Probabilistic Proofs

This column is devoted to mathematics for fun. What better purpose is there for mathematics? To appear here, a theorem or problem or remark does not need to be profound (but it is allowed to be); it may not be directed only at specialists; it must attract and fascinate.

We welcome, encourage, and frequently publish contributions from readers—either new notes, or replies to past columns.

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In this issue I present a collection of nice proofs that are based on some kind of a probabilistic argument, though the statement doesn't mention any probabilities. First a simple geometric example.

(1) It is known that ocean covers more than one half of the Earth's surface. Prove that there are two symmetric points covered by water.

Indeed, let X be a random point. Consider the events "X is covered by water" and "$-X$ is covered by water". (Here $-X$ denotes the point antipodal to X.) Both events have probability more than $1/2$, so they cannot be mutually exclusive.

Of course, the same (trivial) argument can be explained without any probabilities. Let $W \subset S^2$ be the subset of the sphere covered by water, and let $\mu(X)$ be the area of a region $X \subset S^2$. Then $\mu(W) + \mu(-W) > \mu(S^2)$, so $W \cap (-W) \neq \emptyset$.

However, as we see in the following examples, probability theory may be more than a convenient language to express the proof.

(2) A sphere is colored in two colors: 10% of its surface is white, the remaining part is black. Prove that there is a cube inscribed in the sphere such that all its 8 vertices are black.

Indeed, let us take a random cube inscribed in the sphere. For each vertex the probability of the event "vertex is white" is 0.1. Therefore the event "there exists a white vertex" has probability at most $8 \times 0.1 < 1$, therefore the cube has 8 black vertices with a positive probability.

This argument assumes implicitly that there exists a random variable (on some sample space) whose values are cubes with numbered vertices and each vertex is uniformly distributed over the sphere. The easiest way to construct such a variable is to consider $SO(3)$ with an invariant measure as a sample space.

It seems that here probability language is more important: if we did not have probabilities in mind, why should we consider an invariant measure on $SO(3)$?

Now let us switch from toy examples to more serious ones.

(3) In this example we want to construct a bipartite graph with the following properties:

(a) both parts L and R (called "left" and "right") contain n vertices;

(b) each vertex on the left is connected to at most eight vertices on the right;

(c) for each set $X \subset L$ that contains at least $0.5n$ vertices the set of all neighbors of all vertices in X contains at least $0.7n$ vertices.

(These requirements are taken from the definition of "expander graphs", constants are chosen to simplify calculations.)

We want to prove that for each n there exists a graph that satisfies conditions (a) – (c). For small n it is easy to draw such a graph (e.g., for $n \leq 8$ we just connect all the vertices in L and in R), but it seems that in the general case there is no simple construction with an easy proof.

However, the following probabilistic argument proves that such graphs do exist. For each left vertex x pick eight random vertices on the right (some of them may coincide) and call these vertices neighbors of x. All choices are independent. We get a graph that satisfies (a) and (b); let us prove that it satisfies (c) with positive probability.

Fix some $X \subset L$ that has at least $0.5n$ vertices and some $Y \subset R$ that has less than $0.7n$ vertices. What is the probability of the event "All neighbors of all elements of X belong to Y"? For each fixed $x \in X$ the probability that all eight random choices produce an element from Y, does not exceed $(0.7)^8$. For different elements of X choices are independent, so the resulting probability is bounded by $(0.7^8)^{0.5n} = 0.7^{4n}$.

There are fewer than 2^n different possibilities for each of the sets X and Y, so the probability of the event "there exist X and Y such that $\#X \geq 0.5n$, $\#Y < 0.7n$, and all neighbors of all vertices in X belong to Y" does not exceed $2^n \times 2^n \times 0.7^{4n} = 0.98^{2n} < 1$. This event embodies the negation of the requirement (c), so we are done.

All the examples above follow the same scheme. We want to prove that an object with some property α exists. We consider a suitable probability distribution and prove that a random object has property α with nonzero probability. Let us consider now two examples of a more general scheme: if the expectation of a random variable ζ is greater than some number λ, some values of ζ are greater than λ.

(4) A piece of paper has area 10 square centimeters. Prove that it can be placed on the integer grid (the side of whose square is 1 cm) so that at least 10 grid points are covered.

Indeed, let us place a piece of paper on the grid randomly. The expected number of grid points covered by it is proportional to its area (because this expectation is an additive function). Moreover, for big pieces the boundary effects are negligible, and the number of covered points is close to the area (relative error is small). So the coefficient is 1, and the expected number of covered points is equal to the area. If the area is 10, the expected number is 10, so there must be at least one position where the number of covered points is 10 or more.

(5) A stone is convex, its surface has area S. Prove that the stone can be placed in the sunlight in such a way that the shadow will have area at least $S/4$. (We assume that light is perpendicular to the plane where the shadow is cast; if it is not, the shadow only becomes bigger.)

Let us compute the expected area of the shadow. Each piece of the surface contributes to the shadow exactly twice (here convexity is used), so the shadow is half the sum of the shadows of all pieces. Taking into account that for each piece all possible directions of light are equiprobable, we see that the expected area of the shadow is proportional to the area of the stone surface. To find the coefficient, take the sphere as an example: it has area $4\pi r^2$ and its shadow has area πr^2, so the expected shadow area is $S/4$.

(6) We finish our collection of nice probabilistic proofs with a well-known example, so nice and unexpected that it cannot be omitted. It is the probabilistic proof of the Weierstrass theorem saying that *any continuous function can be approximated by a polynomial*. (As far as I know, this proof is due to S.N. Bernstein.)

Let $f: [0, 1] \to \mathbb{R}$ be a continuous function. Let p be a real number in $[0, 1]$. Construct a random variable in the following way. Make n independent trials, the probability of success in each of them being p. If the number of successes is k, take $f(k/n)$ as the value of the random variable. For each p we get a random variable. Its expectation is a function of p; let us call it $f_n(p)$.

It is easy to see that for each n the function f_n is a polynomial. (What else can we get if the construction uses only a finite number of f-values?) On the other hand, f_n is close to f, because for any p the ratio k/n is close to p with overwhelming probability (assuming n is big enough), so in most cases the value of $f(k/n)$ is close to $f(p)$, since $f(p)$ is uniformly continuous.

The formal argument requires some estimates of probabilities (Chernoff bound or whatever), but we omit the details.

Colorings Revisited

In the 1997, no. 4 issue of *The Intelligencer* I discussed a homotopic proof of the following fact: *If in a triangulation of the sphere S^2 each vertex is incident to an even number of edges, then there is a 3-coloring of the vertices such that endpoints of any edge have different colors.*

David Gale of Berkeley writes in response:

As you may know, the condition that each vertex lies on an even number of edges is also a necessary and sufficient condition for the faces to be 2-colorable. Using that fact I found the following homological, or rather cohomological, proof of the theorem.

Color the faces red and blue. Then give the red triangles the clockwise, the blue ones the counter-clockwise orientation. This gives a unique orientation to all edges of the polyhedron, meaning we can put arrows on the edges so that as one goes around a triangle, arrows point in the same direction. Now use cohomology mod 3 and define the 1-cochain which assigns 1 to each edge in the direction of its arrow and -1 in the opposite direction. By the property above this is a 1-cocycle (its coboundary on any triangle is $1 + 1 + 1 = 0$), hence because we are on the sphere it must be the coboundary of a 0-cochain C, and this C must assign different integers mod 3 to adjacent vertices. Otherwise its coboundary would be zero on some edge.

The proof that the faces are 2-colorable is also homological. Using mod 2 homology, let c be the unit 1-chain that assigns 1 to all the edges. By the evenness property, the boundary of this chain is zero, so it is a 1-cycle and hence because we are on the sphere it must be the boundary of a 2-chain. This 2-chain must have distinct values on adjacent faces, for if not, the boundary operator would assign 0 to their common edge.

This finishes the proof of the following theorem.

Theorem 1. *Let T be a triangulation of the 2-sphere. The following statements are equivalent:*

1. Every vertex has even degree.

2. The faces can be 2-colored.

3. The vertices can be 3-colored.

There is a similar theorem that also has a nice homological proof.

Theorem 2. *The vertices can be 4-colored $\iff$ the edges can be 3-colored (meaning all three colors appear around every triangle).*

The proof uses cohomology with coefficients in the Klein Four Group K_4 (whereas the proof of Theorem 1 used $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$).

$\Rightarrow$

Assume that vertices are 4-colored. Identify the colors with the elements of $K_4 = \{0, A, B, C\}$. For any triangle its coloring is either of the form $0, A,$

B or A, B, C, and the coboundary is either $(A, A + B = C, B)$ or $(A + B = C, B + C = A, C + A = B)$, so the edges are 3-colored with the *non-zero* elements of K_4

$\Leftarrow$

Assume that edges are 3-colored Identify the colors with the non-zero elements of K_4 Then this is a 1-cocycle C_1, since the sum of the three non-zero elements of K_4 is zero Hence on the 2-sphere C_1 is the coboundary of a 0-cochain C_0, and this must assign different colors to adjacent vertices a and b since $C_1(a,b) = C_0(a) + C_0(b)$ must be non-zero

Letter from Prof. Dr. Hanfried Lenz

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A problem in mathematical entertainment, without any scientific value Find two or more squares or higher powers of integers with the same decimal digits in different order, such as 125 and 512, 256 and 625, 169, 196, and 961, 1024 and 2401, 1296, 2916, and 9216, 1728 and 2781 etc Such numbers can be constructed, e g , 10609, 16900, 19600, 90601, and 96100, or else $3004^3 = 27{,}108{,}144{,}064$ and $4003^3 = 64{,}144{,}108{,}027$

The following example shows a method of construction Let a and b be two integers with one digit, and suppose a^2 and b^2 are either both smaller than 10 or both larger than 10 and ab is smaller than 50 Then the two squares $(100a + b)^2 = 10000a^2 + 200ab + b^2$ and $(100b + a)^2 = 10000b^2 + 200ab + a^2$ have the same digits It is easy to generalize this construction, see 3004^3 and 4003^3 above But I am more interested in random examples such as the first example 125 and 512

Yours sincerely,
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