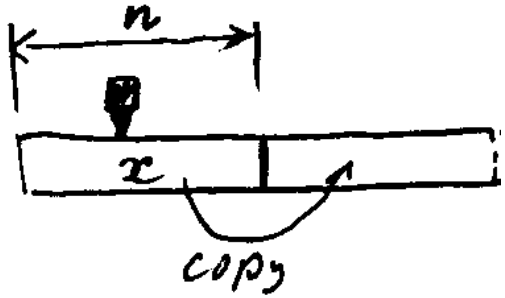

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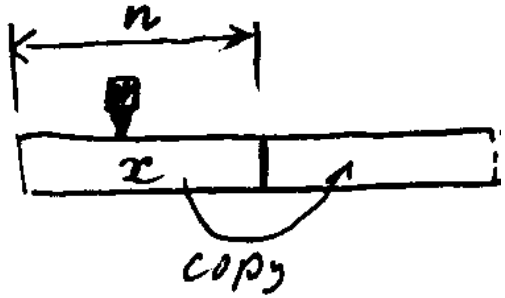

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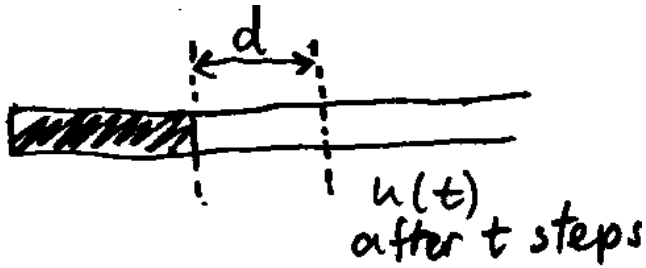

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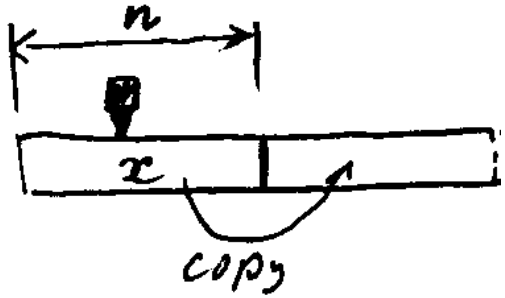

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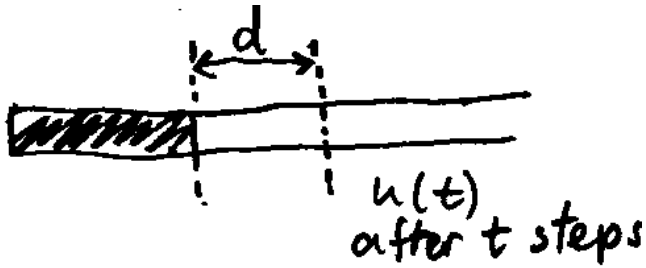

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- proof: border guards in each cell of the border security zone write down the contents of the head of TM; each of the records is enough to reconstruct $u(t)$ so the length of it should be $\Omega(K(u(t))$; the sum of lengths does not exceed time

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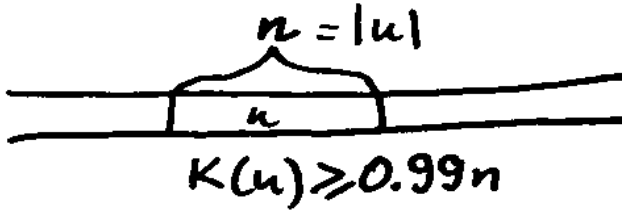

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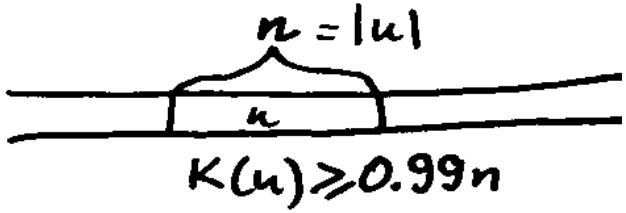

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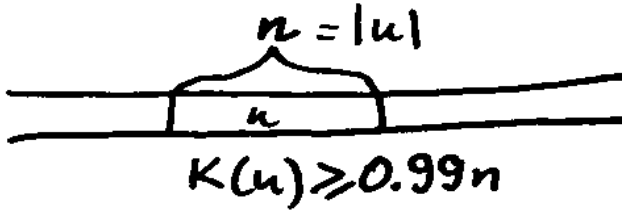

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- combinatorial and complexity proofs not just translations of each other (Lovasz lemma, Rumyantsev, Miller, Muchnik)

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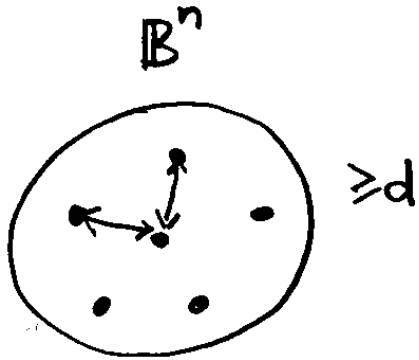

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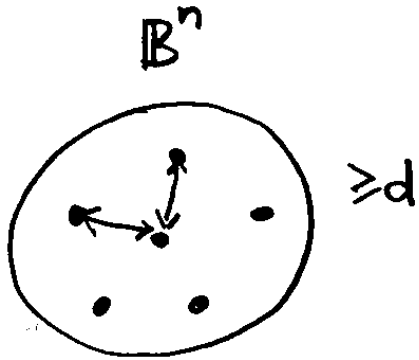

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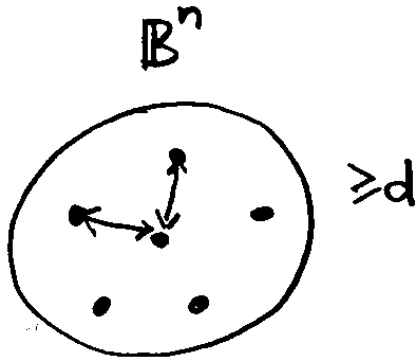

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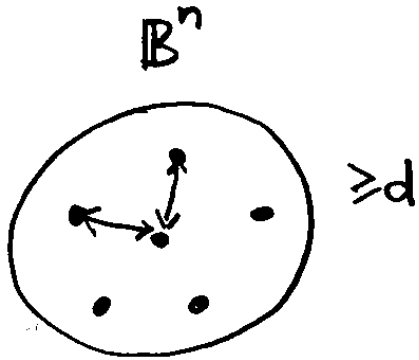

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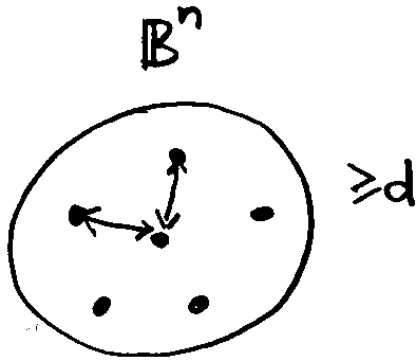

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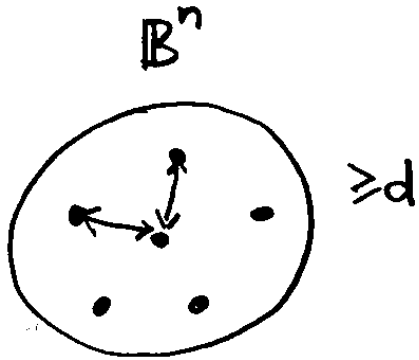

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- generalization of GV bound: d-separated family of size $\Omega(2^{n-d})$

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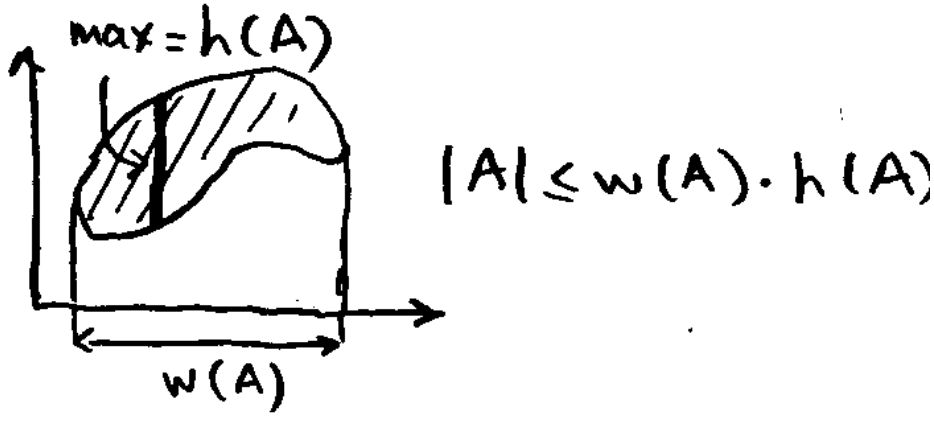

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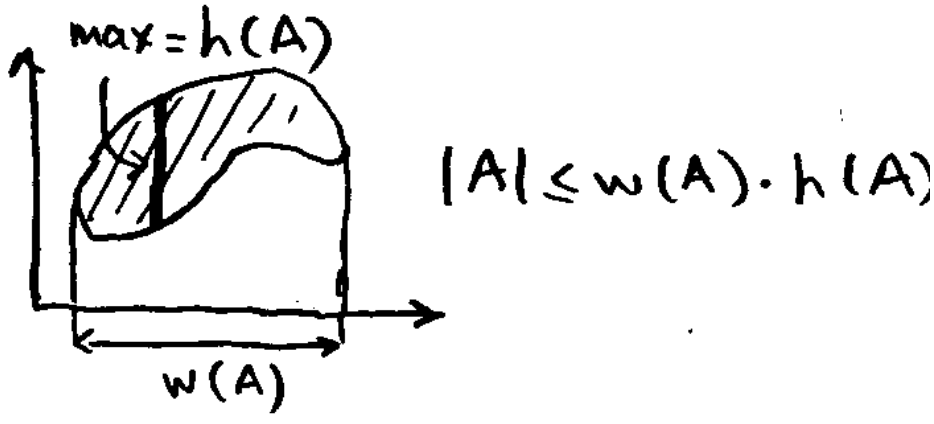

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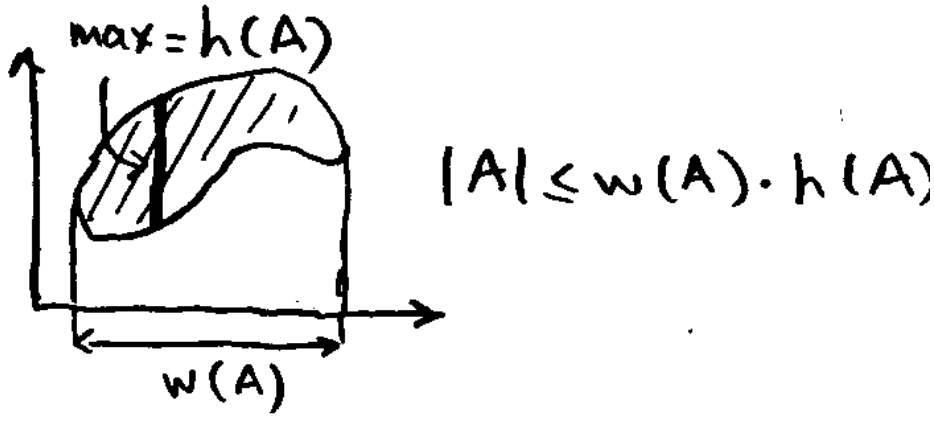

- $C(x, y) \geq C(x) + C(y|x) + O(\log)$
- $C(x, y) < k + l \Rightarrow C(x) < k + O(\log)$ or $C(y|x) < l + O(\log)$

Inequalities for complexities and combinatorial interpretation

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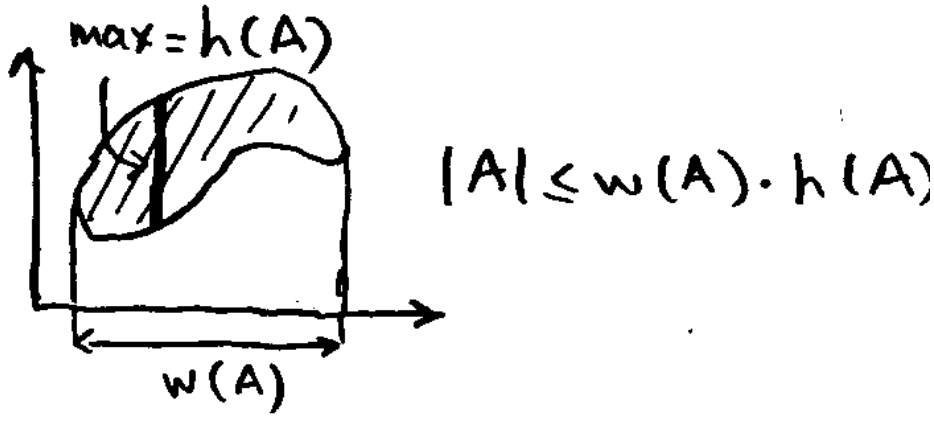

- $C(x, y) \geq C(x) + C(y|x) + O(\log)$
- $C(x, y) < k + l \Rightarrow C(x) < k + O(\log)$ or $C(y|x) < l + O(\log)$
- every set A of size $< 2^{k+l}$ can be split into two parts $A = A_1 \cup A_2$ such that $w(A_1) \leq 2^k$ and $h(A_2) \leq 2^l$

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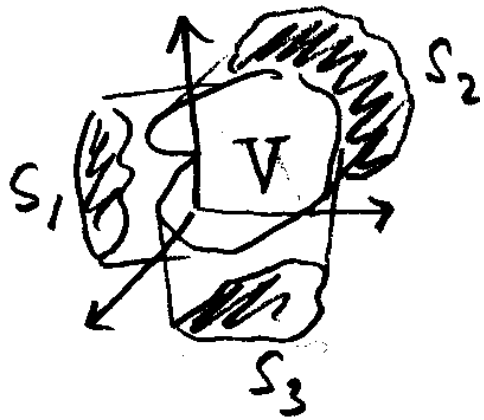

- $V^2 \leq S_1 \times S_2 \times S_3$

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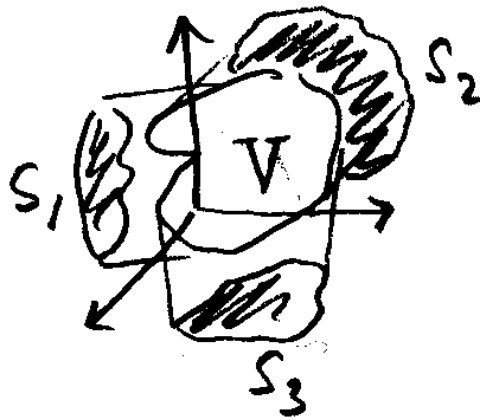
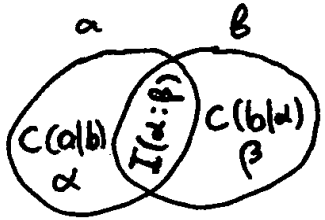


- $V^2 \leq S_1 \times S_2 \times S_3$
- Also for Shannon entropies; special case of Shearer lemma

Common information and graph minors

Common information and graph minors

- mutual information: $I(a : b) = C(a) + C(b) - C(a, b)$



$C(a) = \alpha$
$C(b) = \beta$
$C(a,b) = \gamma$

$I(a;b) =$
$= \alpha + \beta - \gamma$

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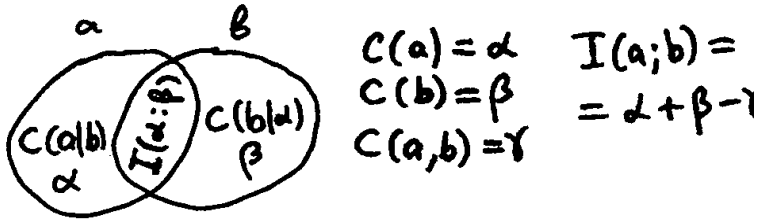

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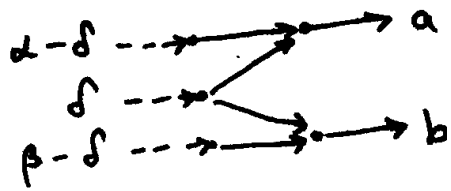

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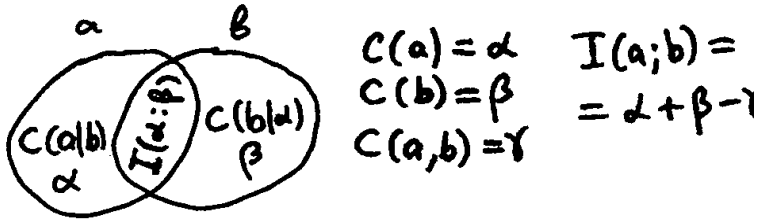

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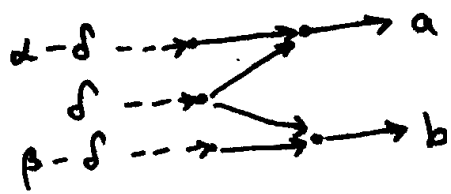
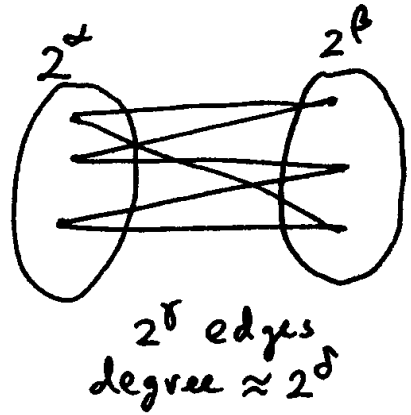


- combinatorial: graph minors



can the graph be covered by 2^δ minors of size $2^{\alpha-\delta} \times 2^{\beta-\delta}$?

Almost uniform sets

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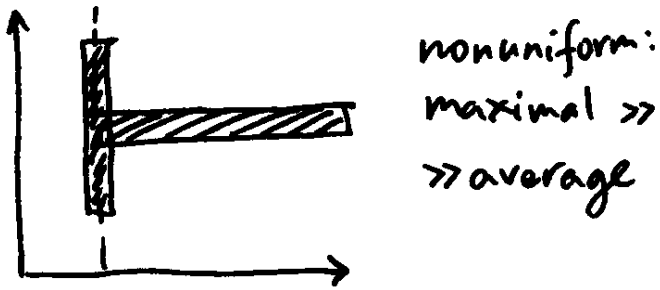

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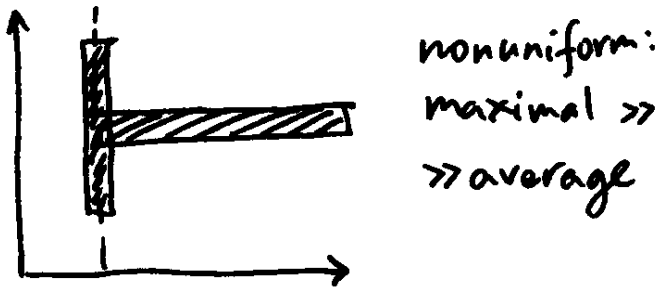

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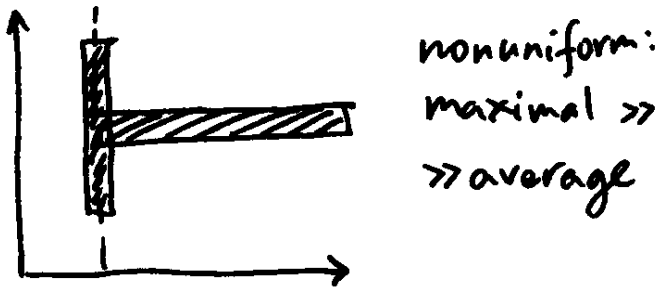

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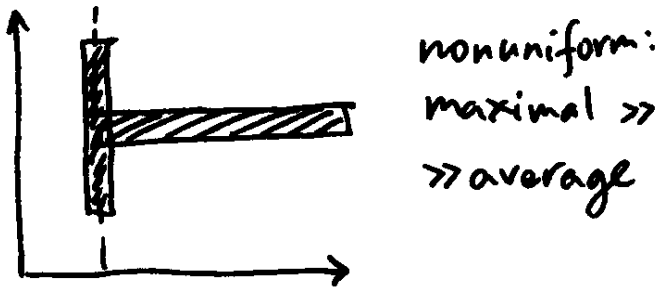

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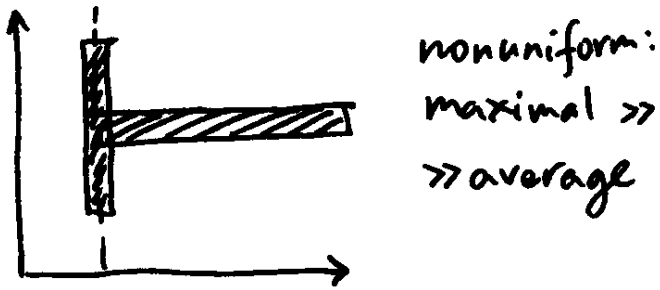

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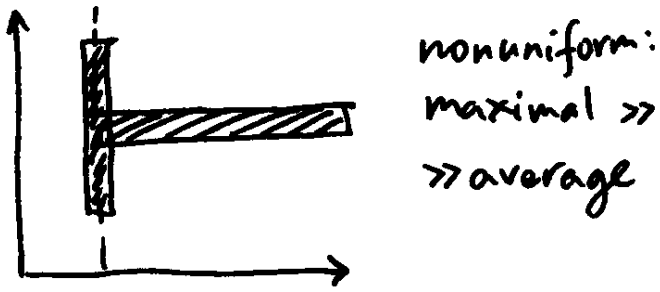

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- but combinatorial argument exists (and gives even a stronger result)

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- Complexity version: with high probablity the value of ξ^N has complexity close to $NH(\xi)$.

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- Related to Romashchenko's theorem: if three last terms are zeros, one can extract common information from a, b, e.

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- (Romashchenko, Musatov): how to use explicit extractors and derandomization to get space-bounded versions

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- "Minimal integer that cannot be described in ten English words" (Berry)

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- maximal elements: $N(2^{-n}) > BP(n - O(1))$, where $BP(k)$ is the maximal integer whose prefix complexity is k or less.

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- Recent breakthrough (Moser): this algorithm with high probability terminates quickly

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- $\Rightarrow$ CNF is satisfiable
- Non-constructive proof: lower bound for probability, Lovasz local lemma
- Naïve algorithm: just resample false clause (while they exist)
- Recent breakthrough (Moser): this algorithm with high probability terminates quickly
- Explanation: if not, the sequence of resampled clauses would encode the random bits used in resampling making them compressible

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- (Gödel second theorem, Kritchman–Raz proof): the "unexpected test paradox" (a test will be given next week but it won't be known before the day of the test)

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- Kolmogorov complexity version: we have three strings a, b, c on a blackboard. It is allowed to write (add) a new string if it simple relative to two of the strings on the board. Which strings we can obtain in $O(1)$ steps? Only strings of small complexity relative to a, b, c, but not all of them (for random a, b, c)

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- Some relation between Kolmogorov and traditional settings (Romashchenko, Kaced)

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- Full version: Eve knows some c and we want to send message of a minimal possible length $C(b|a)$

Andrej Muchnik (1958-2007)