Small cycle transversals in tournaments

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Abstract

We prove that every tournament T with no three disjoint cycles contains a set X of at most four vertices such that $T - X$ is acyclic.

Keywords: Disjoint cycles, tournaments, cycle transversal.

1 Introduction

In this paper, we are interested in computing the size of a minimal cycle transversal in tournaments. First, we precise notations. The notation not given below can be found in [3].

We denote the vertex set and arc set of a digraph D by $V(D)$ and $A(D)$, respectively and write $D = (V, A)$ where $V = V(D)$ and $A = A(D)$. If xy is an arc of D we say that x **dominates** y and that y is **dominated** by x. Extending this to disjoint subsets of vertices $X, Y \subset V(D)$, we say that X dominates Y when x dominates y for every choice of $x \in X$ and $y \in Y$. For a digraph $D = (V, A)$ the **out-neighbourhood** $N^+_D(x)$ (resp. **in-neighbourhood** $N^-_D(x)$) of a vertex $x \in V$ is the set of vertices y in $V - x$ such that xy (resp. yx) is an arc of A. The **out-degree** of x, denoted by $d^+_D(x)$ is the cardinality of $N^+_D(x)$, and the **in-degree** of x, denoted by $d^-_D(x)$ is the cardinality of $N^-_D(x)$. For $X \subseteq V$, we shall also write $d^+_X(x)$ to denote the number of vertices in X that are dominated by x.

In the present paper, paths and cycles are always assumed to be directed unless other qualified. A k-cycle is a cycle of length k. For convenience we will use the shorthand notation xyz to mean a 3-cycle on vertices x, y, z and arcs xy, yz, zx. A digraph D is **acyclic** if it does not contain any cycle. An (s, t)**-path** in a digraph D is a directed path from the vertex s to the vertex t. A digraph $D = (V, A)$ is **strongly connected** (or just **strong**) if there exists an (x, y)-path and a (y, x)-path in D for every choice of distinct vertices x, y of D, and D is k**-strong** if $D - X$ is strong for every subset $X \subseteq V$ of size at most $k - 1$. A subset $Y \subseteq V$ of a digraph D is a **vertex-cut** of D if $D - Y$ is not strong. A **strong component** (or when there is no confusion a **component**) of a digraph D is a maximal set of vertices X such that $D\langle X\rangle$ is strong. If a digraph is not strong then we can order its strong components $D_1, D_2, \ldots, D_p$ in such a way that there is no arc from a vertex in D_j to a vertex in D_i when $i < j$ (or equivalently, the digraph induced on the components D_i is acyclic). A strong component with no arcs entering (resp. leaving) is called an **initial** (resp. **terminal**) component of D. Moreover, a strong component is **trivial** if it contains a unique vertex.

For a subset X of $V(D)$ we denote by $D\langle X\rangle$ the subdigraph induced by the vertices in X. The **underlying graph** of a digraph D, denoted $UG(D)$, is obtained from D by suppressing the orientation of each arc and deleting multiple edges. In a digraph D, if X and Y are two disjoint subsets of vertices of D or subdigraphs of D, we say that there is a k**-matching** from X to Y if the arcs from X to Y contain a matching (in $UG(D)$) of size at least k. A **tournament** is an orientation of a complete graph (and so, does not contain any 2-cycle). We denote by TT_k the unique acyclic tournament on k vertices. This is also called the **transitive tournament** on k vertices.

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A **cycle transversal** X of a digraph D is a set of vertices of D which intersects all the cycles of D, or equivalently, such that $D - X$ is acyclic. We denote by $\tau(D)$ the size of a minimum cardinality cycle transversal of D. A digraph D is **intercyclic** if D does not have a pair of vertex-disjoint cycles. The problem of deciding whether a digraph is intercyclic is highly nontrivial for general digraphs. McCuaig [8] found a very complex polynomial algorithm for testing whether a given input digraph is intercyclic and he also proved the following.

Theorem 1.1 *[8] If a digraph D is intercyclic then $\tau(D) \leq 3$ and this is best possible.*

This result was generalized some years later by Reed, Robertson, Seymour and Thomas [10] who positively answered to an old-standing conjecture from Younger and proved the following.

Theorem 1.2 *[10] For every natural number k there exists a natural number $f(k)$ such that every digraph D which has no set of $k+1$ vertex-disjoint cycles satisfies $\tau(D) \leq f(k)$.*

In this paper we are focusing on tournaments, and give bounds on the parameter τ for this class of digraphs. First, remark that this parameter is hard to compute, even for tournaments.

Theorem 1.3 *[4] It is NP-hard to find a minimum cycle transversal in a tournament.*

A natural lower bound on the size of a minimum cardinality cycle transversal is the maximum number of vertex-disjoint cycles. So, as in the statement of Theorem 1.2, for tournaments we define the following.

$f_t(k) = \min\{p$: every tournament with no $k{+}1$ vertex-disjoint cycles has a cycle transversal of size $p\}$

The following special case of Moon's theorem allows us to restrict our interest to vertex-disjoint 3-cycles when we consider tournaments without many vertex-disjoint cycles.

Theorem 1.4 *[9] Every vertex of a strong tournament T is contained in a 3-cycle. In particular, T has k disjoint cycles if and only if it has k disjoint 3-cycles.*

Thus if a tournament T has no set of $k+1$ disjoint cycles, then, by Theorem 1.4, T has at most k disjoint 3-cycles and the vertex set of these has size at most $3k$ and forms a cycle transversal of T. Thus, we obtain an easy bound on f_t.

Corollary 1.5 *We have $f_t(k) \leq 3k$.*

A lower bound on f_t has been known for a long time. Indeed, Erdős and Moser ([7] or see Alon [1] for a short probabilistic proof) show that for every n there exists a tournament on n vertices containing no transitive subtournament on more than $2\log_2 n{+}1$ vertices. So, such a tournament on $3k{+}2$ vertices has no $k+1$ vertex-disjoint 3-cycles and no cycle transversal with less than $3k - 2\log_2(3k+2) + 1$ vertices. We then obtain the following.

Theorem 1.6 *[7] For $k \geq 2$, we have $f_t(k) \geq 3k - 2\log_2 k - 3$.*

So, the gap between the lower and the upper bound on f_t is not large. Our intuition is that it is possible to be as far as desired from the upper bound of $3k$.

Conjecture 1.7 *For every $p \geq 1$, there exists a value k_p such that for all $k \geq k_p$, every tournament without $k+1$ disjoint cycles has a cycle transversal of size $3k - p$.*

The main purpose of the paper is to compute the value of $f_t(k)$ for small values of k, and then give some evidence for Conjecture 1.7. For $k = 1$, by Theorem 1.1, we know that $f_t(1) \leq 3$, but it is possible to sharpen this bound.

Theorem 1.8 *Every intercyclic tournament T has a cycle transversal of size 2. In particular, we have $f_t(1) = 2$.*

Proof: The rotational tournament on five vertices RT_5 has vertex set $\{1,2,3,4,5\}$, and $ij \in A(RT_5)$ if $j-i=1$ or 2 modulo 5. This tournament is intercyclic and has no cycle transversal of size one, so $f_t(1) \geq 2$. It is also possible to give an infinite family of strong tournaments at which this bound is attained. For instance, consider a transitive tournament T' and add four vertices, x_1, x_2 and x_3 which form a 3-cycle and y. The remaining arcs are given by: $\{x_1, x_2, x_3\}$ dominates y and is dominated by T' and y dominates T'. It is straightforward to prove that the the tournament obtained in result is intercyclic and has no cycle transversal of size one.

To prove the reverse inequality, consider a minimal counter-example T, i.e. an intercyclic tournament with no cycle transversal of size two. First, if T' is a subtournament of T which is not strong, T' has at most one non trivial strong component, otherwise we could find two disjoint cycles in T' and then in T. In particular, as T has no vertex with out-degree or in-degree 0 (otherwise, if x is such a vertex, then $T-x$ forms a smaller counter-example than T), T is strong. Similarly, we show that T has no vertex with out-degree or in-degree 1. If not, assume that x is a vertex of T with out-degree 1. Let y be the only out-neighbour of x. Obviously, any 3-cycle containing x also includes y. Consider any such cycle xyz. Since T is intercyclic, any 3-cycle not containing x includes either y or z. Hence $\{y,z\}$ is a cycle transversal of T, a contradiction. Now, if T has a vertex-cut of size one, say $\{x\}$, the tournament $T-x$ is no more strongly connected, and then, by the initial remark, has one of its initial or terminal component with size one. Thus, we find a vertex with in- or out-degree 1 in T, which is not possible. So, T is a 2-strong tournament.

Now, if T contains a transitive subtournament of order 4, say $T\langle x_1, x_2, x_3, x_4\rangle$ with an arc from x_i to x_j for all $i<j$, then using Menger's Theorem (see e.g. [3, Theorem 5.4.1]) and the fact that T is 2-strong, we can find two vertex disjoint paths from $\{x_3, x_4\}$ to $\{x_1, x_2\}$. Then, we can add two arcs to form two disjoint cycles from these paths. Now, if C is a 3-cycle of T, then $T-C$ is acyclic and thus contains at most three vertices. So, $|T| \leq 6$. Assume that $|T|=6$. In this case, as T has no transitive subtournament of order 4, both the out-neighbourhood and in-neighbourhood of any of its vertices contain at most three vertices (note that any tournament of order 4 includes a transitive tournament of order 3). Let X and Y be the sets of vertices of out-degree 3 and 2, respectively. Since $|X|+|Y|=6$ and $3|X|+2|Y|=15$, we have $|X|=|Y|=3$. As T is intercyclic, at most one of $|X|$ and $|Y|$ induces a 3-cycle. By duality, we can assume that $|X|$ is transitive. Let x_1, x_2, x_3 be the unique Hamiltonian path in X. Obviously, the vertex x_3 dominates any vertex in Y and the vertex x_2 dominates two vertices, say, y_1 and y_2, in Y. For this case, the vertex-set $\{x_2, x_3, y_1, y_2\}$ induces a transitive subtournament of order 4, a contradiction.

The sole remaining possibility for a counter-example is to have size five, but it is then easy to exhibit a cycle transversal of size two. $\diamond$

The main result of this paper is the computation of $f_t(2)$. We obtain the following.

Theorem 1.9 *Every tournament with no three vertex-disjoint cycles has a cycle transversal of size four.*

Observe that Theorem 1.9 is optimal, in the sense that there exist tournaments with no three disjoint cycles and no cycle transversal of size three. For instance, the Paley tournament P_7 has these properties. For each prime power $q=3$ modulo 4, the Paley tournament P_q with q vertices is the tournament whose vertices are the elements of the finite field with q elements. There is an arc from x to y if and only if $y-x$ is a nonzero square in the field. In the case $q=7$, the vertex set of P_7 is $\{0,\dots,6\}$ and ij is an arc of P_7 if $j-i=1,2$ or 4 modulo 7. Once again, it is possible to obtain an infinite family of strong tournament with the same properties. For this, we add a transitive tournament T' to P_7 with the following adjacencies: $\{0,1,2,3,4,5\}$ dominates T' which dominates 6. As at most one 3-cycle can contain a vertex of T', it is straightforward to verify that this tournament contains no three disjoint cycles.

So, a first corollary of Theorem 1.9 is the following.

Corollary 1.10 *We have $f_t(2)=4$.*

Let us now show by induction on k that for any $k \geq 2$, we have $f_t(k) \leq g(k)$, where $g(k)=3k-2$. Corollary 1.10 implies that this inequality holds for $k=2$. Let $k \geq 3$. If a tournament T admits no

k vertex-disjoint cycles, then, by the induction hypothesis, T has a cycle transversal of size at most $g(k-1) < g(k)$. Assume now that T contains k vertex-disjoint 3-cycles $C_1, \dots, C_k$ but has no $k+1$ such cycles. Then $T - \cup_{i=1}^{k-2} C_i$ admits no three vertex-disjoint cycles and hence, by Theorem 1.9, has a cycle transversal of size at most 4. This means that T includes a cycle transversal of size at most $3(k-2)+4 = 3k-2 = g(k)$. So, we obtain a second corollary of Theorem 1.9.

Corollary 1.11 *For all $k \geq 2$, we have $f_t(k) \leq 3k-2$.*

Observe that this is best possible for $k=3$ also. Indeed, P_{11} the Paley tournament has no TT_5 as a subtournament (no vertex x of P_{11} can be the first vertex of such a TT_5 as the subtournament induced by $N^+_{P_{11}}(x)$ on P_{11} is isomorphic to RT_5 the rotational tournament on 5 vertices which does not contain any TT_4 as subtournament). So, a cycle transversal of P_{11} contains at least seven vertices.

Corollary 1.12 *We have $f_t(3) = 7$.*

In the next section, we present the proof of Theorem 1.9. It is similar but longer than the one of Theorem 1.8 : First, we show that the strong connectivity of a counter-example must be large enough, and then, we have to conclude on some finite cases. We conclude the paper with some remarks and problems.

2 Proof of Theorem 1.9

As we are looking for a cycle transversal or vertex-disjoint cycles, throughout this section, we will use the word 'disjoint' instead of 'vertex-disjoint'.

So, we assume that Theorem 1.9 does not hold and consider a minimum counter-example T to this statement. Each following subsection establishes a result on the strong connectivity of T, eventually leading to a contradiction.

The following lemma is a classical corollary of König's Theorem (see e.g. [3, Theorem 4.11.2]), and we will use it several times.

Lemma 2.1 *Let D be an r-strong digraph and let R, $r = |R|$, be a minimum vertex-cut of D. There exist two matchings of size r, one from R to $D-R$ and one from $D-R$ to R. More precisely, if X is an initial (resp. terminal) non-trivial component of $D-R$, then there exists a matching of size $\min\{|X|, r\}$ from R to X (resp. from X to R), and every vertex of R dominates at least one vertex of X (resp. is dominated by at least one vertex of X). In particular, if $X = \{x_1\}$ is an initial (resp. terminal) trivial component of $D-R$, then x_1 is dominated by every vertex of R (resp. dominates every vertex in R).*

2.1 T is 2-strong

First, it is easy to see that T has to be strong.

Claim 1 *T is strong.*

Proof: Assume that T is not strongly connected and denote respectively by T_1 and T_f its initial and terminal components. If T_1 or T_f contains only one vertex x, then $T-x$ would be a smaller counter-example to Theorem 1.9. As every non-trivial component contains a cycle, only T_1 and T_f are non-trivial. Now, if T_1 or T_f contains two disjoint cycles, we find three disjoint cycles in T. Otherwise, by Theorem 1.8, T_1 and T_f have cycle transversals of size two, and the union of these two cycle transversals form a cycle transversal of T of size four, contradiction. ◇

The two next claims show that T has strong connectivity at least two.

Claim 2 *Every vertex x of T satisfies $d^+_T(x) \geq 2$ and $d^-_T(x) \geq 2$.*

Proof: Assume, for instance, that a vertex x of T satisfies $d^+_T(x) = 1$ and denote by y its unique out-neighbour. As T is strong, then y has an out-neighbour z in T, and xyz forms a 3-cycle of T. By choice of T, $T - \{x, y, z\}$ does not contain two disjoint cycles, and then by Theorem 1.8, has a cycle transversal of size two. We add y and z to this transversal and obtain a cycle transversal of T of size four, as every cycle containing x has to contain y.
The case $d^-_T(x) = 1$ is similar. ◇

Claim 3 *T is 2-strong.*

Proof: Assume that T is not 2-strong and denote by r a vertex of T such that $T - r$ is not strong. So, we denote respectively by T_1 and T_f the initial and terminal components of $T - r$. By Claim 2, T_1 and T_f are not trivial, and as every non-trivial component contains a cycle, the other components of $T - r$ are trivial. We denote by r_1 (resp. r_f) an out-neighbour of r in T_1 (resp. in-neighbour of r in T_f). If none of $T_1 - r_1$ and $T_f - r_f$ contains a cycle, then $\{r, r_1, r_f\}$ is a cycle transversal of T, contradiction. So, assume that $T_f - r_f$ contain a cycle C, then $T_1 - r_1$ does not contain a cycle, otherwise this cycle with C and rr_1r_f would be three disjoint cycles. So, $\{r_1\}$ is a cycle transversal of T_1. Now, T_f does not contain two disjoint cycles, otherwise adding a cycle of T_1 we would find three disjoint cycles in T, contradiction. By Theorem 1.8, T_f contains a cycle transversal of size two, but adding r and r_1 to this set, we obtain a cycle transversal of size four of T, contradiction. ◇

2.2 T is 3-strong

Assume that T is not 3-strong and consider a minimum vertex-cut $\{r, s\}$ of size two (by Claim 3), that is $T - \{r, s\}$ is not strong. As T has no set of three disjoint cycles $T - \{r, s\}$ cannot have three or more non-trivial components. On the other hand, if $T - \{r, s\}$ has only trivial components, then $\{r, s\}$ is a cycle transversal of T. First, we deal with the case where $T - \{r, s\}$ has two non-trivial components.

Claim 4 *If $T - \{r, s\}$ has two non-trivial components T_1 and T_2, then they are its initial and terminal components.*

Proof: If none of T_1 and T_2 is an extremal component of $T - \{r, s\}$, then denote by x_1 (resp. x_2) the vertex of the initial (resp. terminal) component of $T - \{r, s\}$. As T_1 and T_2 both contain a cycle, x_1x_2r is a third cycle of T, contradiction. So, assume that T_1 is the initial component of $T - \{r, s\}$, and that T_2 is not its terminal component. We denote by x_2 the vertex of the last component of $T - \{r, s\}$. Let $\{rx, sy\}$ be a 2-matching from $\{r, s\}$ to T_1 (which exists by Lemma 2.1). Then x is a cycle transversal of T_1. Indeed, if there is a cycle C in $T_1 - x$, then C, a cycle in T_2 and xx_2r are three vertex-disjoint cycles in T. Now, let z be a vertex of T_2. We claim that z is a cycle transversal of T_2. Indeed, assume that there exists a cycle C in $T_2 - z$. Suppose first that $zr \in A(T)$. Then C, xzr and yx_2s are three disjoint cycles in T, which is impossible. In turn, if $rz \in A(T)$, then rzx_2 is the third (after C and a cycle of T_1) cycle in T, which is impossible, again. Hence, z is a cycle transversal of T_2 and $\{x, z, r, s\}$ is a cycle transversal of T, contradiction. ◇

Claim 5 *$T - \{r, s\}$ cannot have two non-trivial components.*

Proof: Assume that $T - \{r, s\}$ has two non-trivial components and note that, by Claim 4, these must be the initial and terminal non-trivial components of $T - \{r, s\}$, respectively denoted by T_1 and T_f. If $\tau(T_1) = 1$ and $\tau(T_f) = 1$ then denote respectively by $\{t_1\}$ and $\{t_2\}$ a cycle transversal of T_1 and T_f and observe that $\{t_1, t_2, r, s\}$ forms a cycle transversal of T. So, $\max\{\tau(T_1), \tau(T_f)\} \geq 2$. Now, if $\tau(T_1) \geq 2$ and $\tau(T_f) \geq 2$ then denote by x_1 an out-neighbour of r in T_1 and by x_2 an in-neighbour of r in T_f (which exist by Lemma 2.1). We could find three disjoint cycles in T, rx_1x_2, a cycle of $T_1 - x_1$ and a cycle of $T_f - x_2$, a contradiction. So, we can assume (by reversing all arcs of T if necessary) that $\tau(T_1) = 1$ and $\tau(T_f) \geq 2$.

Suppose first that $f > 2$ and let $T_i = \{x\}$ be an internal component of $T - \{r, s\}$. If $rx \in A(T)$, then a cycle of T_1, rxx_2, where x_2 is an in-neighbour of r in T_f, and a cycle of $T_f - \{x_2\}$ form three disjoint cycles of T, contradiction. So, $xr \in A(T)$. Let $\{ra, sb\}$ be a 2-matching from $\{r, s\}$ to T_1 (which exists by Lemma 2.1). Then rax, sbx_2, where x_2 is an in-neighbour of s in T_f, and a cycle of $T_f - \{x_2\}$ form three disjoint cycles of T, contradiction.

So, $f = 2$ and T_1 and T_2 are the only strong components of $T - \{r, s\}$.
Let $C = abc$ be a 3-cycle of T_1. As $\tau(T_1) = 1$, the subtournament $T' := T_1 - C$ of T is acyclic and furthermore, there is no arc from $\{r, s\}$ to T' otherwise if st' is such an arc, we consider the cycles C, $st'x_2$, where x_2 is an in-neighbour of s in T_2 , and a cycle of $T_2 - x_2$ to form three disjoint cycles in T, contradiction. So, by Lemma 2.1, there is a 2-matching from $\{r, s\}$ to T_1, and then, the ends of this 2-matching belong to C. Thus, by symmetry, we can assume that ra and sb are arcs of T. Now, if $cs \in A(T)$ then, sbc, rax_2 where x_2 is an in-neighbour of r in T_2 , and a cycle of $T_2 - x_2$ would form three disjoint cycles in T, which cannot be. Thus $sc \in A(T)$, and similarly, we prove in this order that $rb \in A(T)$, $sa \in A(T)$ and that $rc \in A(T)$.

Now, if $T' \neq \emptyset$ then, as T_1 is strong, there is an arc from C to T', say at' for instance. But, $at's$, rbx_2 where x_2 is an in-neighbour of r in T_2 , and a cycle of $T_2 - x_2$ would form three disjoint cycles in T, which cannot be. So $T_1 = C$ and $\{r, s\}$ entirely dominates T_1. To conclude, we study the structure of T_2. By Lemma 2.1, there is a 2-matching from T_2 to $\{r, s\}$. We denote by $\{dr, es\}$ such a matching, and remark that $\{d, e\}$ is a cycle transversal of T_2, otherwise, a cycle of $T_2 - \{d, e\}$, dra and esb would form three disjoint cycles of T. Then, $T'' = T_2 - \{d, e\}$ is a transitive subtournament of T. If there is no 2-matching from T'' to $\{r, s\}$, then there is a vertex x of $T'' \cup \{r, s\}$ which is contained in all the arcs going from T'' to $\{r, s\}$. In this case, $\{a, x, d, e\}$ would form a cycle transversal of T. Hence, there is a 2-matching from T'' to $\{r, s\}$ and we denote it by $\{d'r, e's\}$. As $\tau(T_2) \geq 2$, we have $|T_2| \geq 5$ (the unique strong tournament on 4 vertices has a cycle transversal of size one) so T_2 contains a vertex x different from d, e, d' and e'. As T_2 is strong, by Theorem 1.4 there exists a 3-cycle C' which contains x. If $V(C') \cap \{u, v\} = \emptyset$ for some 2-matching $\{ur, vs\}$ from T_2 to $\{r, s\}$ then, as previously, C', aur and bvs would form three disjoint cycles in T. In particular, C' has to intersect all the following pairs: $\{d, e\}$, $\{d', e'\}$, $\{d, e'\}$ and $\{d', e\}$, and thus $V(C') = \{x, d, d'\}$ or $V(C') = \{x, e, e'\}$. Without loss of generality, we can assume that $V(C') = \{x, d, d'\}$. Now, if $er \in A(T)$, then $\{er, e's\}$ would be a 2-matching from T_2 to $\{r, s\}$ which avoids C', what is forbidden. Thus, $re \in A(T)$ and similarly $re' \in A(T)$. This implies that $rs \in A(T)$, otherwise, sre, C' and C would be three disjoint cycles in T. Furthermore, we have $ds \in A(T)$, otherwise, C, sdr and a cycle of $T_2 - d$ would form three disjoint cycles in T. Similarly, we have $d's \in A(T)$. Now, we conclude by considering a 3-cycle C'' of $T_2 - d$. As $dr \in A(T)$ and $ds \in A(T)$, $\{d, e\}$, $\{d, e'\}$ and $\{d, d'\}$ are beginnings of 2-matchings from T_2 to $\{r, s\}$, and then C'' has to contain e, e' and d', implying that $V(C'') = \{e, e', d'\}$. Now, if $xr \in A(T)$, then C'' does not intersect the 2-matching $\{xr, ds\}$ from T_2 to $\{r, s\}$ and we can conclude. So, $rx \in A(T)$. If $xd \in A(T)$ then rxd, C'' and C form three disjoint cycles in T, contradiction, so C' is the 3-cycle dxd'. We have $sx \in A(T)$ as otherwise C'' avoids $\{dr, xs\}$. This implies that $de, de' \in A(T)$ as sxd' and C are disjoint from $\{d, e, e', r\}$ so this set cannot contain a 3-cycle. Suppose first that $C'' = e'd'e$. Then $ex \in A(T)$ or C, xes, $e'd'd$ are disjoint 3-cycles. But then the 3-cycle $xd'e$ avoids $\{dr, e's\}$, contradicting the conclusion above. Thus $C'' = ed'e'$ and since $\{x, d', e'\}$ avoids $\{dr, es\}$ we have $xe' \in A(T)$ and then $xe's$, $ed'd$ and C are disjoint 3-cycles, contradiction. ⋄

So, the last case to establish is when $T - \{r, s\}$ has only one non-trivial component. Again, we will see that this case is not possible. To see it, assume that T_1 is the only non-trivial component of $T - \{r, s\}$. Then, we depict the situation (see also Figure 1). First, remark that if T_1 has a cycle transversal of size two, then, with r and s we would obtain a cycle transversal of size four for T, which is impossible. So, by Theorem 1.8, T_1 contains two disjoint cycles. Now, if T_1 is neither initial nor terminal, then we denote by x_1 (resp. x_2) the vertex that forms the (trivial) initial (resp. terminal) component of $T - \{r, s\}$ and with two disjoint cycles of T_1, x_1x_2r would form a third disjoint cycle in T. So, T_1 is either the initial component of $T - \{r, s\}$ or its terminal component. By symmetry, assume that T_1 is the initial component of $T - \{r, s\}$. We denote by $\{x_2\}$ the trivial terminal component of $T - \{r, s\}$. We claim that $T - \{r, s\}$ only contains the vertices of T_1 and x_2. Indeed, if there is a vertex t in $T - (\{r, s\} \cup T_1)$ which is different from x_2, assume first that there is an arc from r to t, then we could form the disjoint cycles rtx_2, C and C', where C and C' are two disjoint cycles of T_1.

So, there is an arc from t to r. By Lemma 2.1, there is a 2-matching $\{ru, sv\}$ from $\{r, s\}$ to T_1, and as T_1 has no transversal of size two, there is a cycle C in $T_1 - \{u, v\}$. So, we can form the disjoint cycles C, rut and svx_2. Thus $T - \{r, s\}$ only contains T_1 and x_2 and we denote by $C = abc$ and $C' = a'b'c'$ two disjoint 3-cycles of T_1, and by T' the acyclic subtournament of T_1 induced by T on $T_1 - (C \cup C')$. Finally, observe that there is no arc from $\{r, s\}$ to T' otherwise we could form a 3-cycle with this arc and x_2 disjoint from C and C'.

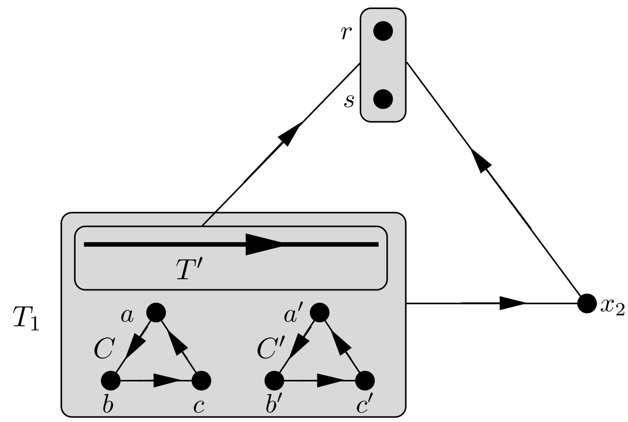

Figure 1: The situation in the case where T is 2-strong and $T - \{r, s\}$ has a unique non-trivial component. The arcs between two boxes stand for all the arcs between these boxes.

Claim 6 *There is no 2-matching from $\{r, s\}$ to C.*

Proof: Assume w.l.o.g. that $\{ra, sb\}$ is such a 2-matching. Then, $sc \in A(T)$, otherwise, we form the three disjoint cycles csb, rax_2 and C'. Using the same argument, we prove, in this order, that $rb \in A(T)$, $sa \in A(T)$ and that $rc \in A(T)$. Finally, with the hypothesis that there is a 2-matching from $\{r, s\}$ to C, we prove that $\{r, s\}$ entirely dominate C. In particular, there is then no arc from C to T', otherwise if at is such an arc, we could form the three disjoint cycles atr, sbx_2 and C'. It follows that there is no 2-matching from $\{r, s\}$ to C', otherwise, similarly $\{r, s\}$ and T' would entirely dominate C' and the only out-neigbour of $C \cup C'$ would be x_2, which contradicts the fact that T is 2-strong.

So, there is no 2-matching from $\{r, s\}$ to C' and hence there is a vertex x which belongs to all the arcs going from $\{r, s\}$ to C'. We have two cases to consider:

1. Case $x \in \{r, s\}$. Without loss of generality, assume that $x = s$, which means that there is no arc from r to C'. We will use the following properties to conclude the proof of Claim 6 in this case:

   -($\mathcal{P}_1$): $\tau(T_1) \geq 3$. Otherwise, a cycle transversal of T_1 of size two and r and s would form a cycle transversal of size four of T.

   -($\mathcal{P}_2$): as previously remarked, T' dominates C.

   -($\mathcal{P}_3$): there is no 3-cycle S of T_1 which intersects C on only one vertex and an arc from C to C' disjoint from S. Otherwise, assume that a 3-cycle S of T_1 contains only c among $\{a, b, c\}$, that $a' \notin S$ and that $aa' \in A(T)$ then, we could find three disjoint cycles in T: S, $aa'r$ and sbx_2.

   Now, if there is no arc from C to C', the only out-neigbour of C would be x_2, which contradicts the fact that T is 2-strong. So, w.l.o.g. assume that aa' is an arc of T. Using ($\mathcal{P}_1$), we know that $T\langle V(T') \cup \{b, c, b', c'\}\rangle$ contains a 3-cycle S. By ($\mathcal{P}_3$), S has to contain b and c, and through ($\mathcal{P}_2$), we know that S has to contain b' or c'. In particular, if $S = bcc'$ then, we have $cc' \in A(T)$, and if $S = bcb'$ then, by ($\mathcal{P}_3$), $cb'c'$ is not a 3-cycle and we have $cc' \in A(T)$ also. Now, considering the arc cc' and the three vertices a, a' and b', we know, through ($\mathcal{P}_3$) that $ab' \in A(T)$. Finally, by ($\mathcal{P}_1$), we know that $T\langle V(T') \cup \{a, b, a', b'\}\rangle$ contains a 3-cycle S. By ($\mathcal{P}_3$) and as $cc' \in A(T)$, S has to contain a and b, and by ($\mathcal{P}_2$), S has to contain a' or b', contradiction because a dominates both a' and b'. This concludes the case $x \in \{r, s\}$.

2. Case $x \in C'$. Without loss of generality, we can assume that $x = a'$, what means that there is no arc from $\{r, s\}$ to $\{b', c'\}$. Furthermore, as we are not in the previous case, $sa' \in A(T)$ and $ra' \in A(T)$ hold. Now, if $T\langle V(T') \cup \{c', b'\}\rangle$ contains a 3-cycle S, then S, $ra'x_2$ and C would form three disjoint cycles in T. So, $T\langle V(T') \cup \{c', b'\}\rangle$ is an acyclic subtournament of T dominating $\{x_2, r, s\}$ which does not form a 3-cycle. Thus, $T\langle V(T') \cup \{c', b', x_2, r, s\}\rangle$ is acyclic and $\{a, b, c, a'\}$ is a cycle transversal of T, contradiction.

$\diamond$

Now, we are in the case where there is no 2-matching from $\{r, s\}$ to C. By symmetry, we assume that there is no 2-matching from $\{r, s\}$ to C', and more generally, that there are no two disjoint 3-cycles in T_1 with a 2-matching from $\{r, s\}$ to one of these 3-cycles. Also, as by Lemma 2.1, there is a 2-matching from $\{r, s\}$ to T_1, and as there is no arc from $\{r, s\}$ to T', we can assume that ra and sa' are arcs of T. In this situation we have the following.

Claim 7 *The 3-cycle C dominates s which dominates C', and symmetrically, the 3-cycle C' dominates r which dominates C.*

Proof: We know that $ra \in A(T)$ and $sa' \in A(T)$ and then, by Claim 6, $bs, cs \in A(T)$ and $b'r, c'r \in A(T)$.

First, let us see that $d^+_{T_1}(r) \geq 2$ and $d^+_{T_1}(s) \geq 2$. If it is not the case, assume for instance that $d^+_{T_1}(r) = 1$, and then that $N^+_{T_1}(r) = \{a\}$. If $T_1 - a$ contains two disjoint cycles, then these cycles with rax_2 would form three disjoint cycles in T. So, by Theorem 1.8, $T_1 - a$ has a cycle transversal of size two. We denote this transversal by $\{u, v\}$ and remark then that $\{u, v, s, a\}$ would form a cycle transversal of T, contradiction. So, we have $d^+_{T_1}(r) \geq 2$ and $d^+_{T_1}(s) \geq 2$.

Now, assume that $sa \in A(T)$. As there is no 2-matching from $\{r, s\}$ to C, there is no arc from r to $\{b, c\}$. As $d^+_{T_1}(r) \geq 2$, ra' has to be an arc of T, and then, there is no arc from s to $\{b', c'\}$. So, $T' \cup \{b, c, b', c'\}$ dominates $\{r, s\}$. Furthermore, $T\langle V(T') \cup \{b, c, b', c'\}\rangle$ does not contain two disjoint cycles, otherwise we could form a third one with rax_2 for instance. So, by Theorem 1.8 it has a cycle transversal of size two. If we denote this transversal by $\{u, v\}$, then $\{u, v, a, a'\}$ would form a cycle transversal of T, contradiction.

Then, this means that C dominates s and similarly that C' dominates r. As $d^+_{T_1}(r) \geq 2$ and, similarly, $d^+_{T_1}(s) \geq 2$, r and s have respectively two out-neighbours in C and C'. So, without loss of generality, we can assume that rb and sb' are arcs of T. Moreover, by symmetry, we can also assume that aa' is an arc of T. Now, if $T\langle V(T') \cup \{b, c, c'\}\rangle$ contains a 3-cycle S, then S, $aa'r$ and $sb'x_2$ would form three disjoint cycles in T. So, the subtournament $T\langle V(T') \cup \{b, c, c'\}\rangle$ is acyclic. If $c's$ is an arc of T, then s is dominated by all vertices of this subtournament and $\{a, a', b', r\}$ would be a cycle transversal of T, contradiction. So, s dominates C'. The last point to prove is that rc is an arc of T. Assume that this is not true and that $cr \in A(T)$. Then, a dominates C', otherwise, as $aa' \in A(T)$, there exists a 3-cycle containing a and two vertices of C', say for instance a' and b'. In this case, we could form the disjoint 3-cycles $aa'b'$, bcr and $sc'x_2$. Now, we pick two vertices u and v among $\{a', b', c'\}$ and denote by w the third one. Note first that $T\langle V(T') \cup \{b, c, u, v\}\rangle$ is not acyclic (otherwise, $\{r, s, a, w\}$ would be a cycle transversal of T). Let S be a 3-cycle of $T\langle V(T') \cup \{b, c, u, v\}\rangle$. If S does not contain u and v (say for instance that $u \notin S$), then we could form the three disjoint cycles S, awr and sux_2. Furthermore, if S contains neither b nor c, then we could form the three disjoint cycles S, bcr and swx_2. So, it means that S contains u, v and one of b and c. Since $\{u, v\}$ can be any of three pairs of $\{a', b', c'\}$, this means that two different pairs of $\{a', b', c'\}$ form a 3-cycle with the same vertex (b or c), which is not possible as $a'b'c'$ is a 3-cycle. Finally, we conclude that $rc \in A(T)$ and that r dominates C. $\diamond$

Now, we can show that T has nine vertices, namely we have the following.

Claim 8 *We have $T' = \emptyset$.*

Proof: Assume that T' is not empty and then contains a vertex t'. As T_1 is a strongly connected component, there exists a 3-cycle S of T_1 which contains t'. If S contains one vertex of C and one vertex of C', say that $a \in S$ and $a' \in S$, then, we could form the disjoint 3-cycles S, $bb'r$ and $sc'x_2$ if $bb' \in A(T)$, or S, $b'bs$ and rcx_2 otherwise. So, we may assume that S does not contain any vertex of C', for instance, and contains at most two vertices of C. Without loss of generality, assume that $b \notin S$. Then we could form three disjoint 3-cycles S, C' and rbx_2, a contradiction. ◇

To conclude, we have to study the arcs between C and C'. There are three cases where we can conclude:

-*Case 1*: $T\langle C \cup C' \rangle$ contains a transitive subtournament on four vertices. Then, with x_2 we find a transitive subtournament T_a on five vertices in T and $V(T) - V(T_a)$ is a cycle transversal of T containing four vertices, contradiction.

-*Case 2*: there are a 3-cycle S of T containing two vertices of C and one vertex of C' and an arc from C to C' disjoint from S. For instance, assume that aba' is a 3-cycle and that $cc' \in A(T)$. In this case, we could form the disjoint 3-cycles aba', $cc'r$ and $sb'x_2$.

-*Case 3*: there are a 3-cycle S of T containing two vertices of C' and one vertex of C and an arc from C' to C disjoint from S. This case is the symmetrical case of *Case 2*.

So, we will conclude using these three cases. If every vertex $u \in \{a', b', c'\}$ either dominates or is dominated by C, then we find a transitive subtournament of $T\langle C \cup C' \rangle$ of size four, using two vertices of C and two vertices of $\{a', b', c'\}$ both dominating C or both being dominated by C. This means that one vertex of C', say a' for instance, forms a 3-cycle with two vertices of C, say w.l.o.g. a and b. Avoiding *Case 2* implies that $c'c \in A(T)$ and $b'c \in A(T)$. If $ac'a'$ is a 3-cycle, we are in *Case 3* with the arc $b'c$, so $c'a \in A(T)$. Finally, for any orientation of the arc between c and a', the set $\{a, c, a', c'\}$ induces an acyclic subtournament and, hence, we are in *Case 1*.

2.3 Final cases

Now, we are in the case where T is 3-strong. Using this, we make the following observation which will be very useful.

Observation 2.2 *In T, there are no two sets of vertices X and Y each of size three such that X dominates Y.*

Proof: Assume that there are two such set X and Y. As T is 3-strong, by Menger's Theorem (see e.g. [3, Theorem 5.4.1]), there are three disjoint paths P_1, P_2 and P_3 from Y to X. For every $i = 1, 2, 3$, we denote the initial and terminal vertices of P_i by a_i and b_i respectively. By hypothesis, $b_i a_i \in A(T)$ for every $i = 1, 2, 3$ and so $V(P_i)$ induces a cycle of T. Thus, by Theorem 1.4, we obtain three disjoint 3-cycles in T, contradiction. ◇

We can directly derive from Observation 2.2, that T must have at most 10 vertices. Indeed, if T has at least 11 vertices, let C be a 3-cycle of T and recall that, by Theorem 1.8, $\tau(T - C) \leq 2$ and then, as $T - C$ has at least 8 vertices, it contains a TT_6. In this acyclic subtournament, the three first vertices entirely dominate the three last vertices, which contradicts Observation 2.2.
Obviously, any tournament T of order at least 4 contains a transitive subtournament T' of order 3. Hence, if $4 \leq |T| \leq 7$, then $V(T) - V(T')$ is a cycle transversal with size at most 4. Suppose now that $|T| = 8$. Then T has a vertex x of out-degree at least 4 and, hence, in its out-neighbourhood, we can find a copy of TT_3. As a consequence, T admits a transitive subtournament T' of order 4 and hence, $V(T) - V(T')$ forms a cycle transversal with size 4.Thus we must have $9 \leq |V(T)| \leq 10$.

First, we deal with the case $|V(T)| = 9$.

2.3.1 Case $|T| = 9$

In this case, the vertices of T cannot all have odd out-degree (as $|A(T)|$ is even). So, at least one vertex, x, has even out-degree and as T is 3-strong, we have $d^+_T(x) = d^-_T(x) = 4$. We respectively denote $N^+_T(x)$ and $N^-_T(x)$ by X and Y. First, observe that T does not contain a copy T' of TT_5 as otherwise $V(T) - V(T')$ would be a cycle transversal of size four, contradiction. In particular, this means that neither X nor Y induces an acyclic subtournament of T.

Claim 9 *There exists a 4-matching from X to Y.*

Proof: If this is not the case, then by König's theorem, there is a set Q of three vertices that intersects all the arcs from X to Y. If $|Q \cap X| = 1$ and $|Q \cap Y| = 2$, then $(Y - Q) \cup \{x\}$ has size three and dominates $X - Q$ which has size three and we can conclude using Observation 2.2. The case $|Q \cap X| = 2$ and $|Q \cap Y| = 1$ is analogous. So, this means that we have either $Q \subset X$ or $Q \subset Y$. By reversing all arcs if necessary, we may assume that $Q \subset X$, and denote by v_1 the vertex of $X - Q$. By the choice of Q, Y dominates v_1. But now, as $T\langle Y \rangle$ is a tournament on four vertices, it contains Z, a subtournament on three vertices isomorphic to TT_3. Hence, the vertex-set $V(Z) \cup \{x, v_1\}$ induces a copy of TT_5, implying that $\tau(T) \leq 4$, contradiction. $\diamond$

Now, we fix a labelling $X = \{v_1, v_2, v_3, v_4\}$ and $Y = \{w_1, w_2, w_3, w_4\}$ such that $v_i w_i \in A(T)$ for all $i = 1, 2, 3, 4$.

Claim 10 *In X, there is no vertex with in-degree 3 in X.*

Proof: Suppose w.l.o.g. that $d^-_X(v_1) = 3$. As X does not induce an acyclic subtournament, it means that v_2, v_3 and v_4 form a 3-cycle in X, which is w.l.o.g. $C = v_2v_3v_4$. Since T is 3-strong v_1 has three out-neighbours in Y. If there is a 3-cycle S in Y which does not contain some out-neighbour w' of v_1, then we could form the three disjoint cycles S, C and $v_1w'x$. So v_1 has out-degree exactly three in Y and we may assume w.l.o.g. that $N^+_Y(v_1) = \{w_1, w_2, w_3\}$, and that $C' = w_1w_2w_3$ or $C' = w_3w_2w_1$ is the only 3-cycle of Y, which means that either w_4 dominates C' or C' dominates w_4.

First, assume that w_4 dominates C'. As $d^-_T(w_4) \geq 3$, we know that C dominates w_4. If $v_2w_3 \in A(T)$, then $\{v_2, w_4, v_1, w_2, w_3\}$ induces a TT_5 in T, contradiction. So, we have $w_3v_2 \in A(T)$, and similarly, as $\{v_3, w_4, v_1, w_3, w_1\}$ cannot induce a TT_5, we have $w_1v_3 \in A(T)$. If $C' = w_1w_2w_3$, then we can form the three disjoint cycles $v_2w_2w_3$, v_4w_4x and $w_1v_3v_1$. So we must have $C' = w_3w_2w_1$. Now $w_2v_4 \in A(T)$ as otherwise $v_2w_4w_3, w_1v_3v_1, v_4w_2x$ are disjoint 3-cycles. For this case $w_2v_4w_4, v_2v_3w_3, v_1w_1x$ are disjoint 3-cycles, contradiction.

So w_4 is dominated by C'. We must have $v_4w_2 \in A(T)$ or we could form the disjoint cycles $w_2v_4v_2$, v_3w_3x and $v_1w_1w_4$. Similarly $v_2w_3 \in A(T)$ or we form the disjoint cycles $v_2v_3w_3$, v_4w_2x and $v_1w_1w_4$. Now we must have $w_2v_3 \in A(T)$ since otherwise $\{v_2, v_3, v_1, w_2, w_3\}$ induce a TT_5, no matter what the orientation of the arc between w_2 and w_3 is. Finally we obtain a contradiction by observing that $v_3v_4w_2, v_2w_3x, v_1w_1w_4$ are disjoint 3-cycles. $\diamond$

Claim 11 *In X, there is no vertex with out-degree 3 in X.*

Proof: On the contrary, assume that $d^+_X(v_1) = 3$. As X does not induce an acyclic subtournament, it means that v_2, v_3 and v_4 form a 3-cycle, say w.l.o.g., $C = v_2v_3v_4$ in X. If there is a 3-cycle S in Y which does not contain some out-neighbour w' of v_1, then we could form the three disjoint cycles S, C and $v_1w'x$. So, without loss of generality, we may assume that $C' = w_1w_2w_3$ or $C' = w_3w_2w_1$ is a 3-cycle, and that $w_4v_1 \in A(T)$. Furthermore we also have that $W = \{w_2, w_3, w_4\}$ induce a copy of TT_3. As T is 3-strong and has 9 vertices there is at most one arc from v_1 to $\{w_2, w_3\}$. There is also at least one, since otherwise the vertex-set $W \cup \{v_1, x\}$ induces a copy of TT_5. We may assume (since we have not fixed the orientation of C' yet) that $v_1w_2, w_3v_1 \in A(T)$. Suppose first that $C' = w_1w_2w_3$. Then we conclude in that order that the following arcs are in $A(T)$: $v_3w_1 \in A(T)$, or $w_1v_3w_3, v_2w_2x, v_4w_4v_1$ are disjoint 3-cycles, $v_2w_3 \in A(T)$, or $w_3v_2w_2, v_3w_1x, v_4w_4v_1$ are disjoint 3-cycles, $v_3w_2 \in A(T)$, or $w_2v_3w_1, v_2w_3x, v_4w_4v_1$ are disjoint 3-cycles, and $v_2w_1 \in A(T)$, or $w_1v_2v_3, v_1w_2w_3, v_4w_4x$ are disjoint 3-cycles. Now $\{v_1, v_2, v_3, w_1, w_2\}$ induces a copy of TT_5, contradiction. So we must have $C' = w_3w_2w_1$. Then, we must have $v_2w_1 \in A(T)$, or $w_1v_2w_2, v_3w_3x, v_1v_4w_4$ are disjoint 3-cycles and $v_3w_2 \in A(T)$ or $v_3w_3w_2, v_1v_4w_4, v_2w_1x$ are disjoint 3-cycles. Now $w_1v_3 \in A(T)$, or $\{v_1, v_2, v_3, w_1, w_2\}$ is a TT_5. Then $w_3v_2 \in A(T)$, or $v_2w_3x, v_1v_4w_4, v_3w_2w_1$ are disjoint 3-cycles. Finally we get the contradiction that

$v_2w_1w_3, v_1v_4w_4, v_3w_2x$ are disjoint 3-cycles. $\diamond$

So, now, we can conclude that we cannot have $|T| = 9$: By symmetry, we assume that there is no vertex in Y with in or out-degree three inside Y, which means that X and Y both induce strongly connected subtournaments of T. For instance, we assume that $v_1v_2v_3v_4$ is a 4-cycle and that $v_1v_3 \in A(T)$ and $v_2v_4 \in A(T)$ (all strong tournaments on 4 vertices are isomorphic). If there is a 3-cycle S in Y which does not contain at least one of w_2 or w_3, say $w_2 \notin S$ for instance, then we form the cycles S, v_2w_2x and $v_1v_3v_4$. This means that the two 3-cycles of $T\langle Y\rangle$ have vertex set $\{w_2, w_3, w_4\}$ and $\{w_2, w_3, w_1\}$. Now, if there exists an arc uv from $\{v_2, v_3\}$ to $\{w_1, w_4\}$, then we can find a 3-cycle in $T\langle X\rangle$ which does not contain u, a 3-cycle in $T\langle Y\rangle$ which does not contain v, and we obtain the third 3-cycle uvx. Finally, we conclude that $\{w_1, w_4\}$ dominates $\{v_2, v_3\}$, but now $\{w_1, w_4, x, v_2, v_3\}$ induces an acyclic subtournament of T, contradiction.

2.3.2 Case $|T| = 10$

Let R be a minimal vertex-cut of T. We have $|R| \geq 3$ and since every vertex has out-degree at least $|R|$, $|T| \geq 2|R| + 1$ and we have $|R| \leq 4$, so, $|R| \in \{3, 4\}$. Note that $T - R$ cannot contain two or more non-trivial components, because in this case, two of such components both contain at least three vertices and we conclude using Observation 2.2. In particular either the initial or the terminal component of $T - R$ is trivial. From now on, we assume w.l.o.g. that the last component is trivial, and we denote it by $\{x_2\}$. By Lemma 2.1, we know that x_2 dominates R. On the other hand, $T - R$ has at least one non-trivial component, otherwise R is cycle transversal of T with size at most four. So, we denote by T_1 the non-trivial component of $T - R$.

Claim 12 *The initial component of $T - R$ is T_1.*

Proof: On the contrary, assume that the initial component of $T - R$ is not T_1, so it is a trivial component and denote it by $\{x_1\}$. Fix a 3-cycle C in T_1. By Lemma 2.1, we know that R dominates x_1. First, we deal with the case $|R| = 3$. In this case, we denote by v and v' the two vertices of $T - (R \cup C)$ different from x_1 and x_2, and by r, s and t the vertices of R. If there is an arc from $\{v, v'\}$ to R, say vr, disjoint from an arc from R to $\{v, v'\}$, say sv', then we form the disjoint cycles C, x_1vr and x_2sv'. It means that either $\{v, v'\}$ dominates R or R dominates $\{v, v'\}$. In the first case, $\{v, v', x_2\}$ dominates R and in the second one, R dominates $\{x_1, v, v'\}$. In both cases, we can conclude using Observation 2.2.

So, now we look at the case $|R| = 4$. We denote by v the vertex of $T - (R \cup C)$ different from x_1 and x_2, and by r, s, t and u the vertices of R. If $T\langle R \cup v\rangle$ contains a 3-cycle S, this cycle avoids some vertex of R, say r, and we could form the disjoint cycles C, S and x_1x_2r. Otherwise, $T\langle R \cup v\rangle$ is an acyclic subtournament of T. In this case, its initial vertex cannot be v, otherwise v, x_2 and the initial vertex of $T\langle R\rangle$ would dominate the three other vertices of R and we could conclude by using Observation 2.2. Similarly, v is not the terminal vertex of $T\langle R \cup v\rangle$. So, assume that the initial and terminal vertex of $T\langle R \cup v\rangle$ are respectively r and u. As u must have at least four out-neighbours, u dominates C, and similarly, C dominates r. Now, if v dominates C, then $\{x_1, v, u\}$ dominates C and once again, we conclude by using Observation 2.2. Using x_2, we also see that C does not dominate v. This means that there exist a 3-cycle S which contains v and two vertices of C, say, a and b. To conclude, we can form the three disjoint cycles S, cru and x_1x_2s. $\diamond$

Now, we denote by T' the acyclic subtournament $T - (R \cup V(T_1))$ (with last vertex x_2). We show that T_1 and $\{x_2\}$ are the only components of $T - R$.

Claim 13 *We have $T' = \{x_2\}$.*

Proof: On the contrary, assume that there is a vertex x_2' in T' different from x_2. We have $V(T') = \{x_2', x_2\}$ as otherwise T_1 dominates three vertices and we conclude using Observation 2.2. Observe that since T is $|R|$-strong, x_2' dominates R except possibly one vertex. We study the different cases $|R| = 3$ and $|R| = 4$. First, assume that $|R| = 3$, and we have then $|T_1| = 5$. No vertex of T_1 has in-degree 3 in T_1, otherwise the in-neighbourhood of such a vertex dominates three vertices, using that vertex and x'_2 and x_2 and we conclude with Observation 2.2. So, every vertex of T_1 has exactly in- and out-degree 2, and, hence, T_1 is the unique regular tournament on 5 vertices (it is also isomorphic to RT_5). The diameter of this tournament equals 2 and hence, each of its arcs lies on a 3-cycle. Now, by Lemma 2.1, there is a 3-matching $\{ra, sb, tc\}$ from R to T_1 and we denote by d and e, with $de \in A(T)$, the two vertices of T_1 not involved in this matching. The arc de lies on a 3-cycle of T_1, say dea. On the other hand, x'_2 has to dominate one of the two vertices s and t, say, $x'_2s \in A(T)$. Then, we form the disjoint cycles dea, sbx'_2 and tcx_2, contradiction.

So $|R| = 4$ and then we have that $|T_1| = 4$. By Lemma 2.1, we know that there is a 4-matching from R to T_1, we denote it by $\{qa, rb, sc, td\}$ and assume that $abcd$ is a 4-cycle of T_1. If there is a 3-cycle involving one vertex of R and two of T_1, say qab for instance, disjoint from a 2-matching from R to T_1, say $\{sc, td\}$, then we can find three disjoint cycles in T. Indeed, x'_2 is the tail of an arc to s or to t, say that $x'_2s \in A(T)$, and we can form the cycles qab, x'_2sc and tdx_2. So, we have $qb \in A(T)$, and similarly $rc \in A(T)$, $sd \in A(T)$ and $ta \in A(T)$. Continuing that way, we see that R dominates T_1 and that $T - \{x_2, x'_2\}$ is no more strongly connected, contradicting T is 3-strong. ⋄

Now, we focus on the case $|R| = 4$. In this case, we have $|T_1| = 5$ and we denote by $C = abcde$ a Hamilton cycle of T_1.

Claim 14 *If $|R| = 4$, then R is an acyclic subtournament of T.*

Proof: Otherwise, denote by $C_R = rst$ a 3-cycle of R and by q its fourth vertex. First, assume that q forms a 3-cycle with two consecutive vertices of C, say that qab is a 3-cycle. If $T\langle b, c, d, e\rangle$ contains a cycle, this cycle, C_R and qax_2 form three disjoint cycles of T. So $T\langle b, c, d, e\rangle$ is an acyclic subtournament of T and $bd, be, ce \in A(T)$. So, abe is a 3-cycle of T and if $qc \in A(T)$ (resp. $qd \in A(T)$) we form the three disjoint cycle C_R, abe and qcx_2 (resp. qdx_2). Thus, we have $cq \in A(T)$ and $dq \in A(T)$. But now, $\{b, c, d\}$ dominates $\{q, e, x_2\}$ and we conclude using Observation 2.2.
So, we can assume that q does not form any 3-cycle with two consecutive vertices of C, it means that either C dominates q or q dominates C. As q has at least one out-neighbour in T_1 (otherwise $R - q$ is also a vertex-cut of T), we have that q dominates T_1. To conclude, let S be a 3-cycle of T_1 and y a vertex of $T_1 - S$. We form the three cycles C_R, S and qyx_2. ⋄

So, using that no vertex-cut of T with size four contains a 3-cycle, we can now conclude the case $|R| = 4$.

Claim 15 *We have $|R| = 3$.*

Proof: Assume that $|R| = 4$. By Lemma 2.1, we know that there exists a 4-matching from R to T_1. We are looking for such a matching with a special property involving C. To obtain it, consider the following procedure on 4-matchings from R to T_1 starting from a 4-matching $\{r_1c_1, r_2c_2, r_3c_3, r_4c_4\}$, where $c_1c_2c_3c_4c_5$ forms a Hamiltonian cycle of T_1: if $c_5r_4 \in A(T)$ then we stop the procedure, otherwise, we repeat the procedure on the matching $\{r_4c_5, r_1c_1, r_2c_2, r_3c_3\}$. We start with a 4-matching M from R to T_1 and the Hamiltonian cycle C and we apply recursively the procedure. If we stop at some step, then we are done, otherwise there is a loop in the procedure and after some number of steps the procedure is back to the initial matching M. In this case, it is easy to see that R dominates T_1 and that $T - x_2$ is no more strongly connected, contradicting T is 3-strong.
So, without loss of generality, we can assume that $\{qa, rb, sc, td\}$ forms a 4-matching from R to T_1 and that $et \in A(T)$. If we can find a 3-cycle S of T which contains one vertex of $\{q, r, s\}$ and two vertices of $\{a, b, c\}$ and which is disjoint from an arc uv with $u \in \{q, r, s\}$ and $v \in \{a, b, c\}$, then we can form the three cycles S, det and uvx_2. Using this argument, we can see that $rc \in A(T)$ (otherwise rbc is a 3-cycle disjoint from qa). If $ca \in A(T)$, then using the above argument, we successively see that sa and qb are arcs of T and finally that qc, ra and sb are also arcs of T. Then, $\{q, r, s\}$ dominates $\{a, b, c\}$ and we conclude with Observation 2.2. So, we have $ac \in A(T)$ and as above we also see that $qc \in A(T)$ must hold. Now, using that T is 4-strong (by the assumption that $|R| = 4$), we know that the out-neighbourhood of c is exactly $R' = \{x_2, e, d, t\}$, which then forms a minimum vertex-cut of T. But, R' contains the cycle det which contradicts Claim 14. ⋄

From now on, we assume that $|R| = 3$, and then that $|T_1| = 6$. We denote by $C = abcdef$ a Hamiltonian cycle of T_1. Using a similar procedure as in the previous proof, as R does not dominate T_1 (otherwise $T - x_2$ is no more strongly connected, contradicting T is 3-strong) we can see that there exists $M = \{rr_1, ss_1, tt_1\}$ a 3-matching from R to T_1 such that r_2 the out-neighbour of r_1 on C is different from s_1 and t_1 and dominates r. So we assume that $r_1 = a$ and $r_2 = b$. This means that rab is 3-cycle.

We study the possible positions of s_1 and t_1. We will intensively use the fact that if there exists a 3-cycle S in $T\langle\{c, d, e, f, s, t\}\rangle$ disjoint from an arc uv with $u \in \{s, t\}$ and $v \in \{c, d, e, f\}$ then we obtain the three disjoint cycles rab, S and uvx_2.

Case $s_1 = c$, $t_1 = d$. We denote this case by [Δ 1 2] to mean that the 3-cycle rab is directly followed on C by the end of one arc of the 3-matching M which is also directly followed on C by the end of the last arc of M. If $et \in A(T)$, then we use the 3-cycle tde and the arc sc to conclude. So, we have $te \in A(T)$ and similarly, $tf \in A(T)$. Now, if s is contained in a 3-cycle S with two vertices of $\{c, d, e, f\}$, then there exists an arc from t to $\{d, e, f\} - V(S)$ and we find the desired three disjoint cycles. This means that s dominates $\{c, d, e, f\}$. To conclude, if there exists a 3-cycle S on $\{c, d, e, f\}$ then the vertex of $\{c, d, e, f\}$ not belonging to S is dominated by s and we also conclude. So, $T\langle\{c, d, e, f\}\rangle$ is an acyclic subtournament of T and we have that $\{s, t, c\}$ dominates $\{d, e, f\}$. Thus, we conclude this case using the Observation 2.2.

Case $s_1 = c$, $t_1 = e$. We denote this case by [Δ 1 . 2] to mean that there is a vertex along C between the ends of the two last arcs of the matching M from R to C. As we assume that we are not in the previous case, we have that $dt \in A(T)$. To avoid having a 3-cycle with one vertex in $\{s, t\}$ and two in $\{c, d, e, f\}$ disjoint from an arc from $\{s, t\}$ to $\{c, d, e, f\}$, we see in this order that we have $sd \in A(T)$, $tf \in A(T)$, $se \in A(T)$ and $ct \in A(T)$. Now, as every vertex of $\{c, d, e, f\}$ is dominated by s or by t, we know that $T\langle\{c, d, e, f\}\rangle$ is acyclic, and then we have $ce \in A(T)$, $cf \in A(T)$ and $df \in A(T)$. So, we have $sf \in A(T)$, otherwise we could form the 3-cycle scf disjoint from the arc te. If $st \in A(T)$, then $T\langle\{s, c, d, t, e, f\}\rangle$ is acyclic and we conclude using Observation 2.2. Then, we have $ts \in A(T)$ and thus $bf \in A(T)$, otherwise we obtain the disjoint 3-cycles bcf, rax_2 and tsd. This means that fab is a 3-cycle of T and if there is an arc from r to c, d or e, then we form a 3-cycle with this arc and x_2 and complete a family of three disjoint cycles with one of tsd or tsc. Finally, $cr \in A(T)$, $dr \in A(T)$ and $er \in A(T)$ and $\{c, d, e\}$ dominates $\{r, f, x_2\}$. We conclude with Observation 2.2.

Case $s_1 = c$, $t_1 = f$. Following our notation, we denote this case by [Δ 1 . . 2]. We assume that we are not in one of the previous cases and then that $dt \in A(T)$ and $et \in A(T)$. As, $tf \in A(T)$, neither scd nor sde is a 3-cycle of T, and we have $sd \in A(T)$ and $se \in A(T)$. Now, every vertex of $\{c, d, e, f\}$ is dominated by s or t, so we know that $T\langle\{c, d, e, f\}\rangle$ is acyclic, and then we have $ce \in A(T)$, $cf \in A(T)$ and $df \in A(T)$. To conclude, observe that if $tc \in A(T)$, then tcd, sex_2, rab are disjoint 3-cycles so, we have $ct \in A(T)$ and as $\{c, d, e\}$ dominates $\{f, t, x_2\}$, we conclude using Observation 2.2.

Case $s_1 = d$, $t_1 = e$. This case is denoted by [Δ . 1 2]. As we assume that we are not in one of the previous cases, we have in particular that $cs \in A(T)$ and $ct \in A(T)$. If $ft \in A(T)$, then we form the cycles eft, sdx_2 and rab. So, we have $tf \in A(T)$. As $sd \in A(T)$ and rab is a 3-cycle, similarly, neither cte nor ctf is a 3-cycle of T, and we have $ce \in A(T)$ and $cf \in A(T)$. Now, c has in-degree at least 3 and so $ac, rc \in A(T)$. If $tb \in A(T)$, then the 3-cycle tbc and the arcs sd and ra form a configuration [Δ 1 . . 2] that we settled before. So, we have $bt \in A(T)$. Similarly, if $ta \in A(T)$, then tab, rc and sd are in position [Δ 1 2]. So, we have $at \in A(T)$. Now we see, in that order, that $se \in A(T)$ or sde, tfa, rcx_2 are disjoint 3-cycles; $rd \in A(T)$ or rcd, tfa, sex_2 are disjoint 3-cycles; $bs \in A(T)$ or sbc, tfa, rdx_2 are disjoint 3-cycles; and $sa \in A(T)$ or Observation 2.2 applied to $\{a, b, c\}$ and $\{s, t, x_2\}$ gives the desired contradiction. Now sab, rc, te is of type [Δ 1 . 2] which we have already handled.

Case $s_1 = d$, $t_1 = f$. This case is denoted by [Δ . 1 . 2]. As we assume that we are not in one of the previous cases, we have in particular that $cs \in A(T)$, $ct \in A(T)$ and $et \in A(T)$. If $es \in A(T)$, then we form the cycles sde, tfx_2 and rab. So, we have $se \in A(T)$ and then $dt \in A(T)$ otherwise rab,

td and se are in position [Δ . 1 2] treated before. Finally, as $se \in A(T)$ and rab is a 3-cycle, neither fdt nor fct is a 3-cycle of T, and we have $df \in A(T)$ and $cf \in A(T)$. But now, the set of vertices $\{c, d, e\}$ dominates the set $\{f, t, x_2\}$ and we conclude using Observation 2.2.

Case $s_1 = e$, $t_1 = f$. This case is the last one and is denoted by [Δ . . 1 2]. Assuming that we are not in a previous case, we know that dt, ds, ct and cs are arcs of T. Now, if $tb \in A(T)$, then bct, se and ra are in position [Δ . 1 . 2] which was done before. So, we have $bt \in A(T)$ and if $bs \in A(T)$, then $\{b, c, d\}$ would dominate $\{s, t, x_2\}$ and we could conclude with Observation 2.2. Thus, we have $sb \in A(T)$ and sbc, tf and ra are in position [Δ . . 1 2]. So, repeating the arguments, we will conclude that tcd, ra and sb are in position [Δ . . 1 2], contradiction because we have seen that $ct \in A(T)$.

3 Concluding remarks

In this paper, we were interested in tournaments with few disjoint cycles. Though it seems quite hard to compute the number of disjoint cycles in tournaments, some sufficient conditions are known to say that this number is large.

Using the straightforward generalization of Observation 2.2 is easy to see (and was remarked in [6]) that every k-strong tournament with at least $5k - 3$ vertices has k-disjoint 3-cycles: Let p be the maximum number of disjoint 3-cycles. If $p \leq k - 1$, then removing the vertices of p such cycles yields a transitive tournament of order at least $5k - 3 - 3(k - 1) = 2k$ and hence, there are two vertex-sets X and Y of size k such that X dominates Y. However, the fact that T is k-strong taken together with Observation 2.2 implies that T contains k vertex-disjoint 3-cycles including all the vertices of X and Y, which contradicts the original assumption $p \leq k - 1$.

Another situation possibly leading to the existence of k disjoint cycles is the case when the minimum degree of the digraph is large enough. Indeed, Bermond and Thomassen [5] conjectured that every digraph with minimum out-degree at least $2k - 1$ has k vertex-disjoint cycles. Let us mention that recently, with Thomassé, we proved that this conjecture holds for tournaments [2].

Finally, it is NP-complete to decide whether a given digraph has k disjoint cycles (k is part of the input) [3, Theorem 13.3.2]. We conjecture that this holds even for tournaments. Recall that finding a minimum cycle transversal is NP-complete by Theorem 1.3.

Conjecture 3.1 *It is NP-hard to find the maximum number of disjoint cycles in a given tournament.*

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