On the expressive power of quasi-periodic SFT

Bruno Durand and Andrei Romashchenko

MFCS 2017

Subshits: motivations and central notions

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Subshifts: finite type

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  (crystals, quasicrystals, etc.)

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Central idea: local rules $\Longrightarrow$ global properties

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Subshift on $\mathbb{Z}^d$ over an alphabet Σ :

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Follows from the definition:

- A **subshift** is shift-invariant

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- A **subshift** is topologically closed

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In $\dim = 1$ all SFT are pretty simple

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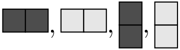
- $\dim = 2$ and $\mathcal{F} = \{$  $\}$

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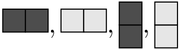
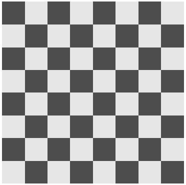
- $\dim = 2$ and $\mathcal{F} = \{$  $\}$
- configurations: checkerboard, 

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So for $\dim \geq 2$ the **SFT**s are not that trivial!

SFTs simulate more sophisticate structures: (1) sofic subshifts

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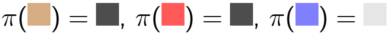
Simple example. 

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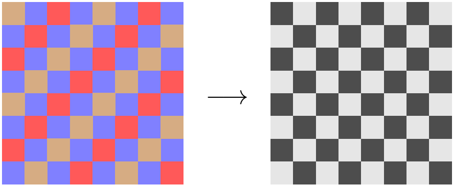

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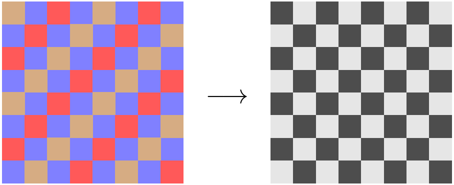

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$\dim = 1$:

sofic subshift = subshift recognized by a nondeterministic finite automata

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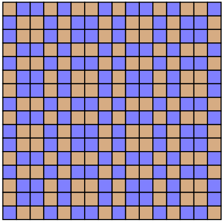
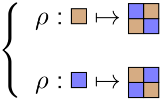
A typical pattern corresponding to the Thue–Morse system:



Example: Thue–Morse substitution rule



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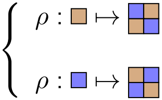

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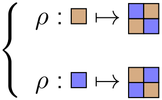

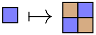

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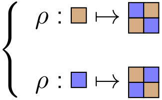

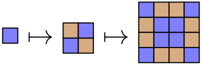

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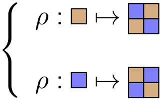

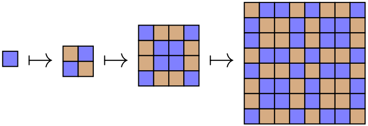

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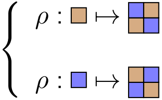

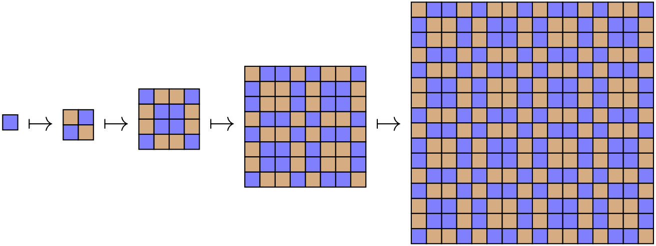

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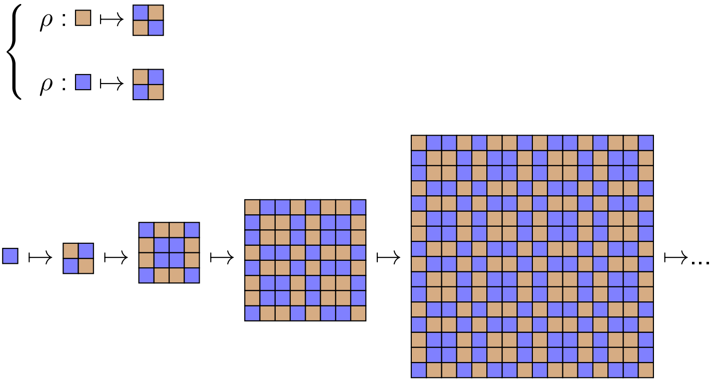

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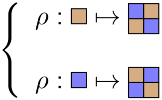

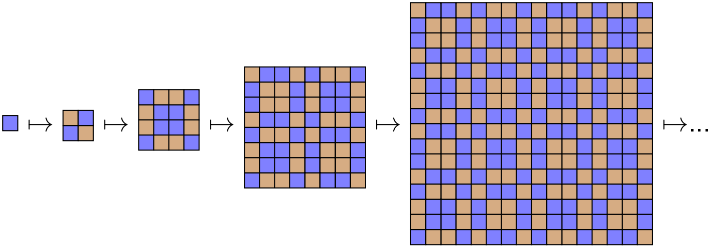

The limit of this sequence results in a sofic subshift!

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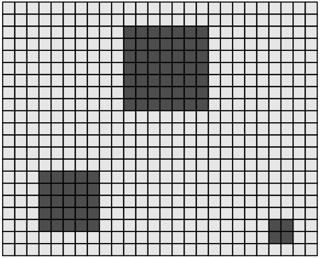

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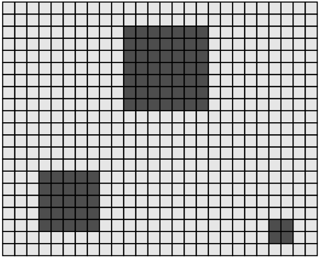

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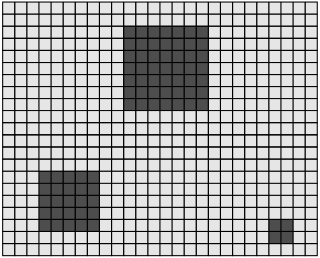

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This subshift is sophic [L. B. Westrick 2016]

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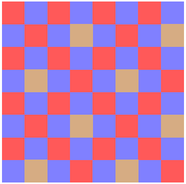

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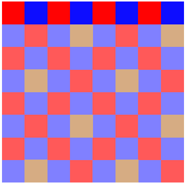

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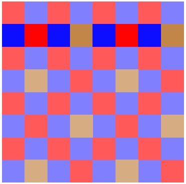

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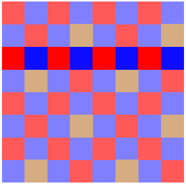

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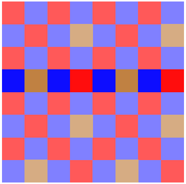

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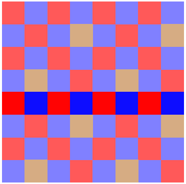

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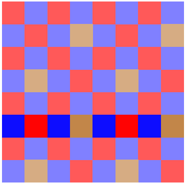

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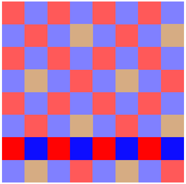

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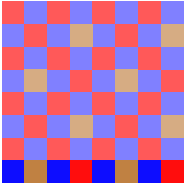

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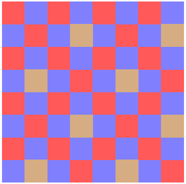

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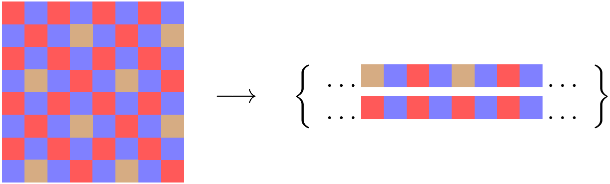

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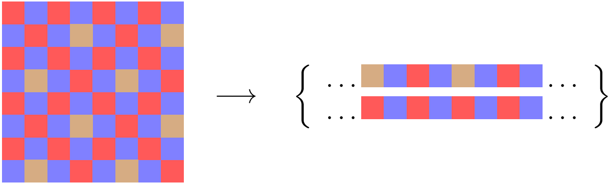

restriction to a subspace can make things **much** more complex

SFT simulates any effective subshift

Theorem [Hochman, D-R-Shen, Aubrun-Sablik]

For every effective subshift S_1 there exists an SFT S_2 that **simulates** S_1.

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Partial answer:

Yes, we can! We can transfer this result to **minimal** subshifts.

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Definition.

A subshift S is **minimal** if there is no nonempty subshifts $S' \subsetneq S$.

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One More Theorem (for experts).

For every **quasiperiodic** effective subshift S_1 there exists a **quasiperiodic** SFT S_2 such that S_2 **simulates** S_1.

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Corollary.

There exists a **quasiperiodic** 2-dim SFT where Kolmogorov complexity of all $n \times n$ patterns is equal to $\Omega(n)$.

under the hood: gear wheels in the proof

Algorithmic part:

recursive programming (a program handling its own text)

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Combinatorial part:

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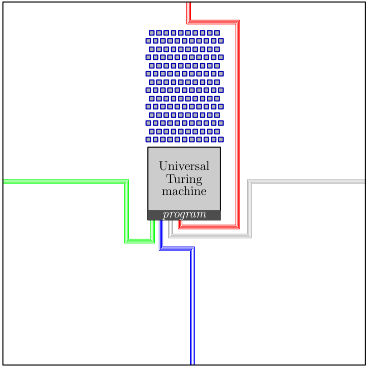

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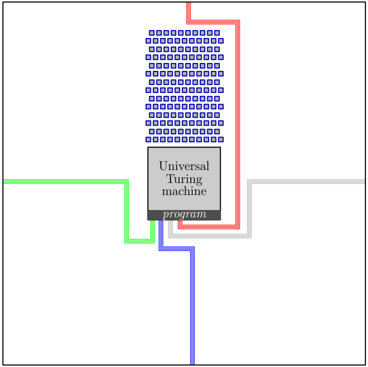

... + enforced quasi-periodicity + combinatorial lemmas

under the hood: gear wheels in the proof

Combinatorial part (folklore?): combinatorics on quasiperiodic words

Lemma 1.

If $\mathbf{x} = (x_n)$ is *recurrent* (quasiperiodic) and $\mathbf{y} = (y_n)$ is *periodic*, then the product $\mathbf{x} \otimes \mathbf{y}$

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is also recurrent.

Lemma 2.

If a subshift $\mathcal{S}$ is *minimal* and a sequence $\mathbf{y}$ is *periodic*, then the subshift

$$\left\{ \ldots \begin{array}{|c|c|c|c|c|} \hline x_0 & x_1 & x_2 & x_3 & x_4 \\ \hline y_0 & y_1 & y_2 & y_3 & y_4 \\ \hline \end{array} \ldots \quad \text{where} \quad \ldots x_0 x_1 x_2 x_3 x_4 \ldots \text{ belongs to } \mathcal{S} \right\}$$

is also *minimal*.