Partitioning a graph into a cycle and an anticycle, a proof of Lehel's conjecture

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Abstract

We prove that every graph G has a vertex partition into a cycle and an anticycle (a cycle in the complement of G). Emptyset, singletons and edges are considered as cycles. This problem was posed by Lehel and shown to be true for very large graphs by Łuczak, Rödl and Szemerédi [7], and more recently for large graphs by Allen [1].

Many questions deal with the existence of monochromatic paths and cycles in edge-colored complete graphs. Erdős, Gyárfás and Pyber asked for instance in [3] if every coloring with k colors of the edges of a complete graph admits a vertex partition into k monochromatic cycles. In a recent paper, Gyárfás, Ruszinkó, Sárközy and Szemerédi [5] proved that $O(k \log k)$ cycles suffice to partition the vertices. This question was also studied for other structures like complete bipartite graphs by Haxell [6]. One case which received a particular attention was the case $k = 2$, where one would like to cover a complete graph which edges are colored blue and red by two monochromatic cycles. A conjecture of Lehel, first cited in [2], asserts that a blue and a red cycle partition the vertices, where emptyset, singletons and edges are considered as cycles. This was proved for sufficiently large n by Łuczak, Rödl and Szemerédi [7], and more recently by Allen [1] with a better bound. Our goal is to completely answer Lehel's conjecture.

Our starting point is the proof of Gyárfás of the existence of two such cycles covering the vertices and intersecting on at most one vertex (see [4]). For this, he considered a longest path consisting of a red path followed by a blue path. The nice fact is that such a path P is hamiltonian. Indeed, if a vertex v is not covered, it must be joined in blue to the origin a of P and in red to the end b of P. But then, one can cover the vertices of P and v using the edge ab. Consequently, there exists a hamiltonian cycle consisting of two monochromatic paths. Hence, there exists a monochromatic cycle C, of size at least two, and a monochromatic path P with different colors partitioning the vertex set. This is the key-structure for the proof of our main result:

Theorem 1 *Every complete graph with red and blue edges has a vertex partition into a red cycle and a blue cycle.*

Proof. Assume that C and P are chosen as above in such a way that C has maximum size and has color, say, blue. We will show that we can either increase the length of C or prove the existence of our two cycles. If P has less than three vertices, we are done. We denote by x and y the endvertices of P. Note that if x and y are joined by a red edge, we have our two cycles. We then assume that xy is a blue edge. A vertex of C is *red* if it is joined to both x and y by red edges. The other vertices of C are *blue*. Observe that C cannot have two consecutive blue vertices, otherwise we would extend C. Moreover, the two neighbors in C of a red vertex v cannot be joined by a blue edge, since we could add v to the path P to form a red cycle. Similarly, if C has two or three vertices, one of them is red and could be added to P to form a red cycle. So, in particular, C has at least four vertices. Observe also that $|C| > |P|$ since we

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could give a red vertex to P to form a better partition into a cycle and a path. In this proof, removing a vertex x from a path or a cycle Q is denoted by $Q \setminus x$, whereas removing an edge xy is denoted by $Q - xy$.

Claim 1 *There are no successive blue, red, blue, red, blue vertices in C.*

Proof. Assume that b, r, b', r', b'' is such a sequence (with possibly $b = b''$) and that, say, xb' is a blue edge. Let x' be the successor of x on the red path P. We claim that $x'r$ is a blue edge. Otherwise, either bx is a blue edge in which case $(C \setminus r) \cup x$ is a blue cycle and $(P \setminus x) \cup r$ is a red cycle, or by is a blue edge and then $(P \setminus \{x, y\}) \cup r$ is a red path and $(C \setminus r) \cup \{x, y\}$ is a blue cycle longer than C, a contradiction. Similarly, $x'r'$ is a blue edge. Since b'' is blue, there exists a blue path $P' = b'xb''$ or $P' = b'xyb''$. Replacing the path $brb'r'b''$ in C by $brx'r'b'P'b''$ would increase the length of C, a contradiction. ■

When acb are consecutive vertices of C and c is a red vertex, we call ab a *special edge*. Observe that special edges are red. We denote by G_s the graph on the same vertex set as C whose edges are the special edges. Observe that the maximum degree of G_s is two. It appears that the proof is easier if we have several blue vertices in C. Let us prove for a start that there exists at least one.

Claim 2 *There exists a blue vertex in C.*

Proof. If not, G_s is either a cycle or the union of two cycles, depending if C has an odd or an even number of vertices. If C contains a red hamiltonian path, we can form, with P, a hamiltonian red cycle of the whole graph. Therefore G_s is the union of two red cycles W and Z, alternating along C, with the same cardinality and no red edge between them. We denote by x' and y' the respective neighbors of x and y in P, with possibly $x' = y'$ if $|P| = 3$. There is no red edge from x' to W, otherwise, $(P \setminus x)WxZ$ forms a hamiltonian red cycle. Similarly, there is no red edge from x' to Z or from y' to $W \cup Z$.

- If $|P| = 3$, then $W \cup x \cup Z \cup y$ is spanned by a red cycle and x' forms a blue one.
- If $|P| = 4$, then pick a vertex w of W and a vertex z of Z, consecutive along C, and form a red cycle $wxzy$. To conclude, partition the remaining blue path of C into two subpaths P' and P'' and form the blue cycle $x'P'y'P''$.

Now we assume that P has at least five vertices. We denote by x'' and y'' the respective neighbors of x' and y' on P. There is no red edge from x'' to W, otherwise $(P \setminus \{x', x\})WxZ$ forms a red cycle and x' forms a blue one. Similarly, there are all blue edges from x'' to Z and from y'' to $W \cup Z$. Observe that xx'' is a blue edge, otherwise $P \setminus x'$ forms a red path and $C \cup x'$ is spanned by a blue cycle longer than C, a contradiction. Similarly, yy'' is a blue edge.

- Assume that $|P| = 5$, in particular $y'' = x''$ and $|C| \geq |P|+1 = 6$. Pick a vertex w in W and a vertex z in Z. Form the red cycle $wxzy$, a blue cycle covering the blue bipartite graph $(W \setminus w) \cup (Z \setminus z)$ and finally insert in this blue cycle, of length more than three, the vertices x', y' and x''.
- If $|P| \geq 6$, we insert the three blue paths x', y' and $x''xyy''$ in C to form a blue cycle longer than C, a contradiction. ■

Now, fix an orientation of the cycle C. We define the set L of *left* vertices as the vertices which are left neighbors in C of some blue vertex. We define similarly the set R of right vertices. Note that L and R are not empty, may intersect and contain only red vertices.

Claim 3 *The set of left (resp. right) vertices spans a red clique.*

Proof. Assume for contradiction that there exists a blue edge joining two left red vertices u and v. We denote by u' and v' their respective right blue neighbors in C. There exists a path Q from u' to v' in $\{u', x, y, v'\}$ with length at least two. Now $(C - \{uu', vv'\}) \cup Q \cup uv$ is a blue cycle which is longer than C, a contradiction. ■

Every connected component of G_s which is a path has an endvertex in L and the other endvertex in R. Furthermore, if G_s has a cycle Z, it is unique and it contains all the blue vertices of C. In this case, Z cannot contain all the vertices of C, for instance, the neighbor of a blue vertex in C does not belong to Z. Indeed, if it exists, Z is simply obtained by taking all the vertices of C at even distance of some blue vertex, so Z contains every other vertex on C, and $|C|$ is even. Hence, the vertices of the whole graph are partitioned into a red clique L, a red clique R, a set $S = C \setminus (R \cup L \cup Z)$ which is covered by a set $\mathcal{S}$ of $|R|$ disjoint RL paths of red edges, the original path P, and (possibly) the cycle Z.

Claim 4 *There exists a red path which spans $S \cup R \cup L$. More precisely:*
- If $|\mathcal{S}|$ is even, for all distinct vertices of R (resp. L) x and y there is a red path from x to y which spans $S \cup R \cup L$.
- If $|\mathcal{S}|$ is odd, then for all $x \in R$ and $y \in L$ such that x and y are not the endvertices of a same path of $\mathcal{S}$, there is a red path from x to y which spans $S \cup R \cup L$.

Proof. We give a constructive proof. Denote by P_x (resp. P_y) the path of $\mathcal{S}$ which contains x (resp. y). Starting from x, we follow the path P_x until its end. At the end of a path of $\mathcal{S}$, R and L being red cliques, we go to the beginning of a unvisited path of $\mathcal{S}$, which is not P_y, and follow it. When the process stops, using a red edge of R or L, we go to the endvertices of P_y, which is not y (because of the parity of $|\mathcal{S}|$, and terminate the spanning path on y. ■

A direct corollary of Claim 4 is that if Z does not exist, one can cover C, whose vertices are exactly $S \cup R \cup L$, by a red path P' ending in two red vertices. Hence $P \cup P'$ forms a red hamiltonian cycle. Thus we can assume that Z exists. Observe that by Claim 1, every blue vertex of Z is the neighbor in Z of a red vertex.

Claim 5 *Every blue vertex is joined in blue to $R \cup L$.*

Proof. Indeed, assume for contradiction that br is a red edge where b is a blue vertex and r belongs to R. Let z be a red vertex which is consecutive to b in Z. By Claim 4, there exists a red path P' starting at r, covering $S \cup R \cup L$, and terminating on a red vertex of C. Now, $(Z - zb) \cup P \cup P'$ forms a hamiltonian red cycle. ■

Claim 6 *There is a red cycle W spanning $S \cup R \cup L$.*

Proof. If $|\mathcal{S}|$ is even, then Claim 4 directly gives the result. So, we assume that $|\mathcal{S}|$ is odd.
If there is a unique blue vertex b in C, the graph G_s consists of the union of Z and a unique path P' ($|\mathcal{S}| = 1$) whose endvertices u and v are the neighbors of b in C. If uv is a red edge, we are done. So, assume that uv is a blue edge, in particular $|C| > 4$ otherwise uv would be a (red) edge of G_s. Denote by u' the second neighbor of u in C and by u'' the second neighbor of u' in C (and thus the successor of u in P'). If bu'' is a blue edge, then replacing in C the path $vbuu'u''$ by $vubu''$ forms a blue cycle and $P \cup u'$ forms a red cycle. Thus bu'' is a red edge, in which case we form a red cycle $(Z - bu') \cup (P' \setminus u) \cup P$ and the singleton u as a blue cycle.

Assume now that C has at least two blue vertices. As $|\mathcal{S}|$ is odd, by Claim 4, we just have to prove that there exists a red edge between a vertex of R and a vertex of L which are not the endvertices of the same path of $\mathcal{S}$. For this, we consider a subpath I of C containing two blue vertices forming the endvertices of I. By Claim 1 and the fact that there exists at least two blue vertices, there is such an $I = br_1 \dots r_k b'$, wihere $k > 1$ and $r_1, \dots, r_k$ are red vertices. By Claim 5, br_k and $b'r_1$ are blue edges, we can replace I by $br_k r_{k-1} \dots r_1 b'$. Hence, r_1 becomes a left vertex. Thus by Claim 3, r_1 is joined in red to all the vertices of $R \cup L$, except possibly r_k. ■

Now, if a blue vertex is joined in red to any vertex of $S \cup R \cup L$, we can conclude as in Claim 5.

Claim 7 *There is no red edge between W and Z.*

Proof. Assume that zw is a red edge with $z \in Z$ and $w \in W$. Let z' be the first vertex to the right of z in Z which is joined to W with at least a red edge (here z' can be z). By the above remark, z' is a red vertex. Let A be the set of vertices between z and z' in Z. Let B be the set of $|A|$ consecutive vertices to the right of w on the cycle W (recall that Z and W have the same size, hence B does not contain w). Now, $A \cup B$ is a complete bipartite blue graph hence it has a blue spanning cycle. Moreover, $(Z \setminus A) \cup (W \setminus B)$ is spanned by a path P' starting at z' and ending in W. Both endvertices of P' are red, thus $P \cup P'$ forms a red cycle. ■

We now achieve the proof of the theorem. Let W be the red cycle $w_1 \dots w_k$ and Z be the red cycle $z_1 \dots z_k$, where z_1 is a blue vertex such that, say, the edge xz_1 is blue. Denote by y' the neighbor of y on P. There is no red edge between y' and a vertex of Z. Otherwise, letting z be this vertex and Q be a minimal path in Z from z to a red vertex of Z (Q has length zero or one). Denote by Q' a red path of W with same length as Q. Then, $P \cup Q \cup Q'$ is spanned by a red cycle (by inserting Q between y' and y and inserting Q' between x and y) and $C \setminus (Q \cup Q')$ is spanned by a blue one. Similarly, there is no red edge between y' and W, otherwise denote by w a red neighbor of y' on W and by z a red vertex on Z. Then, $P \cup w \cup z$ is spanned by a red cycle and $(Z \setminus z) \cup (W \setminus w)$ is spanned by a blue one. Hence y' is linked in blue to $W \cup Z$.

If $|P| = 3$, we choose two red vertices z and w respectively in Z and in W. Now, $xzyw$ is a red cycle, and $((W \cup Z) \setminus \{w, z\}) \cup y'$ is spanned by a blue cycle. Finally, if $|P| \geq 4$, we denote by y'' the second neighbor of y' on P. The edge yy'' is a blue one, otherwise, $P \setminus y'$ is a red path and $C \cup y'$ is spanned by a blue cycle longer than C. The edge $y''w_1$ is a blue one, otherwise for any red vertex z of Z, we would span $(P \setminus y') \cup w_1 \cup z$ by a red cycle and $(C \setminus \{w_1, z\}) \cup y'$ by a blue one. Now, starting with any blue cycle covering $W \cup Z$ which contains the subpath z_1, w_1, z_2, we replace this path by $z_1, x, y, y'', w_1, y', z_2$, a contradiction to the maximality of C. ■

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