JOURNAL OF ALGORITHMS 4, 35–44 (1983)

NP Completeness of Finding the Chromatic Index of Regular Graphs

DANIEL LEVEN AND ZVI GALIL

School of Mathematical Sciences, Tel-Aviv University, Tel-Aviv, Israel

Received July 2, 1981; revised March 1, 1982

We show that it is NP complete to determine whether it is possible to edge color a regular graph of degree k with k colors for any $k \geq 3$. As a by-product of this result, we obtain a new way to generate k-regular graphs which are k-edge colorable.

1. INTRODUCTION

The chromatic index of a graph is the minimum number of colors required to color the edges of the graph in such a way that no two adjacent edges have the same color. Vizing [5] showed that the chromatic index is either k or $k + 1$, where k is the maximal degree of the vertices of the graph.

The problem of finding the chromatic index of a given graph has been proved to be NP complete by Holyer [3], who proved that it is NP complete to decide whether the chromatic index of a cubic graph is 3 or 4. In his thesis [2] Holyer showed that it is NP complete to decide whether the chromatic index of a graph with maximal degree 4 is 4 or 5. He conjectured that the analogous problem for graphs with maximal degree k is NP complete.

We follow the work of Holyer and prove his conjecture:

THEOREM 1. *For any fixed k, the problem of deciding whether the chromatic index of a regular graph of degree k is k or $k + 1$ is* NP *complete.*

Usually in proving NP-completeness the proof becomes more difficult when some restriction is imposed on the problem. Here to the contrary it seems that the problem is well suited to the cubic case and proving the analogous result for regular graphs of degree k is not immediate, as we shall see.

In our proof we show how to construct regular graphs of degree k from Boolean formulas. In particular the graphs generated from satisfiable formulas are k-colorable. We hope this method will be found useful elsewhere.

0196-6774/83/010035-10$03.00/0

The terminology and results of NP-completeness are given [4]. It is clear that the chromatic index problem is in the class of NP. To prove that it is NP-complete, we give, like Holyer, a reduction from the 3-CNF problem, which is known to be NP complete and which is defined as follows. A set of clauses $C = \{c_1, \ldots, c_n\}$ in variables $u_1, \ldots, u_m$ is given. Each clause c_j consists of three literals $l_{j_1}, l_{j_2}, l_{j_3}$, where a literal l_{j_k} is either a variable u_i or its complement $\bar{u}_i$. A truth assignment to the variables assigns each variable one of two values "true" (T) or "false" (F). The value of $\bar{u}_i$ is "true" iff the value of u_i is "false." Given a truth assignment, a clause is said to be satisfied if at least one of its literals is true. The problem is to determine whether there is a truth assignment to the variables which simultaneously satisfies all clauses.

In the next sections we shall use the term coloring for edge-coloring.

2. Proof of Theorem 1 for $k = 3$

In this section we give a sketch of Holyer's proof of Theorem 1 for $k = 3$ (i.e., for cubic graphs). First we define the graph H (Fig. 1):

Notation. Let a, b be two edges, and let $A \equiv b$ denote the fact that they are colored in the same colors, $a \not\equiv b$ otherwise.

Lemma 1. A. *In any* 3 *coloring of* H, 1 *and* 2 *hold*:

1. *Either* $a \equiv b$ *or* $c \equiv d$.
2. $a \equiv b \ (c \equiv d) \Rightarrow c \not\equiv d \wedge c \not\equiv e \wedge d \not\equiv e \ (a \not\equiv b \wedge a \not\equiv e \wedge b \not\equiv e)$.

B. *Any coloring of* a, b, c, d, e *in* 3 *colors which satisfies* 1 *and* 2 *above can be completed to a* 3*-coloring of* H.

Holyer's proof of Theorem 1 for cubic graphs is as follows. Given an instance C for the problem 3-CNF, we now construct a cubic graph G which is 3-colorable iff C is satisfiable. The graph G is put together from components.

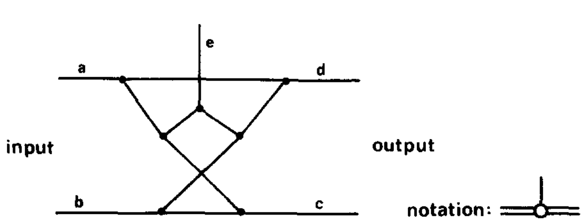

Fig. 1. An inverting component.

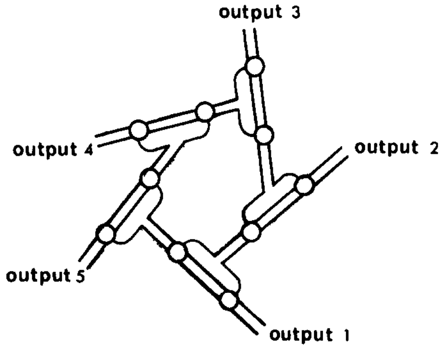

FIG. 2. A variable-setting component with five outputs.

Information is carried between components by pairs of edges. Such pairs are said to carry the value "true" if they are colored with equal colors, and "false" if they are colored with distinct colors. With such a definition the graph H may be called an "inverting component."

For each variable u_i there is a variable setting component U_i. This component has as many pairs of outputs as there are occurrences of the variable u_i or $\bar{u}_i$ in the clauses of C. (The number of occurrences is assumed to be greater than 1 without loss of generality.) The variable-setting component (see Fig. 2 for the instance of five outputs) has the following property:

LEMMA 2. *In a 3-coloring of a variable-setting component, all the outputs are either all "true" or all "false." Both settings are possible.*

The truth of each clause c_j is tested by a satisfaction-testing component C_j, shown in Fig. 3. This component has the following property:

LEMMA 3. 1. *In any 3-coloring of a satisfaction-testing component at least one pair of inputs must be colored with the same color.*

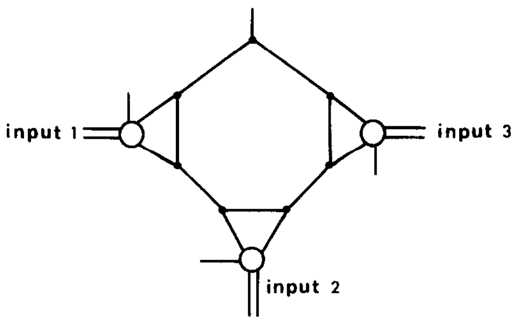

FIG. 3. A satisfaction-testing component.

2. *Any 3-coloring of the inputs which satisfies* 1 *above may be completed to a 3-coloring of the component.*

The connection between a variable-setting component U_i and a satisfaction testing component C_j is done as follows. Suppose literal l_{j_k} in clause c_j is u_i and is the rth occurrence of u_i or $\bar{u}_i$ in C, then the kth input of C_j is identified with the rth output of U_i. If on the other hand l_{j_k} is $\bar{u}_i$ then an inverting component is inserted.

To account for the remaining edges, two copies of the graph are taken and corresponding edges in the two graphs which have as yet unspecified endpoints are identified to give a cubic graph. This graph may be 3-colored iff C is satisfiable. Also it may be constructed from C in polynomial time which gives the required result.

3. Proof of Theorem 1 for General k

The proof of Theorem 1 for general k is similar to the proof for $k = 3$. A set of inverting components is defined, as well as a variable-setting and a satisfaction-testing component. Generalizations of Lemmas 1, 2, and 3 are also proved. The main difference between the two constructions is in the connection between the variable-setting and the satisfaction-setting components.

Notation. (a) $\underline{\ (i)\ }$ denotes an edge of multiplicity i;
(b) $\underline{\ \ i\ \ }$ denotes an edge whose name is i.

To deal with the multiple edges we use the following lemma.

Lemma 4. *For any regular multigraph G of degrees k (≥ 3) there is a regular graph G' of degree k such that G may be k-colored iff G' may be k-colored.*

Proof. We show that it is possible to replace any multiple edge of multiplicity i in G by a subgraph I_i, with i inputs and i outputs such that I_i may be k-colored iff the inputs and outputs are colored with the same set of i distinct colors. In I_i different inputs (outputs) are connected to different vertices. Hence, the resulting graph contains no multiple edges. We use a construction by Izbicki [1] to define a graph $D_k = D_k(V_k, E_k)$ for $(k \geq 3)$ (D_k is of degree k except for k vertices of degree 1) as follows:

$$V_k = \{R_s, Q_t, P_t \mid s = 1,\ldots,k-3, t = 1,\ldots,k\}.$$
$$E_k = \{(R_s, Q_t), (Q_t, Q_{t+1}), (Q_t, P_t) \mid s = 1,\ldots,k-3, t = 1,\ldots,k\},$$

all indices are taken modulo k.

One k-coloring of D_k such that edges (Q_t, P_t) $t = 1,\dots,k$ have distinct colors is:

(R_s, Q_t) is colored with color $t + s - 1 \pmod k$
for $t = 1,\dots,k$, $s = 1,\dots,k - 3$.
(Q_t, P_t) is colored with color $t + k - 1 \pmod k$ for $t = 1,\dots,k$.
(Q_t, Q_{t+1}) is colored with color $t + k - 2 \pmod k$ for $t = 1,\dots,k$.

By symmetry any coloring of the edges (Q_t, P_t) $t = 1,\dots,k$ in k colors can be completed to a k-coloring of D_k.

Also in any k-coloring of D_k, (Q_t, P_t), $t = 1,\dots,k$ must be colored in k distinct colors. This is proved by contradiction.

Suppose there is a color, say l, which appears at least at two of the edges (P_t, Q_t), $t = 1,\dots,k$. Either this color also appears at an edge (Q_i, Q_{i+1}) for some i, or there is a color, say l', which appears at least at two edges of type (Q, Q). In both cases there is a color (l or l') which appears at least at four vertices of type Q on edges of type (Q, Q) or (P, Q). This color also appears at $k - 3$ vertices of type Q on edges of type (R, Q). But since there are only k vertices of type Q, this color must appear twice at one such vertex. This is a contradiction.

Taking two copies of D_k we define i edges of type P in one copy as inputs, i edges of type P in the second copy as outputs, and identify the remaining $k - i$ edges of type P in both copies. This gives the required subgraph I_i (for any i). □

We shall use the following lemma given in [1]:

LEMMA 5. (Parity condition). *Let G be a graph whose vertex degrees are either k or 1. Let the set of edges incident on vertices of degree 1 be E'. Then in any k coloring of G if the number of edges colored i in E' is r_i we have* $r_1 \equiv r_2 \equiv \cdots \equiv r_k \pmod 2$.

Proof. If E_{ij} is the set of edges of G which are colored in colors i, j then E_{ij} consists of either closed cycles of edges not in E' or of open paths which contain exactly two edges in E' (first and last). Therefore E_{ij} meets E' in an even number of edges so that $r_i + r_j = 0 \pmod 2$. □

We define $k - 2$ inverting components H_i $i = 1,\dots,k - 2$ as in Fig. 4.

LEMMA 6. *If H_i ($i = 1,\dots,k - 2$) is colored in k colors, then all colors must appear at a, b, c, d, e. $k - 1$ colors appear once and one color appears three times.*

Proof. We first show that all colors must be present at a, b, c, d, e. So suppose color l is missing. Consider Fig. 5.

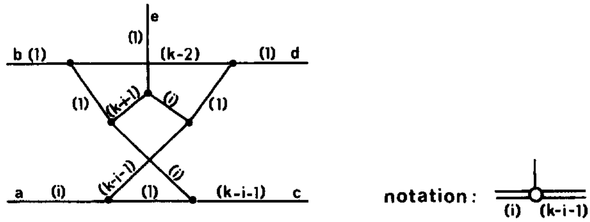

FIG. 4. Inverting component H_i ($i = 1, \ldots, k-2$).

Since the number of edges at o and p together is $k-1$, and l does not appear at e then l must appear at either o or p. Without loss of generality we may assume that it appears at p and does not appear at o. By following the edges marked q, r, s, t, u, v in this order we note that l cannot appear at q, s, u and must appear at r, t, v. We get a contradiction since color l must appear both at p and at v.

Since all colors must appear with the same parity at edges a, b, c, d, e and since there are $k+2$ such edges; we get that for $k > 2$ all colors must appear an odd number of times ($2k > k+2$ and all k colors must appear). Since the number of edges is $k+2$ the only possibility is that one color appears three times and all other colors appear once. □

NOTATION. Given two multiple edges a, b then $a \equiv b$ denotes that some color appears at both a and b. $a \not\equiv b$ denotes that all colors at a and b are distinct.

LEMMA 7. (Generalization of Lemma 1). A. *In any k coloring of the graphs* H_i 1 *and* 2 *hold*:

1. Either $a \equiv b$ or $c \equiv d$.
2. $a \equiv b$ $(c \equiv d) \Rightarrow c \not\equiv d \wedge c \not\equiv e \wedge d \not\equiv e$ $(a \not\equiv b \wedge a \not\equiv e \wedge b \not\equiv e)$.

B. *Any k coloring of the edges at a, b, c, d, e which satisfies the conclusion of Lemma* 6 *and* 1 *and* 2 *above can be completed to a k-coloring of* H_i.

Proof. A-1. Suppose without loss of generality that color 1 appears three times and all other colors appear once. Also suppose that $a \not\equiv b \wedge c \not\equiv d$ and consider Fig. 5. There are four possible cases.

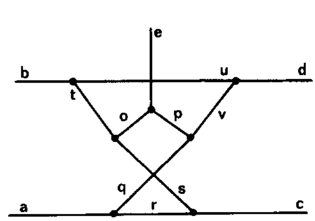

FIGURE 5

Case 1: Color 1 appears at edges b, d, and e. Then edges t and v will be colored with an equal color, say 2. Therefore colors 1 and 2 are missing at o and p. Therefore the $k-1$ adjacent edges at o and p are colored with at most $k-2$ colors, a contradiction.

Case 2: Color 1 appears at b, c, and e. Therefore color 1 is missing at s, t, and o. We get that k adjacent edges are colored with at most $k-1$ colors, a contradiction.

Case 3: Color 1 appears at a, c, and e. Similar to case 1.

Case 4: Color 1 appears at a, d, and e. Similar to case 2.

A–2. Since one color appears three times and all others only once, then

$$a \equiv b \Rightarrow c \not\equiv d \wedge c \not\equiv e \wedge d \not\equiv e \qquad (\text{or } c \equiv d \Rightarrow a \not\equiv b \wedge a \not\equiv e \wedge b \not\equiv e).$$

B. We show a reduction to the case of $k = 3$. The graph H_i is a superposition of the graph H defined for $k = 3$, and the graph B_i which is a regular graph of degree $k-3$ (see Fig. 6).

Suppose we are given a k-coloring of the $k+2$ edges at a, b, c, d, e which satisfies the assumptions of B. So one color, say 1, appears three times and all other colors appear once. We also may suppose that edges at c, d, and e are colored with distinct colors. Color 1 must therefore appear at a and b and also at one edge at one of c, d, or e. From each one of the two edges where 1 does not appear, we choose one more color, together two colors, say colors 2 and 3. By Lemma 1 we may color the graph H in colors $1, 2, 3$ so that a', b' are colored with color 1 and colors $1, 2, 3$ appear in edges c', d', e' according to the order dictated by their appearance in c, d, and e. We now complete the k coloring of H_i by $k-3$-coloring of B_i with colors 4 to k. Let A be the $k-i-2$ colors at c with numbers greater than 3, and let B be the $i-1$ colors at a with numbers greater than 3 (obviously $A \cap B = \varnothing$). B_i is colored so that $\alpha, \beta, \gamma, \delta, \varepsilon, \lambda, \mu$ are colored in $B, A, B, A, B, A, A \cup B$, respectively. □

In a similar way to the cubic case we define the value true if $a \equiv b$ and false if $a \not\equiv b$.

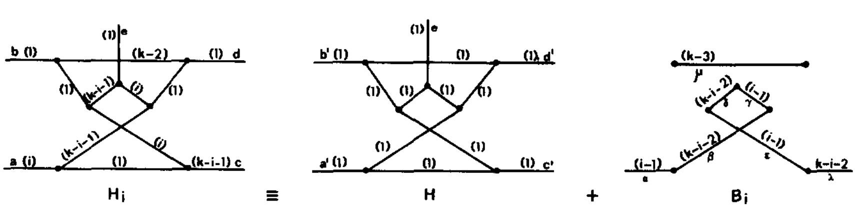

FIG. 6. Decomposition of H_i.

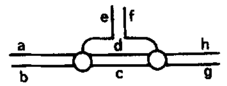

FIGURE 7

The variable-setting component is defined by using H_1 (instead of H) in the same way as in $k = 3$. Consequently all outputs are pairs of edges.

LEMMA 8. (Generalization of Lemma 2). *In the variable-setting component defined by H_1 the outputs are either all true or all false. Both settings are possible in such a way that all ouptputs are colored with just three colors, say* $1, 2,$ *or* 3.

Proof. We first note that in the subgraph of Fig. 7 if $a \equiv b$ then $g \equiv h$ and since the colors at c, d, and e and also at c, d, and f must be distinct then $e \equiv f$ (c and d contain together $k - 1$ edges). By induction it follows that if one output pair is equal then all output pairs must be equal.

Both settings are possible: To get one of them we first color the component in colors $1, 2$, and 3 ignoring multiple edges (considering each as one edge). Such a coloring is possible due to Lemma 2. It is easy to see that we can add $k - 3$ colors as in Lemma 7 to get the desired coloring. □

The satisfaction-testing component is defined as for $k = 3$ but using H_{k-2} instead of H as in Fig. 8.

LEMMA 9. (Generalization of Lemma 3). 1. *In any k-coloring of a satisfaction-testing component in at least one of the inputs one color must appear twice (must be true).*

2. *Any k-coloring of the inputs which satisfies* 1 *above and in which each input is colored so that colors* 4 *to k appear once, each at the multiple edge, and that the remaining two edges are colored with colors* $1, 2,$ *or* 3 *may be completed to a k-coloring of the component.*

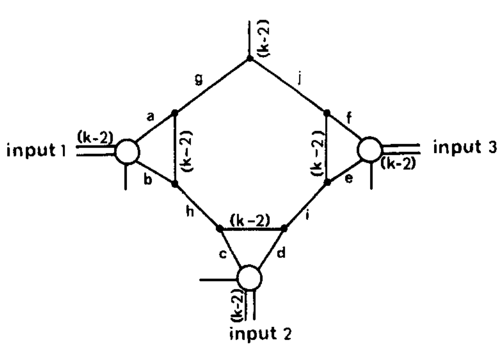

FIG. 8. A satisfaction-testing component.

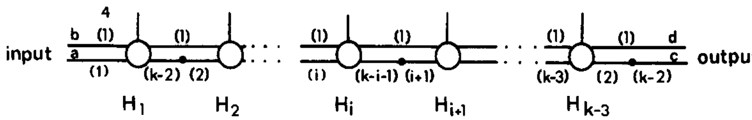

FIG. 9. An edge-adding component.

Proof. 1. If all inputs are distinct, then

$$a \equiv b \wedge c \equiv d \wedge e \equiv f \Rightarrow g \equiv h \equiv i \equiv j, \qquad \text{a contradiction.}$$

2. To get the required coloring we first color the component ignoring multiple edges (considering each as one edge) with colors 1, 2, and 3. This is possible due to Lemma 3. It is easy to see that we may add $k-3$ colors as in Lemma 7 to get the required coloring. □

To connect the variable-setting components to the satisfaction-testing components, we define a new component which we call an "edge-adding component." It is constructed from copies of $H_1, \ldots, H_{k-3}$, as shown in Fig. 9. The new component is required since there is a multiple edge at each input to the satisfaction-testing components and no equivalent multiple edge at the outputs of the variable-setting components, so that we may not identify them as we did for $k = 3$ when a variable u_i appeared in a clause.

LEMMA 10. 1. *Any k-coloring of an edge-adding component satisfies* $a \equiv b$ *iff* $c \equiv d$.

2. *Any coloring of* a, b *in colors from* $\{1, 2, 3\}$ *may be completed to a k-coloring of the component so that* $b \equiv d \wedge a \equiv c$ *and that colors* 4 *to* k *appear at* c.

Proof. The proof follows by induction. For the induction step we consider Fig. 10 and note that $p \equiv q$ iff $x \not\equiv y$ iff $x \equiv z$.

Also assume p is colored in color 1 and q in colors $\{j \mid 4 \leq j \leq i+2\} \cup \{\alpha\}$, $\alpha \in \{1, 2\}$. We color w in color $i+3$, x in color 1 and y in colors $\{j \mid i+4 \leq j \leq k\} \cup (\{1, 2, 3\} - \{\alpha\})$. By Lemma 7 this is possible. Consequently, x will be colored in color 1 and z in colors $\{j \mid 4 \leq j \leq i+3\} \cup \{\alpha\}$, $\alpha \in \{1, 2\}$. □

The connection between a variable-setting component U_i and a satisfaction-testing component C_j is as follows. If literal l_{j_k} of clause c_j is u_i and is the rth occurrence of u_i or $\bar{u}_i$ in C, then we add an edge-adding component between the kth input of C_j and the rth output of U_i. (The rth output of U_i

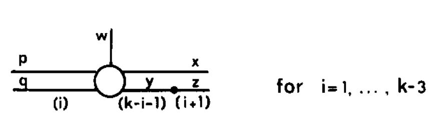

FIGURE 10

is identified with the input of the edge-adding component and the kth input to C_j is identified with the output of the edge-adding component.) If, on the other hand, l_{j_k} is $\bar{u}_i$, then only an inverting component H_1 is added.

To account for the remaining edges two copies of the graph are taken and corresponding edges are identified to give a regular graph of degree k. This graph may be k colored iff C is satisfiable. Also it may obviously be constructed from C in polynomial time which gives the required result. □

Acknowledgment

Dr. Oded Kariv has pointed out to us that Theorem 1 does not seem to follow from the case $k = 3$ in any immediate way.

References

1. H. Izbicki, An edge coloring problem, *in* "Theory of Graphs and its Applications" (M. Fidler, Ed.), pp. 53–61, Academic Press, New York/London, 1963.
2. I. J. Holyer, "The Computational Complexity of Graph Theory Problems," Ph.D. thesis, University of Cambridge, July 1980.
3. I. J. Holyer, The NP-completeness of edge coloring, *SIAM J. Comput.* **10** (1981), 718–720.
4. M. R. Garey and D. S. Johnson, "Computers and Intractability," Freeman, San Francisco, 1979.
5. S. Fiorini and R. J. Wilson, "Edge Coloring of Graphs," Pitman, London, 1977.