finite state AIT, normal numbers

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ultimate test for some notion/theory:

- solve an open problem outside the theory
- make non-trivial proofs trivial

"finite state AIT" and normality

disclaimer: well-known approach in a different envelope (Agafonov, Schnorr,...)

$R(p, x)$

$$K(x) = \min\{|p| : p \text{ is a description of } x\}$$

which "description modes" are allowed?

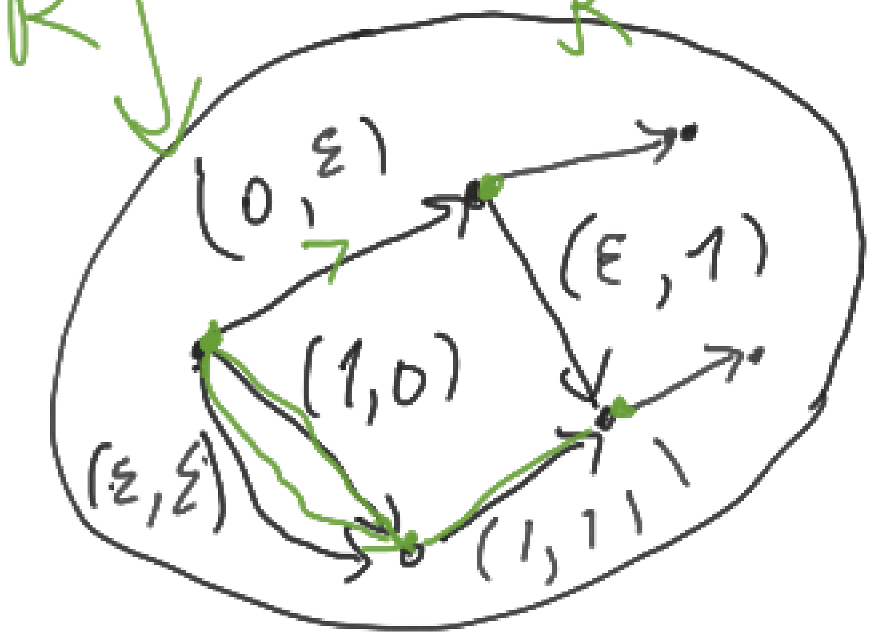
plain: computable functions $p \mapsto x$

or $O(1)$-values c.e. sets of (p, x)

prefix-free: computable functions with prefix-free domain

finite state AIT?

$R \longmapsto K_R$



automatic

$p \longrightarrow x$ O(1)-valued

(u_1, v_1)

$u, v \in \{\varepsilon, 0, 1\}$ (u_2, v_2)

$R = \{(p, x) \mid \text{there is a path}\}$

(11, 01)

(1, 1)

u, v

no initial state

$K_R(xy) \geqslant K_R(x) + K_R(y)$: superadditivity

"locality"

it's OK that $K_R(0) = K_R(1) = 0$.

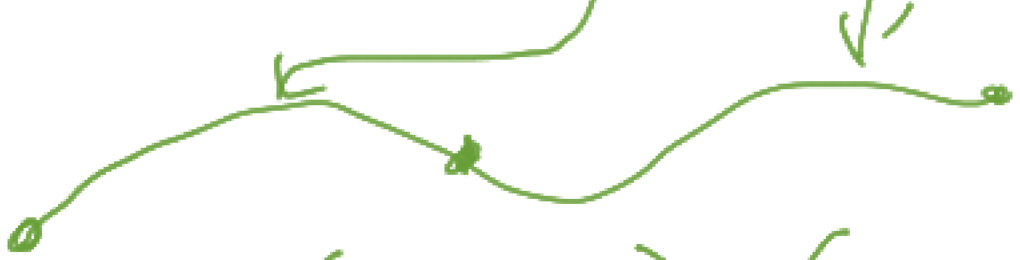
$O(1)$-valued: external semantic restriction

no universality

$C(x) \leqslant K_R(x) + O_R(1)$

$(p, xy) \in R$

$p_x p_y$



$(p_x, x), (p_y, y)$

C, K

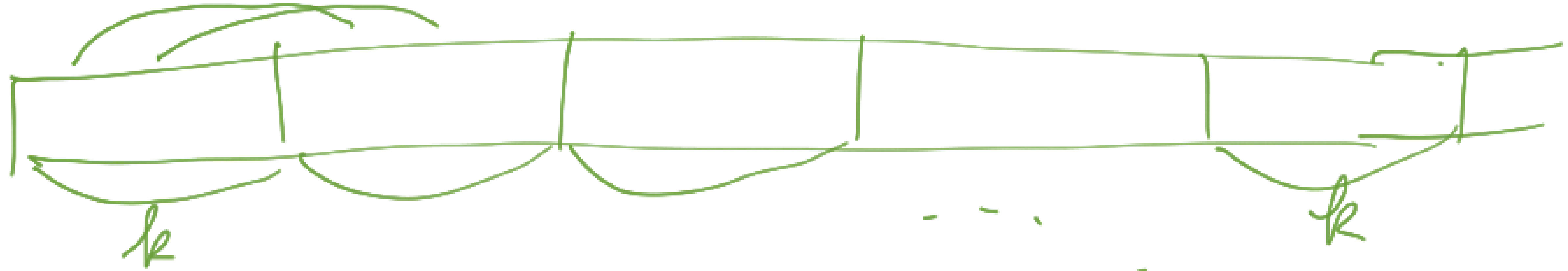
finite state Hausdorff dimension:

$$\inf_R \liminf_n \frac{K_R(x_1 \ldots x_n)}{n}$$

finite state packing dimension:

$$\inf_R \limsup_n \frac{K_R(x_1 \ldots x_n)}{n}$$

f.s. randomness
"



$x_1x_2\dots$ normal:

for all 2^k blocks

freq. $\to 2^{-k}$

$$\underbrace{\forall R \ \liminf_n \frac{K_R(x_1 \dots x_n)}{n} \geqslant 1}$$

$x_1x_2\dots$ normal with respect to p-Bernoulli measure

1	p
0	$1-p$

$$\forall R \ \liminf_n \frac{K_R(x_1 \dots x_n)}{|x_1 \dots x_n|_p} \geqslant 1$$

$$|x|_p = \underbrace{\#_1(x)}_{|1|} \log \frac{1}{p} + \underbrace{\#_0(x)}_{|0|} \log \frac{1}{1-p}.$$

$-\log_2 [B_p(x)]$

$$\forall R \qquad K_R(\underbrace{\qquad\qquad}_{n}) \geq n - o(n)$$

"Champernowne number"

$$K_R\left(\overbrace{0\ 1\ 10\ 11\ 100\ 101\ 110\ 111\ 1000\ 1001\ \ldots}\right) \leftarrow \text{normal}$$

$$\geq$$

$$K_R(0) + K_R(1) + \ldots + K_R(\ldots)$$

$$\geq$$

$$C(0) \quad C(1) + \ldots + C(\ldots)$$

$$C(x) \approx x \quad |0| + |1| + \ldots + |\ldots|$$

standard definition:

for each k all k-block have the same limit frequency ← normality

$X = \mathcal{X}_1$ (classical)

Not normal: distribution on blocks far from normal

Shannon code gives significantly shorter description

block prefix code → automaton

not uniform

Normal: all blocks appear with the same frequency

2^k

superadditivity

increase k

C as lower bound

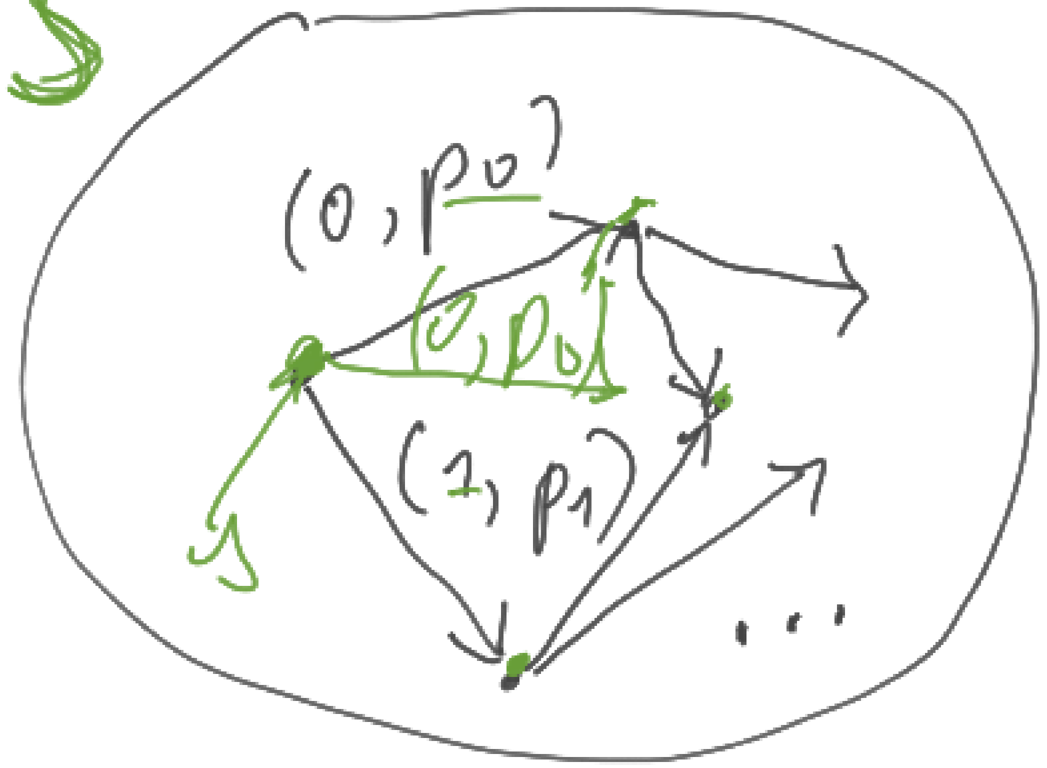
average $C(k\text{-bit block}) \approx k$

constant error vanishes as $k \to \infty$

$K_R(\square \ldots) \geq K_R(\square) + K_R(\square) \ldots$

Finite state a priori probability

S



random walk with output

for each s a measure P_s

$$P_S(x) = \max_s P_s(x)$$

$$KA_S(x) = -\log P_S(x)$$

$p_0 + p_1 = 1$

xy $\quad x \quad y$

superadditive: $KA_S(xy) \geqslant KA_S(x) + KA_S(y)$

can be used for characterization (= definition with gales)

K_R

KA_S

K : strings $\to \mathbb{R}^{\geq 0}$

machine-independent definition

- $K(xy) \geqslant K(x) + K(y)$ [superadditivity]
- $\sum_{x:|x|=k} 2^{-K(x)} \leqslant \mathrm{poly}(k)$ [calibration]

K_R

KA_S

finite state Hausdorff dimension:

$x_1 \ldots x_n \ldots$

$$\inf_K \liminf_n \frac{K(x_1 \ldots x_n)}{n}$$

inf taken over all superadditive calibrated K

- many known results are easy corollaries
- technical tools: Shannon code
- and Shannon's inequality (for B_p case)
- basic facts from AIT

$p_1 \dots p_n$

$H(p)+1 \geq \qquad \geq H(p)$

Shannon

THANKS!