On Parallels Between Shannon's and Kolmogorov's Information Theories (where the parallelism fails and why)

Andrei Romashchenko, LIRMM (Montpellier)

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Outline

1. Parallelism in definitions
2. Perfect parallelism: information inequalities
3. The first threat to the parallelism: conditional inequalities
   - A conditional inequality: why it is so special
   - Parallelism re-established
   - Unexpected profit from the parallelism
4. A more serious threat: conditional inequalities once again
5. Parallels for conditional descriptions: one victory and one defeat

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- to see the same phenomenon from different points of view
- to not miss the cases when the parallelism fails

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- Hartley: "combinatorial entropy", $\chi(A) := \log |A|$ for a finite set A

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- Hartley: $\chi(A_{1|2}) := \max_{y \in \pi_2(A)} \log |\{x \ : \ (x, y) \in A\}|$

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- Hartley: well, it's getting boring...

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Naive philosophy:

- Shannon: average complexity
- Kolmogorov: algorithmic complexity
- Hartley: worst case complexity

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- Hartley: $\chi(A) \leq \chi_1(A) + \chi_{2|1}(A)$
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less standard properties

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- Hartley:
  $2\chi(A) \leq \chi(A_{12}) + \chi(A_{23}) + \chi(A_{13})$
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  $|A|^2 \leq |\pi_{12}(A)| \cdot |\pi_{23}(A)| \cdot |\pi_{13}(A)|$

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Hammer and Shen:
Kolmogorov's or Shannon's inequality implies Hartley's inequality
(*A strange application of Kolmogorov complexity, 1998*)

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- $I_K(a : b) = I_K(b : a) + O(\log C(a, b))$
- $C(a, b) \leq C(a) + H(b) + O(\log \ldots)$

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general inequalities

Equivalence Theorem [Hammer, R., Shen, Vereshchagin]
Exactly the same linear inequality are valid for Shannon's entropy and for Kolmogorov complexity.

for all $(\alpha_1, \ldots, \alpha_n)$

$$\begin{aligned}\lambda_1 H(\alpha_1) + \lambda_2 H(\alpha_2) + \lambda_3 H(\alpha_3) + \ldots \\ +\lambda_{12} H(\alpha_1, \alpha_2) + \lambda_{13} H(\alpha_1, \alpha_3) + \ldots \\ + \lambda_{123} H(\alpha_1, \alpha_2, \alpha_3) + \ldots \ \ldots \ \ldots \geq 0\end{aligned}$$

if and only if

there exists a $C > 0$ *such that for all* $(a_1, \ldots, a_n)$

$$\begin{aligned}\lambda_1 C(a_1) + \lambda_2 C(a_2) + \lambda_3 C(a_3) + \ldots \\ +\lambda_{12} C(a_1, a_2) + \lambda_{13} C(a_1, a_3) + \ldots \\ + \lambda_{123} C(a_1, a_2, a_3) + \ldots \ \ldots \ \ldots \\ + C \log(|a_1| + \ldots + |a_n|) \geq 0\end{aligned}$$

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the intuition behind the information inequalities

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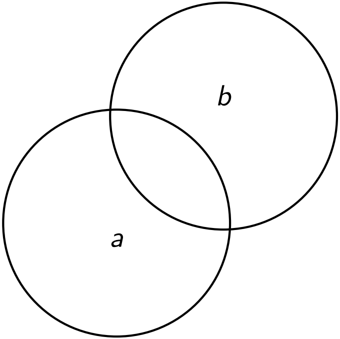
information diagrams



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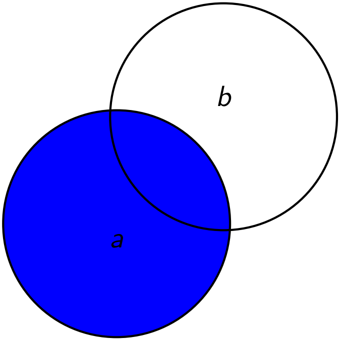
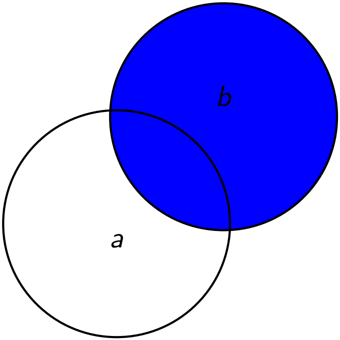


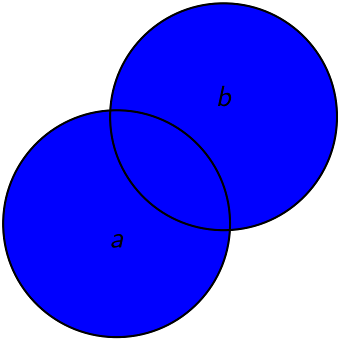
$H(a)$

information diagrams



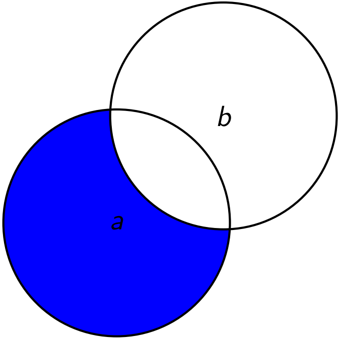
$H(b)$

information diagrams



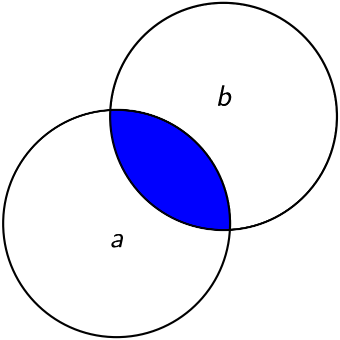
$H(a, b)$

information diagrams



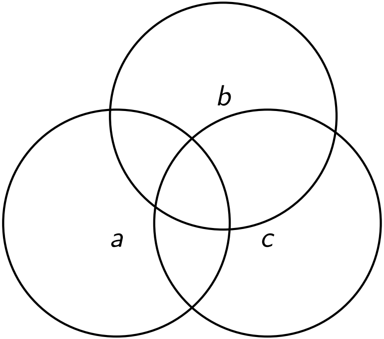
$H(a|b) = H(a, b) - H(b)$

information diagrams



$I_K(a : b) = H(a) + H(b) - H(a, b)$

information diagrams



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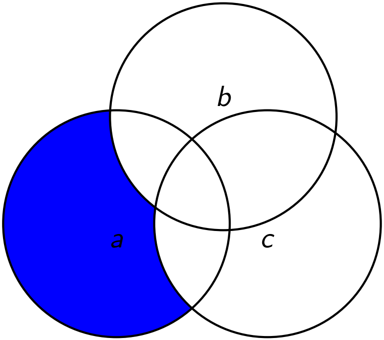
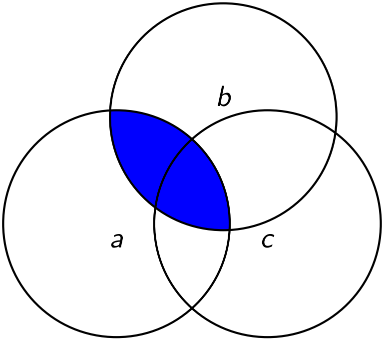


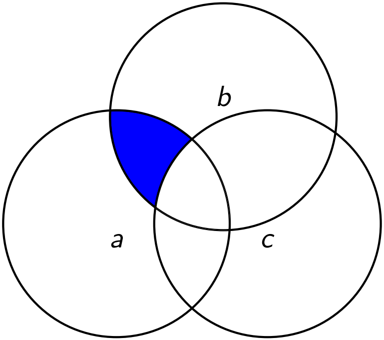
$H(a|b,c) = H(a,b,c) - H(b,c)$

information diagrams



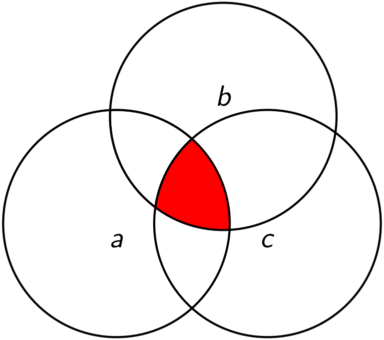
$I_S(a:b) = H(a) + H(b) - H(a,b)$

information diagrams



$I_S(a : b|c) = H(a,c) + H(b,c) - H(a,b,c) - H(c)$

information diagrams



$$I_S(a:b:c) = H(a)+H(b)+H(c)-H(a,b)-H(a,c)-H(b,c)+H(a,b,c)$$

the intuition behind the information inequalities

What are all these linear "information inequalities" ?

We know from Shannon:

- monotonicity: $H(\alpha) \le H(\alpha, \beta)$
- subadditivity: $H(\alpha, \beta) \le H(\alpha) + H(\beta)$
  (a.k.a. $I_S(\alpha : \beta) \ge 0$)
- submodularity: $H(\alpha, \beta, \gamma) + H(\alpha) \le H(\alpha, \beta) + H(\alpha, \gamma)$
  (a.k.a. $I_S(\beta : \gamma|\alpha) \ge 0$)

plus all substitutions

plus all (positive) linear combinations

Anything else?

non Shannon type inequalities

Yes!

non Shannon type inequalities

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Z. Zhang and R. W. Yeung, [1998]:
the first non Shannon type information inequality

non Shannon type inequalities

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Z. Zhang and R. W. Yeung, [1998]:
the first non Shannon type information inequality
$I_S(a:b) \leq 2I_S(a:b|x) + I_S(a:b|y) + I_S(x:y) + I_S(a:x|b) + I_S(b:x|a)$

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Since 1998
we got *other* examples of non Shannon type information inequalities

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F. Matúš [1999]:
For $n > 3$ random variables the cone of all linear information inequalities is not polyhedral !

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For $n > 3$ random variables the cone of all linear information inequalities is not polyhedral !

Applications: lower bounds for secret sharing schemes, for admissible rates in communication networks, etc.

Outline

Outline

conditional inequalities: parallelism under a threat

Another kind of information inequalities for Shannon's entropy:

if $I_S(a:b|c) = I_S(a:c|b) = I_S(b:c|a) = 0$,
then $I_S(a:b) \leq I_S(a:b|x) + I_S(a:b|y) + I_S(x:y)$

conditional inequalities: parallelism under a threat

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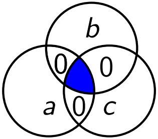
Sketch of the proof:
If $I_S(a:b|c) = I_S(a:c|b) = I_S(b:c|a) = 0$,
then $\exists w$ such that $H(w|a) = H(w|b) = 0$ and $H(w) = I_S(a:b)$



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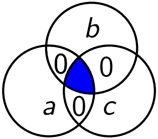

Then we apply a Shannon type inequality

$$H(w) \quad \le \quad H(w|x) \quad + \quad H(w|y) \quad + \quad I_S(x:y)$$

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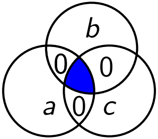

Then we apply a Shannon type inequality

$$\begin{array}{ccccccc} H(w) & \leq & H(w|x) & + & H(w|y) & + & I_S(x:y) \\ || & & || & & || & & \\ I_S(a:b) & & I_S(a:b|x) & & I_S(a:b|y) & & \end{array}$$

conditional inequalities: parallelism under a threat

How to re-formulate it for Kolmogorov complexity?

conditional inequalities: parallelism under a threat

How to re-formulate it for Kolmogorov complexity?

if

$$\begin{cases} I_K(a:b|c) & = & O(\log\ldots), \\ I_K(a:c|b) & = & O(\log\ldots), \\ I_K(b:c|a) & = & O(\log\ldots), \end{cases}$$

then $I_K(a:b) \le I_K(a:b|x) + I_K(a:b|y) + I_K(x:y) + O(\log\ldots)$

How to prove it?

conditional inequalities: soft version

the conjectured inequality for Kolmogorov complexity:

if

$$\left\{ \begin{array}{rcl} I_K(a:b|c) &=& O(?), \\ I_K(a:c|b) &=& O(?), \\ I_K(b:c|a) &=& O(?), \end{array} \right.$$

then $I_K(a:b) \le I_K(a:b|x) + I_K(a:b|y) + I_K(x:y) + O(?)$

conditional inequalities: soft version

the conjectured inequality for Kolmogorov complexity:

if

$$\begin{cases} I_K(a:b|c) &= O(?), \\ I_K(a:c|b) &= O(?), \\ I_K(b:c|a) &= O(?), \end{cases}$$

then $I_K(a:b) \le I_K(a:b|x) + I_K(a:b|y) + I_K(x:y) + O(?)$

Shannon's version with "soft" constraints:

if

$$\begin{cases} I_S(a:b|c) &= \text{[sth small]}, \\ I_S(a:c|b) &= \text{[sth small]}, \\ I_S(b:c|a) &= \text{[sth small]}, \end{cases}$$

then $I_S(a:b) \le I_S(a:b|x) + I_S(a:b|y) + I_S(x:y) + \text{[sth small]}$

Outline

conditional inequalities: coping with the threat

Revisit the Shannon's version of the inequality:

if $I_S(a:b|c) = I_S(a:c|b) = I_S(b:c|a) = 0$,
then $I_S(a:b) \le I_S(a:b|x) + I_S(a:b|y) + I_S(x:y)$

conditional inequalities: coping with the threat

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Alternative proof:

Step 1: prove an unconditional non Shannon type inequality:

$$I_S(a:b) \leq I_S(a:b|x) + I_S(a:b|y) + I_S(x:y) + \\ + I_S(a:b|c) + I_S(a:c|b) + I_S(b:c|a)$$

conditional inequalities: coping with the threat

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Step 2: deduce the conditional inequality

conditional inequalities: coping with the threat

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Step 2: deduce the conditional inequality (even with "soft" constraints!)

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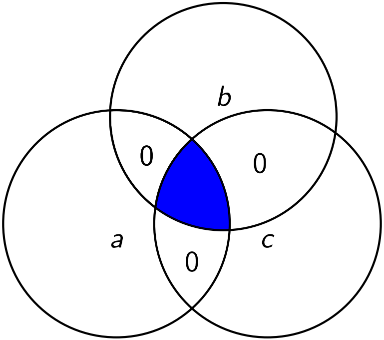
Step 2: deduce the conditional inequality (even with "soft" constraints!)

Towards Kolmogorov's version: the unconditional inequality rewrites to
Step 1': $I_K(a:b) \leq I_K(a:b|x) + I_K(a:b|y) + I_K(x:y) +$
$+ I_K(a:b|c) + I_K(a:c|b) + I_K(b:c|a) + O(\log\ldots)$

Step 2': deduce the conditional version for Kolmogorov complexity

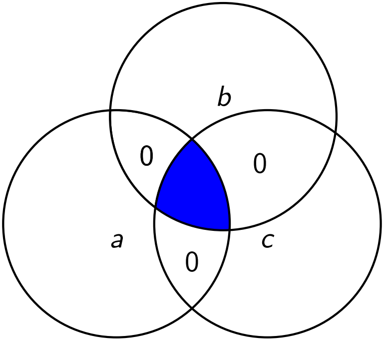
Outline

Ahlswede-Körner and extracting the mutual information



We can *materialize* the mutual information $I_S(a : b : c)$:

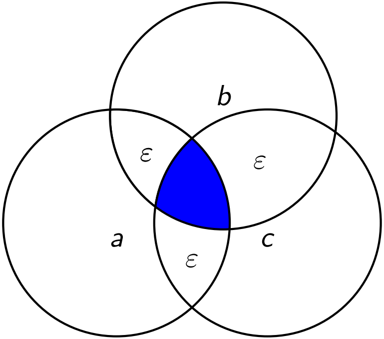
Ahlswede-Körner and extracting the mutual information



We can *materialize* the mutual information $I_S(a : b : c)$: $\exists w$ s.t.

- $H(w) = I_S(a : b : c)$,
- $H(w|a) = 0$,
- $H(w|b) = 0$,
- $H(w|c) = 0$.

Ahlswede-Körner and extracting the mutual information



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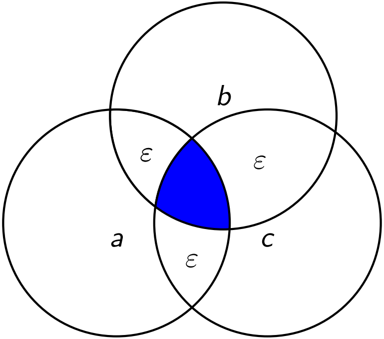
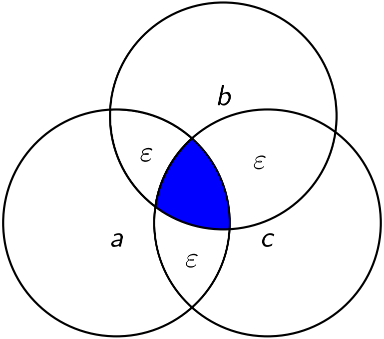


Can we now *materialize* the mutual information $I_S(a : b : c)$?

Ahlswede-Körner and extracting the mutual information



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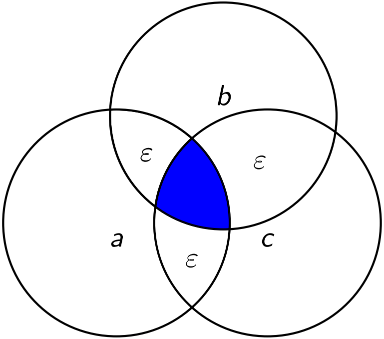
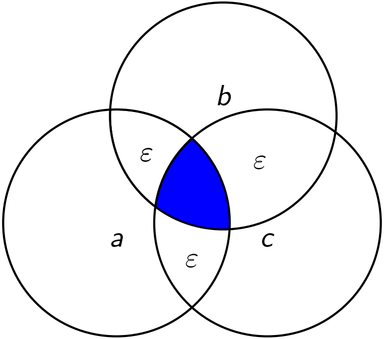
Yes, in some sense...

Ahlswede-Körner and extracting the mutual information



Can we now *materialize* the mutual information $I_S(a:b:c)$?
Yes, in some sense...
Let $(a_1, b_1, c_1), \ldots, (a_n, b_n, c_n)$ be i.i.d., distributed as (a, b, c).

Ahlswede-Körner and extracting the mutual information



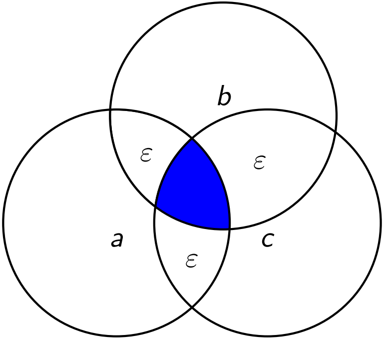
Can we now *materialize* the mutual information $I_S(a : b : c)$?

Yes, in some sense...

Let $(a_1, b_1, c_1), \ldots, (a_n, b_n, c_n)$ be i.i.d., distributed as (a, b, c). $\exists W$ s.t.

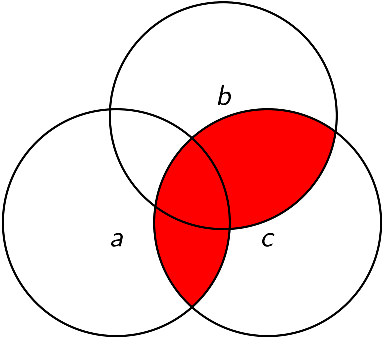
- $H(W) = n \cdot I_S(a : b : c) + O(\varepsilon) + o(n)$,
- $H(W|a_1 \ldots a_n) = O(\varepsilon) + o(n)$,
- $H(W|b_1 \ldots b_n) = O(\varepsilon) + o(n)$,
- $H(W|c_1 \ldots c_n) = O(\varepsilon) + o(n)$.

Ahlswede-Körner and extracting the mutual information



How to *materialize* the mutual information $I_S(a : b : c)$?

Ahlswede-Körner and extracting the mutual information



Let $(a_1, b_1, c_1), \dots, (a_n, b_n, c_n)$ be i.i.d., distributed as (a, b, c).

Ahlswede and Körner [1975]: there exists a W such that

- $H(W) = n \cdot I_S(a, b : c) + o(n)$,
- $H(a_1 \dots a_n | W) = n \cdot H(a|c) + o(n)$,
- $H(b_1 \dots b_n | W) = n \cdot H(b|c) + o(n)$,
- $H(a_1 \dots a_n, b_1 \dots b_n | W) = n \cdot H(a, b|c) + o(n)$.

from Ahlswede-Körner to Zhang-Yeung

Step 0. take n i.i.d. copies of (a, b, c, x, y)

from Ahlswede-Körner to Zhang-Yeung

Step 0. take n i.i.d. copies of (a, b, c, x, y)
$\bar{a} := a_1 \ldots a_n$, $\bar{b} := b_1 \ldots b_n$, $\bar{c} := c_1 \ldots c_n$, $\bar{x} := x_1 \ldots x_n$, $\bar{y} := y_1 \ldots y_n$

from Ahlswede-Körner to Zhang-Yeung

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Step 1. take W from Ahslwede-Körner (for the triple a, b, c)

from Ahlswede-Körner to Zhang-Yeung

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Step 1. take W from Ahslwede-Körner (for the triple a, b, c)

Step 2. Apply a Shannon type inequality

$$H(W) \quad \leq \quad H(W|\bar{x}) \quad + \quad H(W|\bar{y}) \quad + \quad I_S(\bar{x} : \bar{y})$$

from Ahlswede-Körner to Zhang-Yeung

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from Ahlswede-Körner to Zhang-Yeung

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$$\begin{array}{ccccccc} H(W) & \leq & H(W|\bar{x}) & + & H(W|\bar{y}) & + & I_S(\bar{x} : \bar{y}) \\ \wr & & \wr & & \wr & & \\ I_S(\bar{a} : \bar{b}) & & I_S(\bar{a} : \bar{b}|\bar{x}) & & I_S(\bar{a} : \bar{b}|\bar{y}) & & \end{array}$$

Conclusion: if $I_S(a : b|c) \approx I_S(a : c|b) \approx I_S(b : c|a) \approx 0$, then
$I_S(a : b) \leq I_S(a : b|x) + I_S(a : b|x) + I_S(x : y) +$ [small residue term]

from Ahlswede-Körner to Zhang-Yeung

Step 0. take n i.i.d. copies of (a, b, c, x, y)
$\bar{a} := a_1 \dots a_n$, $\bar{b} := b_1 \dots b_n$, $\bar{c} := c_1 \dots c_n$, $\bar{x} := x_1 \dots x_n$, $\bar{y} := y_1 \dots y_n$

Step 1. take W from Ahslwede-Körner (for the triple a, b, c)

Step 2. Apply a Shannon type inequality

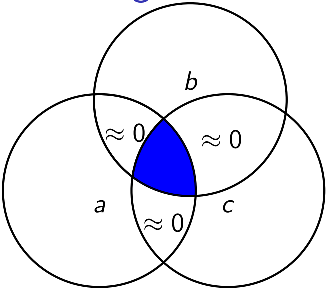
$$\begin{array}{ccccccc} H(W) & \leq & H(W|\bar{x}) & + & H(W|\bar{y}) & + & I_S(\bar{x} : \bar{y}) \\ \wr\wr & & \wr\wr & & \wr\wr & & \\ I_S(\bar{a} : \bar{b}) & & I_S(\bar{a} : \bar{b}|\bar{x}) & & I_S(\bar{a} : \bar{b}|\bar{y}) & & \end{array}$$

Conclusion: if $I_S(a : b|c) \approx I_S(a : c|b) \approx I_S(b : c|a) \approx 0$, then
$I_S(a : b) \leq I_S(a : b|x) + I_S(a : b|x) + I_S(x : y) +$ [small residue term]

More precisely:

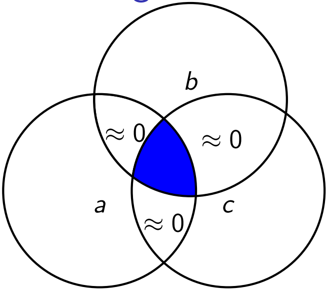
$$I_S(a : b) \leq I_S(a : b|x) + I_S(a : b|y) + I_S(x : y) + \\ + I_S(a : b|c) + I_S(a : c|b) + I_S(b : c|a)$$

extracting mutual information in Kolmogorov's framework



$$\left\{ \begin{array}{lcl} I_K(a:c|b) & \approx & 0, \\ I_K(a:b|c) & \approx & 0, \\ I_K(b:c|a) & \approx & 0. \end{array} \right. \quad \Longrightarrow \quad \text{the mutual information can be "materialized"}$$

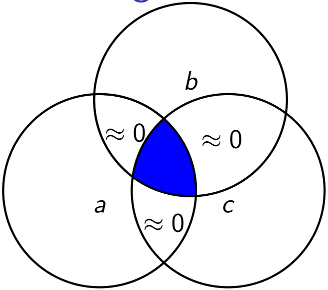
extracting mutual information in Kolmogorov's framework



$$\begin{cases} I_K(a:c|b) & \approx & 0, \\ I_K(a:b|c) & \approx & 0, \\ I_K(b:c|a) & \approx & 0. \end{cases} \implies \text{the mutual information can be "materialized"}$$

No Ahslwede-Körner Lemma for Kolmogorov compl.

extracting mutual information in Kolmogorov's framework



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No Ahslwede-Körner Lemma for Kolmogorov compl. However, we can prove $\exists w$:

- $C(w) \approx I_K(a:b:c)$,
- $C(w|a) \approx 0$,
- $C(w|b) \approx 0$,
- $C(w|c) \approx 0$.

Outline

conditional inequalities: another threat

Another conditional information inequality for Shannon's entropy, F. Matúš:

if $I_S(a : x|b) = I_S(b : x|a) = 0$,
then $I_S(a : b) \leq I_S(a : b|x) + I_S(a : b|y) + I_S(x : y)$

conditional inequalities: another threat

Another conditional information inequality for Shannon's entropy, F. Matúš:

if $I_S(a:x|b) = I_S(b:x|a) = 0$,
then $I_S(a:b) \leq I_S(a:b|x) + I_S(a:b|y) + I_S(x:y)$

Sketch of the proof:
Step 1. if $I_S(a:x|b) = I_S(b:x|a) = 0$,
then $\exists w$ such that $H(w|x) = H(w|y) = 0$ and $I_S(a:w) = I_S(a:xy)$.

conditional inequalities: another threat

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if $I_S(a : x|b) = I_S(b : x|a) = 0$,
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Step 2. apply a suitable Shannon type inequality, etc.

conditional inequalities: another threat

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A version with "soft constraints"?

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A version with "soft constraints"?

A version for Kolmogorov complexity?

conditional inequalities: parallelism undermined

hard-constraints Shannon's entropy:

if $I_S(a:x|b) = I_S(b:x|a) = 0$,
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conditional inequalities: parallelism undermined

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No soft-constraints linear version for Shannon's entropy:
whatever are coefficients λ_1, λ_2, for some distribution (a, b, x, y)
$I_S(a:b) \not\le I_S(a:b|x) + I_S(a:b|y) + I_S(x:y) + \lambda_1 I_S(a:x|b) + \lambda_2 I_S(b:x|a)$

conditional inequalities: parallelism undermined

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However, a *kind of* Kolmogorov's version:
if $I_K(a:x|b) \le \sqrt{n}$ and $I_K(b:x|a) \le \sqrt{n}$, then
$I_K(a:b) \le I_K(a:b|x) + I_K(a:b|y) + I_K(x:y) + O(n^{3/4})$

conditional inequalities: parallelism fails

hard-constraints Shannon's inequality [Kaced, R.]:

if $I_S(x : y|a) = H(a|x, y) = 0$,
then $I_S(a : b) \leq I_S(a : b|x) + I_S(a : b|y) + I_S(x : y)$

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No soft-constraints version Shannon's:
whatever are coefficients λ_1, λ_2, for some distributions
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No Kolmogorov's version: for some strings a, b, x, y
$I_K(x : y|a) = O(\log n)$ and $C(a|x, y) = O(\log n)$
but $I_K(a : b) \gg I_K(a : b|x) + I_K(a : b|y) + I_K(x : y)$

conditional inequalities: parallelism fails

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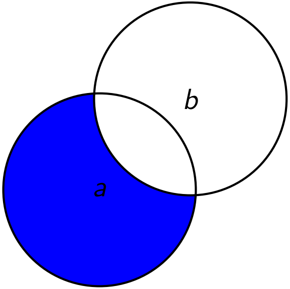
No Kolmogorov's version: for some strings a, b, x, y
$I_K(x:y|a) = O(\log n)$ and $C(a|x,y) = O(\log n)$
but $I_K(a:b) \gg I_K(a:b|x) + I_K(a:b|y) + I_K(x:y)$ (the gap $= \Omega(n)$)

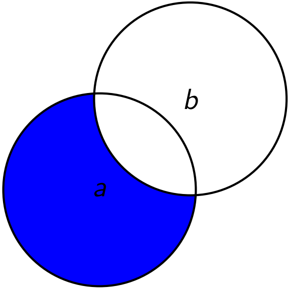
Outline

conditional encoding: slepian-wolf vs muchnik



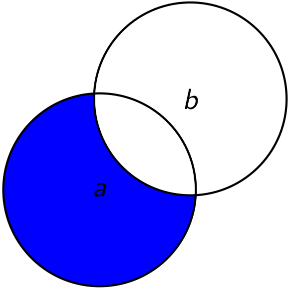
$H(a|b) = H(a,b) - H(b)$.

conditional encoding: slepian-wolf vs muchnik



$H(a|b) = H(a,b) - H(b)$. Can we *materialize* the part $a \setminus b$?

conditional encoding: slepian-wolf vs muchnik



$H(a|b) = H(a,b) - H(b)$. Can we *materialize* the part $a \setminus b$?

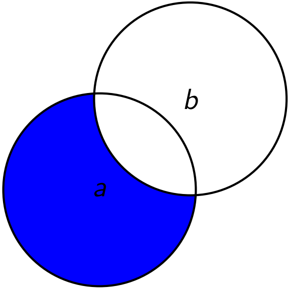
A formal question: can we find a w such that

- $H(w) = H(a|b)$,
- $H(w|a) = 0$,
- $H(a|b,w) = 0$

?

In general, **no**!

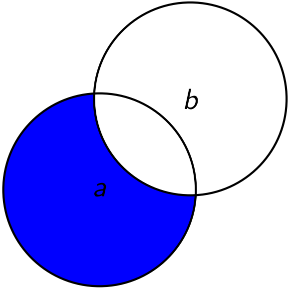
conditional encoding: slepian-wolf vs muchnik



$H(a|b) = H(a,b) - H(b)$. Can we *materialize* the part $a \setminus b$?

Let $(a_1, b_1), \ldots, (a_n, b_n)$ be i.i.d., all with the same distribution as (a, b)

conditional encoding: slepian-wolf vs muchnik



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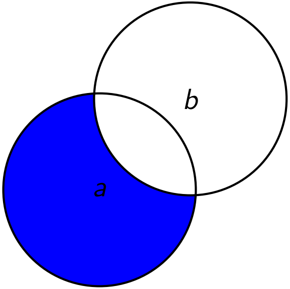
Let $(a_1, b_1), \ldots, (a_n, b_n)$ be i.i.d., all with the same distribution as (a, b)

Selpian and Wolf [1973]: Then there exists a W such that

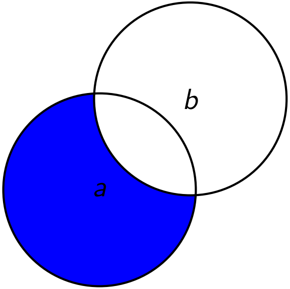
- $H(W) = n \cdot H(a|b) + o(n)$,
- $H(W|a_1 \ldots a_n) = 0$,
- $H(a_1 \ldots a_n | b_1 \ldots b_n, W) = 0$

conditional encoding: slepian-wolf vs muchnik



Kolmogorov's framework:
Can we *materialize* the part $a \setminus b$?

conditional encoding: slepian-wolf vs muchnik

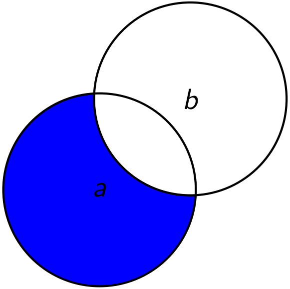


Kolmogorov's framework:
Can we *materialize* the part $a \setminus b$?

Yes, there exists a w such that

- $C(w) = C(a|b) + O(\log \ldots)$,
- $C(w|a) = O(\log \ldots)$,
- $H(a|b, w) = O(\log \ldots)$.

conditional encoding: slepian-wolf vs muchnik



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Can we *materialize* the part $a \setminus b$?

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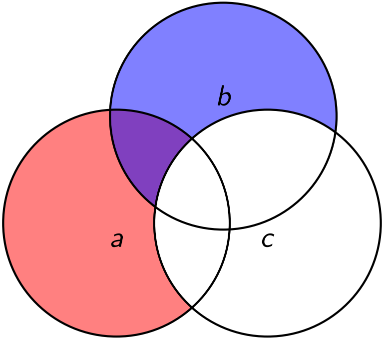
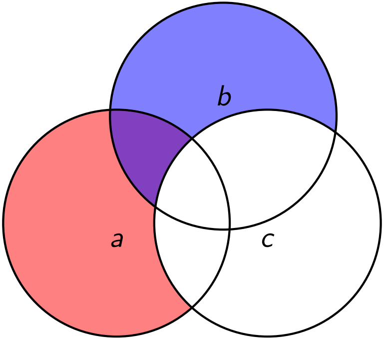
[Bennett-Gács-Li-Vitanyi-Zureck, Fortnow-Laplante, An. Muchnik]

two conditional descriptions: when the parallelism fails



Let $(a_1, b_1, c_1), \ldots, (a_n, b_n, c_n)$ be i.i.d., all distributed as (a, b, c).

two conditional descriptions: when the parallelism fails

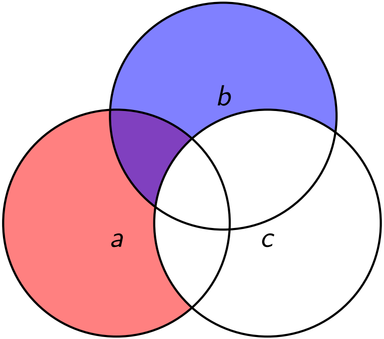


Let $(a_1, b_1, c_1), \ldots, (a_n, b_n, c_n)$ be i.i.d., all distributed as (a, b, c).

Then there exist V and W such that

- $H(V) = n \cdot H(a|c) + o(n)$ and $H(a_1 \ldots a_n | c_1 \ldots c_n, V) = 0$,
- $H(W) = n \cdot H(b|c) + o(n)$ and $H(b_1 \ldots b_n | c_1 \ldots c_n, W) = 0$,
- $H(V, W) = n \cdot H(a, b|c) + o(n)$.

two conditional descriptions: when the parallelism fails



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- $H(V, W) = n \cdot H(a, b|c) + o(n)$. (i.e., $I_S(V, W : c_1 \ldots c_n) = o(n)$)

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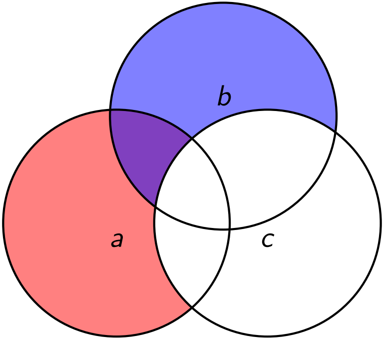

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The counterpart of this statement for Kolmogorov complexity is wrong!