Random noise increases Kolmogorov complexity

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joint work with Gleb Posobin
based on the discussions with Peter Gacs

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- random codeword: no decrease

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- combinatorial tool: Harper's theorem (Hamming balls have minimal neighborhoods)

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- exact lower bound for this increase

Complexity increases with high probability

Theorem

Let $\alpha \in (0,1)$ and $\tau \in (0,1/2)$. There exists some $\beta > \alpha$ with the following property:

$$\mathrm{C}(x) \geqslant \alpha n \Rightarrow \Pr[\mathrm{C}(N_\tau(x)) \geqslant \beta n] \geqslant 1 - \frac{1}{n}$$

for sufficiently large n and for every x of length n

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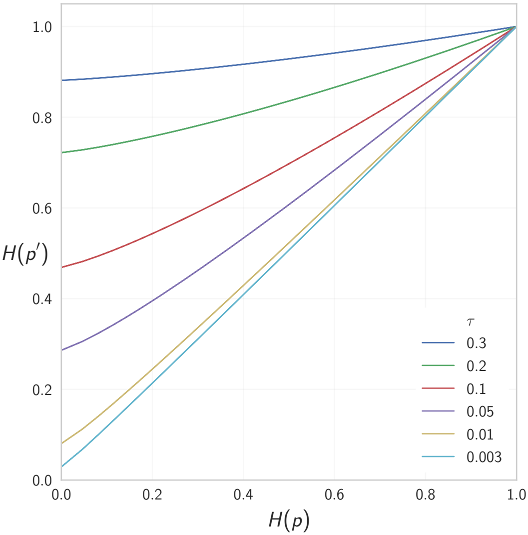
$1/n$ can be replaced by $1/n^d$ for arbitrary fixed d

Optimal lower bound for the complexity increase



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- complexity increase $H(p) \mapsto H(N(p,\tau))$ for Bernoulli random strings

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Let $p \in (0, 1/2)$ *and* $\tau \in (0, 1/2)$*.*
Let $\alpha = H(p)$ *and* $\beta = H(N(\tau, p))$*. Then*

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for $n \to \infty$ *and for every* x *of length* n*. This* β *is the best possible bound.*

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Remark: for some strings (e.g., random codewords) we have better bounds, but the lower bound is optimal: one cannot improve β *for all strings*

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- we need all three

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- Harper's theorem: minimal neighborhoods / maximal interiors happen for Hamming balls

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- so a string of complexity $\geqslant \alpha n + O(\log n)$ is *not* in this interior...

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- so a string of complexity $\geqslant \alpha n + O(\log n)$ is *not* in this interior...
- i.e., it can be changed in at most τn places to get outside X, i.e., to have complexity $\geqslant \beta n$

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- but there are too many of them: contradiction

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- (weak combinatorial) if $\#B \leqslant 2^{\beta n}$, and every element of A get into B with probability at least $1 - \frac{1}{n}$ after τ-noise, then $\#A \leqslant 2^{\alpha n}$.

All equivalent with precision $o(n)$ for complexity (log-cardinality)

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- weak combinatorial $\Rightarrow$ combinatorial: concentration inequality (McDiarmid inequality, a version of Azuma–Hoeffding inequality)

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- "tensorization" + convexity argument

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It remains to check that the curves are convex (computation with power series)

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- the same lower bound curve for the increase
- one may use different noise levels for different positions
- every sequence of dimension α can be changed in a negligible fraction of positions (Besicovitch distance 0) to a strongly α-random sequence. [weakly random: Greenberg et al.]

Thanks!