A simple construction of aperiodic tile set by Kari

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1 Source of aperiodicity

Consider the following game: we have a real number in $[1,6]$; at each step we can either multiply it by 3 or divide it by 2.

Observation: starting with arbitrary number, we can do infinitely many steps (without leaving the interval), but the sequence of operations is aperiodic.

Indeed, the numbers less that 2 can be multiplied by 3, and all others can be divided by 2 without leaving the interval. If the operations are periodic, the cumulative effect of one period will be multiplication by $3^k/2^l$ for some integer k and l. This factor is different from 1, so the numbers will converge either to zero or to infinity.

Of course, the numbers 2 and 3 can be replaced by others; we use only that $\log 3/\log 2$ is irrational (i.e., $2^k \neq 3^l$ except for $k = l = 0$).

2 Idea of the construction

In a tiling, each row will represent some real number in $[1,6]$. If some row represents α, the row below it should represent either $\alpha/2$ or 3α, and this guarantees vertical aperiodicity. (If a tileset has a periodic tiling, it has a double periodic tiling, and this tiling has a vertical period, which is impossible, see the previous section.)

The question is what encoding should be used and what auxiliary information should be added to ensure that this property can be enforced by local rules.

3 Encoding

Each cell contains an integer number in the range 1..6; a line in a tiling encodes the average of the numbers in this line. More formally, the average in the line is α if for every $\varepsilon > 0$ there exists some N such that the average in each substring of length at least N is in $(\alpha - \varepsilon, \alpha + \varepsilon)$.

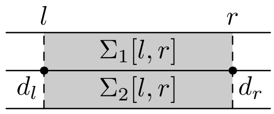
Warning: a line may have no average, so some line mays encode nothing, and we should remember that (see below).

4 Encoding of operations

To check the consistency between two neighbor lines, we need to know which operation (multiplication or division) is performed along this line, and we will have this information (in fact, one bit) everywhere along the line. It is easy to check locally that this bit is the same along the line.

5 Checking densities

The average value is a non-local notion, so we cannot locally check the required property. However, we may check a stronger local property. Imagine that want to check that the numbers in the lower line in average are twice bigger than the numbers in the upper line. Consider the sum of numbers in some interval $[l, r]$ in both lines;



the stronger property that we want to check is that $\Sigma_2[l,r] = 2\Sigma_1[l,r] + O(1)$ (or, vice versa, $\Sigma_1[l,r] = 3\Sigma_2[l,r] + O(1)$, depending on the operation).

This is guaranteed if we have a bounded sequence of integers d_i such that

$$\Sigma_2[l,r] - 2\Sigma_1[l,r] = d_r - d_l$$

for any interval $[l,r]$. Both sides of this equation are additive functions of the interval, so it is enough to check it for intervals of length 1, which is a local check.

The checks for the other operation (division by 3) are similar.

6 What is achieved and what is needed

We have described the information included in the tiling (the 1..6 numbers, the operation bit, the numbers d_i); the only thing that is *not* fixed is the range for d_i: we said that $|d_i|$ are bounded but have not fixed a specific upper bound D.

We need to show that

(1) this tile set (=set of local rules) has no periodic tilings (for every D);

(2) this set has a tiling, if D is large enough.

7 No periodic tilings

The problem here is that (a priori) the line may not have an average.[1] However, if a periodic tiling exist, a tiling with two independent periods also exists, and it has vertical and horizontal periods. In this case all the horizontal averages exist and we get a contradiction.

[1] In fact, for the specific parameters chosen, i.e., for range 1..6, multiplication by 3 and division by 2, the lines are guaranteed to have an average, but this requires an additional argument. See the Appendix for more details.

8 Why tilings exist

To construct a tiling, we first assign the real numbers for each line and corresponding operations. We can start by assigning an arbitrary number in $[1,6]$ for some line and the go in both directions choosing an operation that does not move us outside $[1,6]$. It is always possible (we have discussed this for the forward direction, but a similar argument works for backward direction).

Then each number should be represented as the average of integers in the corresponding line. This can be done in many ways, but we choose a specific one: to represent α, we consider the sequence $\lfloor \alpha m \rfloor$ for integer n; then α equals the average of *differences* between neighbor integers in this sequence:

$$\alpha = \text{average of}(\lfloor (m+1)\alpha \rfloor - \lfloor m\alpha \rfloor).$$

For such a sequence we have

$$\Sigma[l,r] = \lfloor r\alpha \rfloor - \lfloor l\alpha \rfloor;$$

note that the right hand side differs from desired value $(r-l)\alpha$ by $O(1)$ (in fact, at most by 1).

Now look at two neighbor lines that correspond, say, to α and 2α. In the notation used above we have

$$\Sigma_1[l,r] = (r-l)\alpha + O(1)$$
$$\Sigma_2[l,r] = 2(r-l)\alpha + O(1),$$

so

$$\Sigma_2[l,r] - 2\Sigma_1[l,r]$$

in bounded by some (absolute) constant. This difference is a bounded additive function of the interval $[l,r]$, so it can be represented as $d_r - d_l$ for some bounded d_m (e.g., we can let $d_0 = 0$ and then define all other d_m as the function of the interval $[0,m]$; when m is negative, we reverse the sign).

9 Appendix

1. This construction follows the paper of Kari [1]; the only difference is that Kari wanted to beat a record on the minimal number of tiles (and did beat it, getting first 14 tiles and then reducing the number to 13 by noting that one tile is not used [2]). So he explicitly lists all the tiles (using factors 2 and $2/3$ instead of 2 and $1/3$; in this way the interval $1/2..2$ is enough and only digits $0..2$ are needed).

2. As we have mentioned, in our construction one can guarantee the existence of average in every tiling (though it is not needed for the argument); let us sketch the proof.

First, some preparations. Let α and β are two positive real numbers and let $I = [L, R]$ be some interval on the real line (both endpoints L, R are positive). Starting with some point in I, we want to construct a sequence of points in I such that each next point is obtained from the previous one either by multiplication by α or division by β.

Lemma.

(1) If $R/L \geq \alpha\beta$, then it is possible for every starting point; more over, the sequence can be extended to the left.

(2) If $R/L = \alpha\beta$, the choice of the next operation is unique (except for one exceptional case: βL can be both divided by β or multiplied by α).

(3) If $\log\beta/\log\alpha$ is irrational, for $R/L < \alpha\beta$ we cannot stay in $I = [L, R]$ indefinitely.

The statements (1) and (2) are obvious. To prove (3), note that that choice of the next operation is unique in this case. In logarithmic scale, we start with some number and that either add $\log\alpha$ or subtract $\log\beta$ (choosing the operation that does not move the point outside $[\log L, \log R]$). Let us increase the interval in such a way that R/L is $\alpha\beta$ (so in logarithmic scale the length is $\log\alpha + \log\beta$) and the sequence never comes very close to the endpoints of the interval. Then the process is deterministic: we iterate the transformation that is an irrational rotation of a circle. It is well known that the orbits for an irrational rotation are dense, so we cannot avoid coming arbitrarily close to the endpoints, which contradicts our assumption.

It remains to explains how all this is related to the existence of line averages. Assume that some line in some tiling does not have an averages. This means that for some $p < q$ this line has arbitrarily long substrings with average less than p as well as arbitrarily long substrings with average greater than q. The tiling also determines a sequence of operations that are applied to lines (whether multiplication by 3 or division by 2 happens). Let us apply this sequence of operation to some real inbetween p and q; the statement (3) guarantees that we will either almost exceed $R = 6$ or go almost below $L = 1$. Therefore, either we significantly exceed R starting from q (or any greater number) or get significantly below L starting from p (or any smaller number). Assume, for example, that we are in the first case. Now we come to a contradiction: take a very long segment where average is q or higher; the local consistency rules guarantee that the average in the subsequent lines will be very close to the transformation result; if the segment is long enough, at some point we get average more than $R = 6$ which is impossible since all cells are at most 6.

3. Even if we do not want to force the existence of averages, we still can finish the proof without using 2-periodicity: take long fragment of the sequence of reals where the density changes significantly, and for long enough finite intervals we get a contradiction, since boundary effects are negligible (after time interval is fixed and space interval goes to infinity). This assumes that the period is not horizontal; if it is horizontal, we have limit frequency in each line, and we have only finitely many different lines.

References

[1] Kari

[2] Kari and Culik