On the Non-robustness of Essentially Conditional Information Inequalities

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Linear information inequalities

Basic inequality:

$$H(a,b) \leq H(a) + H(b) \qquad [I(a;b) \geq 0]$$
$$H(a,b,c) + H(c) \leq H(a,c) + H(b,c) \qquad [I(a;b\,|\,c) \geq 0]$$

Shannon type inequalities: any positive combination of basic ineq., e.g.,

- $H(a) \leq H(a\,|\,b) + H(a\,|\,c) + I(b;c)$

Non Shannon type inequalities, e.g., [Z. Zhang, R. W. Yeung, 1998] :

- $$\begin{aligned} I(c;d) \quad \leq \quad & I(c;d\,|\,a) + I(c;d\,|\,b) + I(a;b) + \\ & + I(c;d|a) + I(a;c\,|\,d) + I(a;d\,|\,c) \end{aligned}$$

Conditional information inequalities

If [some linear constraints for entropies] then
[a linear inequality for entropies].

- **Example 1 (trivial):** If $I(b;c) = 0$, then $H(a) \leq H(a \mid b) + H(a \mid c)$.
  Explanation:
  $H(a) \leq H(a \mid b) + H(a \mid c) + I(b;c)$.
- **Example 2 (trivial):** If $I(c;d|e) = I(c;e|d) = I(d;e|c) = 0$, then $I(c;d) \leq I(c;d|a) + I(c;d|b) + I(a;b)$.
  Explanation:
  $I(c;d) \leq I(c;d|a) + I(c;d|b) + I(a;b) + I(c;d|e) + I(c;e|d) + I(d;e|c)$.
- **Example 3 (nontrivial) [Zhang–Yeung 1997]:** If $I(a;b) = I(a;b \mid c) = 0$, then $I(c;d) \leq I(c;d \mid a) + I(c;d \mid b)$.
  Any explanation???

Essentially conditional inequalities

- **Z. Zhang, R. W. Yeung 97:**
  if $I(a;b) = I(a;b\,|\,c) = 0$, then $I(c;d) \leq I(c;d\,|\,a) + I(c;d\,|\,b)$.
- **Theorem** [KR 2011] This inequality is *essentially conditional*, i.e., for all λ_1, λ_2 the inequality

  $$I(c;d) \leq I(c;d\,|\,a) + I(c;d\,|\,b) + \lambda_1 I(a;b) + \lambda_2 I(a;b\,|\,c)$$

  does not hold!
- Several other examples of *essentially conditional inequalities*:
  F. Matúš 99, F. Matúš 07, KR 11.

Non robustness of essentially conditional inequalities

- **An example of a robust conditional inequality. (Ex 2)**
  - If $I(c;d|e) \le \varepsilon,\ I(c;e|d) \le \varepsilon,\ I(d;e|c) \le \varepsilon$,
    then $I(c;d) \le I(c;d|a) + I(c;d|b) + I(a;b) + O(\varepsilon)$.
- **An example of a non robust conditional inequality. (ZY97)**
  - Whatever is $\varepsilon > 0$, from $I(a;b) \le \varepsilon$ and $I(a;b\,|\,c) \le \varepsilon$,
    it does **not** follow that

$$I(c;d) \le I(c;d\,|\,a) + I(c;d\,|\,b) + O(\varepsilon),$$

moreover, the ratio

$$\frac{I(c;d)}{I(c;d\,|\,a) + I(c;d\,|\,b)}$$

can be arbitrarily large!

- **In this paper:** We classify the known essentially conditional inequalities.

Footnote for experts: Some conditional inequalities are valid *only* for **entropic points**, the other hold for **almost entropic points** (limits of entropic points).