On Algorithmic Statistics for space-bounded algorithms

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Let $x = \overbrace{01001011\ldots010}$ be a random string of length n i.e. its *Kolmogorov complexity* $\mathrm{C}(x)$ is equal to n. Then $\{0,1\}^n$ is an reasonable explanation for x, however $\{x\}$ is not adequate.

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- $d(x|A) \gtrsim 0$ for every x in A.
- The fraction of elements x in A such that $d(x|A) > k$ is less than 2^{-k}.

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- Consider another parameter instead of $d(x|A)$: just $\log|A|$.
- What can we say about $\mathrm{C}(A) + \log|A|$ for $A \ni x$?
- $\mathrm{C}(A) + \log|A| \gtrsim \mathrm{C}(x)$.
- The difference $\delta(x, A) := \mathrm{C}(A) + \log|A| - \mathrm{C}(x)$ is called *optimality deficiency*.

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- The difference can be large.

Example

Let x be random string of length n (i.e. $\mathrm{C}(x) \approx n$). Let y another independent of x random string of length n. Consider $A := \{0,1\}^n \setminus \{y\}$. Then $d(x|A) \approx 0$ however $\delta(x,A) \approx n$.

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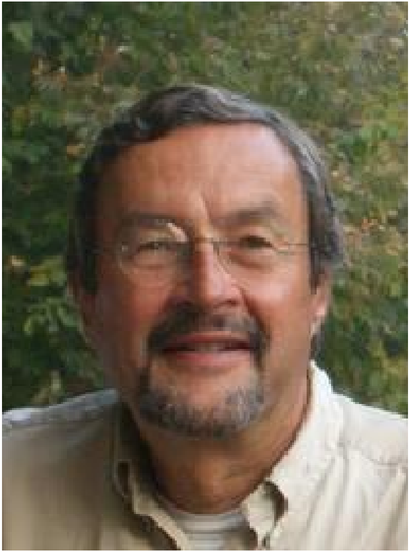
However, the following is true.

Theorem (Vereshchagin, Vitányi)

For every string x and for every $A \ni x$ there exists $B \ni x$ such that $\mathrm{C}(B) \lesssim \mathrm{C}(A)$ and $\delta(x, B) \lesssim d(x|A)$.



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Theorem (Vereshchagin, Vitányi)

For every string x and for every $A \ni x$ from any enumerable family $\mathcal{A}$ there exists $B \in \mathcal{A}$ containing x such that $\mathrm{C}(B) \lesssim \mathrm{C}(A)$ and $\delta(x, B) \lesssim d(x|A)$.

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Definition

The complexity $\mathrm{CD}^m(A)$ of a set A with space bound m is defined as the minimal length of a program p such that

- $p(y) = 1$ if $y \in A$.
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Theorem (Informal)

Let x be a string of length n and let $\mathcal{A}$ be a polynomial-space enumerable family of sets. Then for every set $A \ni x$ from $\mathcal{A}$ there exists a set $B \ni x$ from $\mathcal{A}$ such that $\mathrm{CD}^{poly(n)}(B) \lesssim \mathrm{CD}^{poly(n)}(A)$ and $\delta^{poly(n)}(x, B) \lesssim d^{poly(n)}(x|A)$.

Proof idea

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- Define probability distribution $\mathcal{B}$ as follows. Every set from $\mathcal{A}$ of complexity $\mathrm{CD}^{poly(n)}(A)$ belongs to $\mathcal{B}$ with probability $2^{\mathrm{CD}^{\mathsf{poly}(n)}(A|x)-\mathrm{CD}^{\mathsf{poly}(n)}(A)}$.

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- The same idea was used by Daniil Musatov.

Thank you!