Embedding computations in tilings (Part 3)

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- Tilings are aperiodic, but close to periodic;
- There are periodic configurations that are almost tilings (with a sparse set of tiling errors)

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Such configurations do exist. Moreover, they can be enforced by tiling rules.

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The notion of a **sparse set** is reasonable if for small enough ε a B_ε-random set is **sparse** with prob. 1

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Theorem [Durand-R.-Shen] There exists a tile set τ such that for all small enough ε the following is true for B_ε-almost all sets H:

Every tiling of $\mathbb{Z}^2 \setminus H$ is very aperiodic (every non-zero translation changes $> 10\%$ of tiles).

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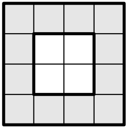
1. introduce some redundancy (every tile "knows" information about its neighbors) $\Rightarrow$ we correct small errors (e.g., 2×2 holes)



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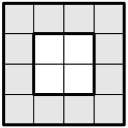

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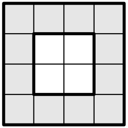

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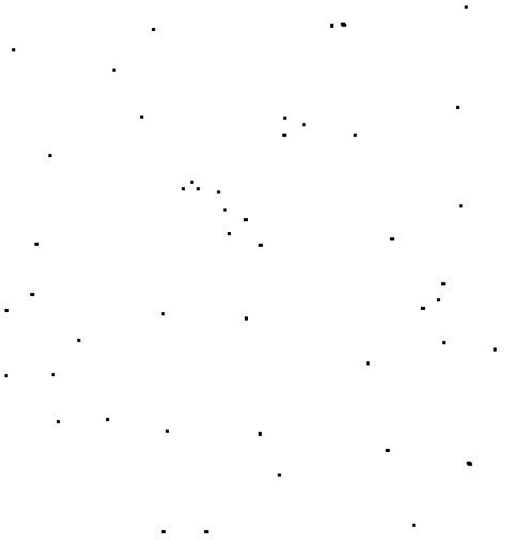
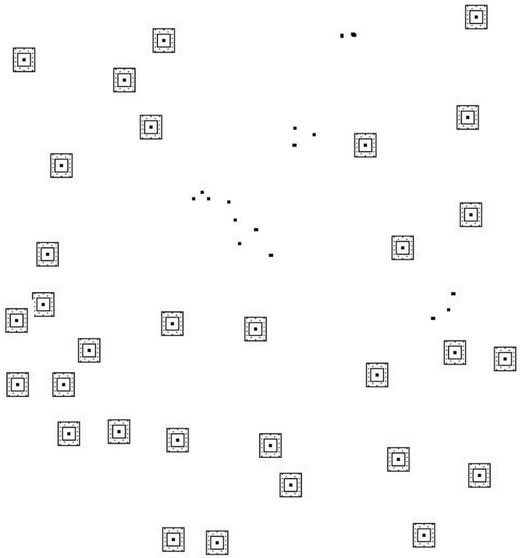
3. a real miracle: we can correct a *random* set of miracles (with prob 1)

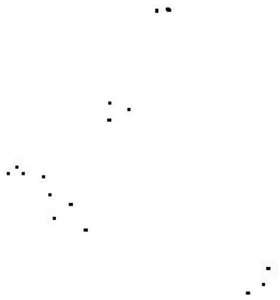
A B_ε-random set consists of isolated “islands” of different levels



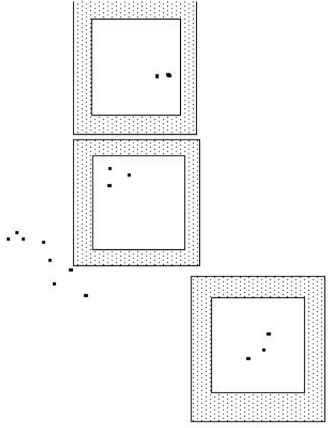
Isolated 0-level islands:



Clean up 0-level islands:

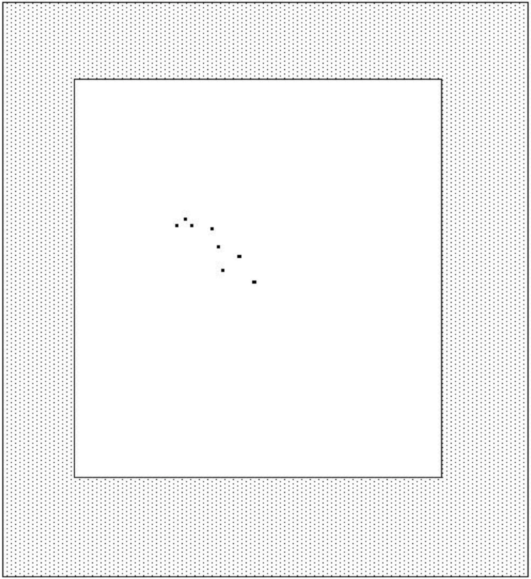


Isolated 1-level islands:



Clean up 1-level islands:

2-level island:



With probability 1 the cleaning procedure converges. Moreover, with probability 1 only the fraction $O(\varepsilon)$ of points is involved in the correcting procedure.

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Lemma. The limit configuration of the Thue–Morse substitution rule is strongly aperiodic.

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And it works!

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Question: How to achieve the "robustness" property without fixed-point?