Disjoint 3-cycles in tournaments: a proof of the Bermond-Thomassen conjecture for tournaments*

Jørgen Bang-Jensen † Stéphane Bessy ‡ Stéphan Thomassé§

July 22, 2011

Abstract

We prove that every tournament with minimum out-degree at least $2k-1$ contains k disjoint 3-cycles. This provides additional support for the conjecture by Bermond and Thomassen that every digraph D of minimum out-degree $2k-1$ contains k vertex disjoint cycles. We also prove that for every $\epsilon > 0$, when k is large enough, every tournament with minimum out-degree at least $(1.5+\epsilon)k$ contains k disjoint cycles. The linear factor 1.5 is best possible as shown by the regular tournaments.

Keywords: Disjoint cycles, tournaments.

1 Introduction

Notation not given below is consistent with [3]. Paths and cycles are always directed unless otherwise specified. In a digraph $D = (V, A)$, a k-cycle is a cycle of length k, and for $k \geq 3$, we denote by $x_1x_2\dots x_k$ the **k-cycle** on $\{x_1, \dots, x_k\}$ with arc set $\{x_1x_2, x_2x_3, \dots, x_{k-1}x_k, x_kx_1\}$. The minimum length of a cycle in D is called the **girth** of D. The **underlying graph** of a digraph D, denoted $UG(D)$, is obtained from D by suppressing the orientation of each arc and deleting multiple edges. For a set $X \subseteq V$, we use the notation $D\langle X\rangle$ to denote the subdigraph of D **induced** by the vertices in X. For two disjoint sets X and Y of vertices of D, we say that X **dominates** Y if xy is an arc of D for every $x \in X$ and every $y \in Y$. In the digraph D, if X and Y are two disjoint subsets of vertices of D or subdigraphs of D, we say that there is a **k-matching** from X to Y if the set of arcs from X to Y contains a matching (in $UG(D)$) of size at least k. A **tournament** is an orientation of a complete graph, that is a digraph D such that for every pair $\{x, y\}$ of distinct vertices of D either $xy \in A(D)$ or $yx \in A(D)$, but not both. Finally, an **out-neighbour** (resp. **in-neighbour**) of a vertex x of D is a vertex y with $xy \in A(D)$ (resp. $yx \in A(D)$). The **out-degree** (resp. **in-degree**) $d^+_D(x)$ (resp. $d^-_D(x)$) of a vertex $x \in V$ is the number of out-neighbours (resp. in-neighbours) of x. We denote by $\delta^+(D)$ the minimum out-degree of a vertex in D.
The following conjecture, due to J.C. Bermond and C. Thomassen, gives a relationship between δ^+ and the maximum number of vertex disjoint cycles in a digraph.

Conjecture 1.1 (Bermond and Thomassen, 1981) *[4] If $\delta^+(D) \geq 2k-1$ then D contains k vertex disjoint cycles.*

Remark that the complete digraph (with all the possible arcs) shows that this statement is best possible. The conjecture is trivial for $k = 1$ and it has been verified for general digraphs when $k = 2$

*Part of this work was done while the first author was on sabbatical at AlGCo, LIRMM, Université Montpellier 2, France whose hospitality is gratefully acknowledged. Financial support from the Danish National Science research council (FNU) (under grant no. 09-066741) is gratefully acknowledged.

†Department of Mathematics and Computer Science, University of Southern Denmark, Odense DK-5230, Denmark (email: jbj@imada.sdu.dk).

‡AlGCo, LIRMM, Université Montpellier 2, France (email: bessy@lirmm.fr). Financial support: ANR GRATOS NANR-09-JCJC-0041-01

§Laboratoire LIP (U. Lyon, CNRS, ENS Lyon, INRIA, UCBL), Lyon, France (email: stephan.thomasse@ens-lyon.fr)

in [8] and for $k = 3$ in [7]. N. Alon proved in [1] that a lower bound of $64k$ on the minimum outdegree gives k disjoint cycles.

It was shown in [5] that every tournament with both minimum out-degree and minimum in-degree at least $2k-1$ has k disjoint cycles each of which have length 3. Very recently Lichiardopol [6] obtained a generalization of this result to the existence of k disjoint cycles of prescribed length q in a tournament with sufficiently high minimum degree.

In this paper we will prove Conjecture 1.1 for tournaments. Recall that by Moon's Theorem [3, Theorem 1.5.1], a tournament has k disjoint cycles if and only if it has k disjoint 3-cycles.

Theorem 1.2 *Every tournament T with $\delta^+(T) \geq 2k-1$ has k disjoint cycles each of which have length 3.*

We also show how to improve this result for tournaments with large minimum out-degree.

Theorem 1.3 *For every value $\alpha > 1.5$, there exists a constant k_α, such that for every $k \geq k_\alpha$, every tournament T with $\delta^+(T) \geq \alpha k$ has k disjoint 3-cycles.*

Remark that the constant 1.5 is best possible in the previous statement. Indeed, a family of sharp examples is provided by the rotative tournaments TR_{2p+1} on $2p+1$ vertices $\{x_1, \dots, x_{2p+1}\}$ with arc set $\{x_i x_j \ : \ j - i \mod 2p+1 \in \{1, \dots, p\}\}$. For $2p+1 = 0 \mod 3$, we denote $2p+1/3$ by k. Then, we have $\delta^+(TR_{2p+1}) = \lfloor 1.5k \rfloor$ and TR_{2p+1} admits a partition into k vertex disjoint 3-cycles and no more.

Theorem 1.3 does not give any result both for small values of k and for tournaments with $\delta^+ \geq 1.5k$, even asymptotically. We conjecture that we could still have k disjoint 3-cycles in these cases. Furthermore, in the light of the sharp examples to Conjecture 1.1 and Theorem 1.3, we extend these questions to digraphs with no short cycles. Namely, we conjecture the following.

Conjecture 1.4 *For every integer $g > 1$, every digraph D with girth at least g and with $\delta^+(D) \geq \frac{g}{g-1}k$ contains k disjoint cycles.*

Once again, the constant $\frac{g}{g-1}$ is best possible. Indeed, for every integers p and g, we define the digraph $D_{g,p}$ on $n = p(g-1)+1$ vertices with vertex set $\{x_1, \dots, x_n\}$ and arc set $\{x_i x_j \ : \ j - i \mod n \in \{1, \dots, p\}\}$. The digraph $D_{g,p}$ has girth g and we have $\delta^+(D_{g,p}) = p = \lfloor \frac{g}{g-1}k \rfloor$. Moreover, for $n = 0 \mod g$, the digraph $D_{g,p}$ admits a partition into k vertex disjoint 3-cycles and no more.

Even a proof of Conjecture 1.4 for large values of k or g (or both) would be of interest by itself. On the other hand, for $g = 3$, the first case of our conjecture which differs from Conjecture 1.1 and which is not already known corresponds to the following question: does every digraph D without 2-cycles and $\delta^+(D) \geq 6$ admit four vertex disjoint cycles?

In Section 2 and Section 3, we respectively prove Theorem 1.2 and Theorem 1.3. Before starting these, we precise notations that will be used in both next sections. Let T be a tournament and $\mathcal{F}$ a maximal collection of 3-cycles of T. The 3-cycles of $\mathcal{F}$ are denoted by $C_1, \dots, C_p$ and their ground set $V(C_1) \cup \dots \cup V(C_p)$ is denoted by W. The remaining part of T, $T \setminus W$ is denoted by U. By the choice of $\mathcal{F}$, U induces an acyclic tournament on T, and we denote its vertices by $\{u_p, u_{p-1}, \dots, u_2, u_1\}$, such that the arc $u_i u_j$ exists if and only if $i > j$.

2 Proof of Conjecture 1.1 for tournaments

In this section, we prove Conjecture 1.1 for tournaments. In fact, we strengthen a little bit the statement and prove the following:

Theorem 2.1 *For every tournament T with $\delta^+(T) \geq 2k-1$ and every collection $\mathcal{F} = \{C_1, \dots, C_{k-1}\}$ of $k-1$ disjoint 3-cycles of T, there exists a collection of k disjoint 3-cycles of T which intersects $T - V(C_1) \cup \dots \cup V(C_{k-1})$ on at most 4 vertices.*

This result implies Theorem 1.2. Indeed, for a tournament T with $\delta^+(T) \geq 2k_0 - 1$, we apply Theorem 2.1 k_0 times, with $k = 1$ to obtain a family $\mathcal{F}$ of one 3-cycle, and then with this family $\mathcal{F}$ and $k = 2$ to obtain a new family $\mathcal{F}$ of two 3-cycles, and so on.

To prove Theorem 2.1, we consider a counter-example T and a family $\mathcal{F}$ of $k-1$ disjoint 3-cycles with k minimum. The chosen family $\mathcal{F}$ is then maximal. So, from now on we use the notation stated in the first Section.

We will say that i 3-cycles of $\mathcal{F}$, with $i = 1$ or $i = 2$ can be **extended** if we can make $i+1$ 3-cycles using the vertices of the initial i 3-cycles and at most four vertices of U. If there one or two 3-cycles in $\mathcal{F}$ can be extended, we say that we could extend $\mathcal{F}$. If this happens, it would contradict the choice of T and $\mathcal{F}$. The following definition will be very useful in all this section. For an arc xy with $x, y \in W$, we say that a vertex z of U is a **breaker** of xy if xyz forms a 3-cycle. By extension, a vertex z of U is a **breaker** of a 3-cycle C_i of $\mathcal{F}$ if it is a breaker of one of the arcs of C_i.

The following claim is fundamental, and we will use it later several times without explicit mention.

Claim 1 *Every 3-cycle C of $\mathcal{F}$ has breakers for at most two of its three arcs, and every arc of C has at most three breakers. As a consequence, C has at most six breakers.*

Proof: Consider a 3-cycle $C_i = xyz$ of $\mathcal{F}$. Assume that C_i has a breaker for each of its arcs. We denote by v_e a breaker of the arc e, for $e \in \{xy, yz, zx\}$. If v_{yz} dominates v_{zx} then we form the 3-cycles xyv_{xy} and $zv_{yz}v_{zx}$, which intersect U on three vertices and we extend $\mathcal{F}$. So, by symmetry, we obtain that $v_{zx}v_{yz}v_{xy}$ forms a 3-cycle. This contradicts that $T\langle U\rangle$ is acyclic.
Now, if an arc xy of C_i has four breakers v_1, v_2, v_3, v_4 in U, then in $T \setminus \{x, y\}$ every vertex has out-degree at least $2(k-1)-1$, and $\mathcal{F} \setminus C_i$ forms a collection of $k-2$ 3-cycles. So, by the choice of T, there exists a collection $\mathcal{F}'$ of $k-1$ 3-cycles of $T \setminus \{x, y\}$ which intersect $U \cup z$ in at most four vertices. Then $\mathcal{F}'$ does not contain one of the vertices v_1, v_2, v_3, v_4, z. If $z \notin V(\mathcal{F}')$, we complete $\mathcal{F}'$ with the 3-cycle xyz, and obtain a collection of k 3-cycles which has the same intersection with U than $\mathcal{F}'$. If $z \in V(\mathcal{F}')$, then one of the v_i, say v_1 does not belong to $V(\mathcal{F}')$ and $\mathcal{F}'$ intersect U on at most three vertices. Then, we complete $\mathcal{F}'$ with the 3-cycle xyv_1, and obtain a collection of k 3-cycles which intersect U on at most four vertices. $\diamond$

Observe that if a 3-cycle xyz of $\mathcal{F}$ has a breaker for two of its arcs, then these breakers are disjoint. Indeed, if x' and y' are respectively breaker of xy and yz then yx' and $y'y$ are arcs of T. As T has no 2-cycle, x' and y' have to be distinct.

Informally, Claim 1 gives that every 3-cycle C of $\mathcal{F}$ can be extended or can be inserted in the transitive tournament $T\langle U\rangle$, that is, there exists a partition (U_2, U_1) of U such that there is no arc from U_1 to U_2, there is few arcs from U_1 to C and few from C to U_2 (otherwise, too roughly many breakers appear). This will be settled at Claim 2. The condition on the minimum out-degree of T will then allow one or two 3-cycles of $\mathcal{F}$ to be extended. Fixing precisely the computation will show, in the following subsection, that k cannot be too large ($k \leq 6$). Then, we treat the small cases in the last subsection.

2.1 A bound on k

For any partition (U_1, U_2) of U with no arc from U_1 to U_2, we have the following.

Claim 2 *For every 3-cycle $C = xyz$ of $\mathcal{F}$, we have:*

1. *If C receives at least four arcs from U_1 then there exists a 2-matching from U_1 to C.*

2. *If C receives at least eight arcs from U_1 then either there exists a 3-matching from U_1 to C or, up to permutation on x, y, z, yz has three breakers, xy has at least two breakers and x has in-degree at least five in U_1. Furthermore, x is dominated by U_2 and both y and z have each at most one out-neighbour in U_2.*

3. Consequently, if C receives at least eight arcs from U_1 then, there is no 2-matching from C to U_2 and, in particular, C sends at most three arcs to U_2.

Symmetrically, the same statements hold if we exchange the role of U_1 and U_2, and the bounds on in- and out-neighbours for every vertex.

Proof: 1. Assume that there is no 2-matching from U_1 to C then one vertex x of $U_1 \cup C$ belongs to all the arcs from U_1 to C. It is clear that $x \in C$. Hence if y is the successor of x in C, then four in-neighbours of x in U_1 form four breakers for the arc xy, which is not possible.

2. If there is no 3-matching from U_1 to C, then two vertices $\{x, y\}$ in $U_1 \cup C$ belongs to all arcs from U_1 to C. If $x \in U_1$ and $y \in C$, then there exists at least four in-neighbours of y different of x which form four breakers for the arc yz, where z is the successor of y in C, which is forbidden. As the case $\{x, y\} \subset U_1$ is not possible, we have $\{x, y\} \subset C$. Assume that x dominates y and call z the third vertex of C. If $d^-_{U_1}(y) \leq 2$, then $d^-_{U_1}(x) \geq 6$ and xy has four breakers, which is not possible. If $d^-_{U_1}(y) \geq 4$, yz has four breakers. So $d^-_{U_1}(y) = 3$ and $d^-_{U_1}(x) \geq 5$ which means that yz has three breakers and that x has at least two in-neighbours in U_1 which are not in-neighbours of y, and so, are breakers of xy. If x has an out-neighbour x' in U_2, we extend C using the 3-cycles $xx'x_1$ and yzy_1 where x_1 and y_1 are breakers of respectively xy and yz. So $U_2 \Rightarrow x$ must hold (that is, there is no arc from x to U_2). Now, if y has two out-neighbours in U_2, they form two more breakers for xy, and xy would have four breakers. Finally, if z has two out-neighbours in U_2, one of these is in-neighbour of y and would form a new breaker for yz, which had already three.

3. Assume that C receives at least eight arcs from U_1 and that there is a 2-matching from C to U_2. If there exists a 3-matching from U_1 to C, then we can extend C using at most four vertices of U. If not, then we are in the case described in the point 2, and C has at least five breakers in U_1, three for yz and at least two for xy. We can conclude except if the 2-matching from C to U_2 starts from y and z. We denote it by $\{yy', zz'\}$. If $z'y$ is an arc of T, then yz would have four breakers. Then yz' is an arc of T, but then, as U_2 dominates x, the vertices y' and z' would be two breakers of xy, which already has two. ◇

The two following claims are useful to extend two 3-cycles of $\mathcal{F}$ in order to form three new 3-cycles.

Claim 3 *There are no two 3-cycles C and C' of $\mathcal{F}$ with a 3-matching from U_1 to C, a 3-matching from C to C' and a 3-matching from C' to U_2.*

Proof: If this happens, we respectively denote these matchings by $\{x_1x, y_1y, z_1z\}$, $\{xx', yy', zz'\}$ and $\{x'x_2, y'y_2, z'z_2\}$, where $V(C) = \{x, y, z\}$, $V(C') = \{x', y', z'\}$, $x_1, y_1, z_1 \in U_1$ and $x_2, y_2, z_2 \in U_2$. If all three of $\{x_2x, y_2y, z_2z\}$ are arcs of T, then we can extend C and C' by x_2xx', y_2yy' and z_2zz'. So, we can assume that xx_2 is an arc of T. If one of the arcs yy_2 or zz_2 exists then, we can extend C. So, xx_2, y_2y and z_2z are arcs of T and we extend C and C' using the 3-cycles xx_2x_1, y_2yy' and z_2zz'. ◇

Claim 4 *There are no two 3-cycles C, C' such that $|E(U_1, C)| \geq 8$, $|E(C, C')| \geq 7$ and $|E(C', U_2)| \geq 8$.*

Proof: Assume that C and C' satisfy the hypothesis of the claim. We denote $V(C) = \{x, y, z\}$ and $V(C') = \{x', y', z'\}$. As $|E(C, C')| \geq 7$ there is a 3-matching between C and C'. By the Claim 3, one cannot both find a 3-matching from U_1 to C and a 3-matching from C' to U_2. By symmetry, two cases arise:

Case 1: there are no 3-matching from U_1 to C and from C' to U_2. We fix the orientations of C and C': $C = xyz$ and $C' = x'y'z'$. By Claim 2, up to permutation, we can assume that yz has three breakers in U_1 and xy at least two, and that $x'y'$ has three breakers in U_2 and $y'z'$ at least two. Furthermore we know, by Claim 2 that U_2 dominates x, z has at most one out-neighbour in U_2, z' dominates U_1 and x' has at most one in-neighbour in U_1. We denote then by x_1 a breaker of

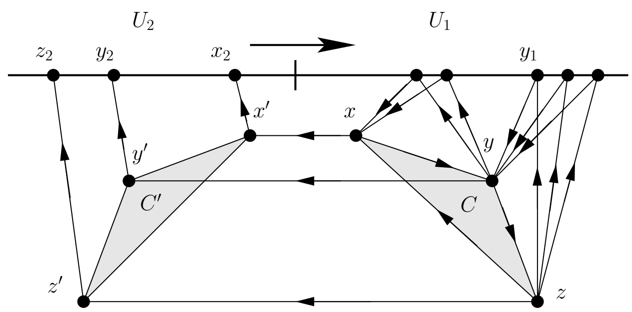

Figure 1: The case 2 of the proof of Claim 4

xy in U_1 which is an out-neighbour of x', and by z_2 a breaker of $y'z'$ in U_2 which is an in-neighbour of z. We denote also by y_2 and y_1 a breaker of respectively $x'y'$ and yz. Now, if xz' is an arc of T, then, we form the 3-cycles $xz'z_2$, y_1yz and $x'y'y_2$. If xx' is an arc of T, then, we form the 3-cycles $xx'x_1$, $y'z'z_2$ and yzy_1. And, if zz' is an arc of T, then, we form the 3-cycles $zz'z_2$, $x'y'y_2$ and xyx_1. As $|E(C, C')| \geq 7$, one of the three arcs xz', xx' and zz' exists and we can extend C and C'.

Case 2: there is no 3-matching from U_1 to C and there is a 3-matching from C' to U_2. We fix the orientation of C, $C = xyz$, but we do not fix the orientation of C'. We just assume that $\{xx', yy', zz'\}$ is a 3-matching between C and C'. We denote by $\{x'x_2, y'y_2, z'z_2\}$ a 3-matching from C' to U_2. By Claim 2, up to permutation, we can assume that yz has three breakers in U_1, we denote by y_1 one of them, and that xy has at least two. Furthermore we know, that $d^-_{U_1}(x) \geq 5$, that U_2 dominates x and that y and z have at most one out-neighbour in U_2. The situation is depicted in Figure 1.

To obtain a contradiction, we follow the next implications:

- zz_2 is an arc of T, otherwise we form the three circuits $zz'z_2$, $xx'x_2$ and $yy'y_2y_1$, which contain three 3-cycles intersecting U on at most four vertices.

- yz_2 is an arc of T, otherwise z_2 is a fourth breaker of yz.

- x_2 and y_2 dominate y and z. Indeed, the only out-neighbour of y and z in U_2 is z_2.

- $\{y', y_2, z, z'\}$ form an acyclic tournament. Indeed if $\{y', y_2, z, z'\}$ contains a circuit, we pick this circuit, $xx'x_2$ and yz_2y_1 to extend C and C'. In particular, the orientation of C' is $x'y'z'$ and $y'z \in A(T)$.

- xy' is an arc of T. Otherwise, $y'z$ and $y'x$ are the only arcs from C' to C and we form the 3-cycles $xz'z_2$, $zx'x_2$ and $yy'y_2$ to extend C and C'.

- $z'x_2$ is an arc of T. Otherwise, we form the 3-cycles $z'x'x_2$, $xy'y_2$ and yzy_1.

Finally, we extend C and C' using the 3-cycles $zz'x_2$, $xy'y_2$ and yz_2y_1. ◇

Now, we will show that $k \leq 6$. For this, we consider the partition (U_2, U_1) of U with $|U_1| = 5$ (as W contains $3k-3$ vertices, and T has at least $4k-1$ vertices, U contains at least $k+2$ vertices, and provided that $k \geq 3$, it is possible to consider such a U_1). So, we denote by $\mathcal{I}$ the set of 3-cycles which receive at least 8 arcs each from U_1 (the **in 3-cycles**), by $\mathcal{O}$ the set of 3-cycles which send at least 8 arcs each to U_2 (the **out 3-cycles**) and by $\mathcal{R}$ the remaining 3-cycles of $\mathcal{F} \setminus (\mathcal{I} \cup \mathcal{O})$. Furthermore, i, o and r respectively denote the size of $\mathcal{I}$, $\mathcal{O}$ and $\mathcal{R}$ (with $i + o + r = k - 1$ as $\mathcal{I} \cap \mathcal{O} = \emptyset$ by Claim 2). First, we bound below and above the number of arcs leaving U_1, and obtain:

$$5(2k-1) - 10 \leq 15i + 7(k-1-i-o) + 3o$$

In the right part, we bound the number of arcs from U_1 to $\mathcal{I}$, to $\mathcal{R}$ and to $\mathcal{O}$ (using Claim 2). Finally, we have:

$$3k + 4o - 8 \leq 8i \tag{1}$$

Now, we bound below and above the number of arcs leaving $\mathcal{F} \setminus \mathcal{O}$ and obtain

$$3(k-1-o)(2k-1) - \frac{1}{2}3(k-1-o)(3(k-1-o)-1) \leq 9ro + 6io + 7r + 3i + (15(i+r) - (10k - 15 - 3o))$$

In the right part, we bound the number of arcs from $\mathcal{R}$ to $\mathcal{O}$, from $\mathcal{I}$ to $\mathcal{O}$ (using Claim 4), from $\mathcal{R}$ to U_2, from $\mathcal{I}$ to U_2 (using Claim 2) and from $\mathcal{I} \cup \mathcal{R}$ to U_1. For the last bound, we know that at least $5(2k-1) - 10 = 10k - 15$ arcs leave U_1 and that at most $3o$ of these arcs go to $\mathcal{O}$. So, at least $10k - 15 - 3o$ arcs go from U_1 to $\mathcal{I} \cup \mathcal{R}$ on the $15(i+r)$ possible arcs between these two parts. Now, we replace r by $k - 1 - i - o$ and obtain:

$$9\,o^2 - 12\,k\,o + 6\,i\,o + 41\,o + 3\,k^2 - 21\,k + 8\,i + 8 \leq 0$$

We bound i from below using (1) to get (after adjusting to get integral coefficients):

$$16\,o^2 - 13\,k\,o + 52\,o + 4\,k^2 - 24\,k \leq 0$$

This inequality admits solution for o only if

$$(52 - 13k)^2 - 4 \cdot 16 \cdot (4\,k^2 - 24\,k) = -87k^2 + 184k + 2704$$

is positive, that is, if $k \leq 6$.

2.2 Small cases

Below we handle the cases $k \leq 6$. The partition (U_1, U_2) is no more fixed by $|U_1| = 5$, we will specify its size later.

2.2.1 Some remarks

We need some more general statements to solve the cases $k \leq 6$. For the following Claim 5 and Claim 6, symmetric statements hold if we exchange the roles of U_1 and U_2, and the bounds on in- and out-neighbours for every vertex.

Claim 5 *If $|E(U_1, C)| \geq 10$, then there exists a 3-matching from U_1 to C.*

Proof: Otherwise, two vertices, $\{x, y\}$, belong to all arcs from U_1 to C. As $\{x, y\} \subset U_1$ is not possible (otherwise only at most 6 arcs go from U_1 to C), either $x \in U_1$ and $y \in C$ or $\{x, y\} \subset C$. In the first case, y has at least seven in-neighbours in U_1 distinct of x, and if z is the out-neighbour of y in C, these seven vertices would be breakers of yz, contradicting Claim 1. So, we have $\{x, y\} \subset C$. We assume that x dominates y and that the orientation of C is $C = xyz$. Then y has at most three in-neighbours in U_1, otherwise yz would have four breakers, and x has at most three in-neighbours in U_1 which are not also in-neighbours of y, otherwise xy would have four breakers. But then there are at most nine arcs from U_1 to C, contradicting the hypothesis. ◇

As for Claim 2, it is possible to obtain the same result by exchanging U_1 and U_2 and the role of in- and out-neighbours for every vertex.

We say that a 3-cycle C has a **3-cover** from U_1 if there is a 3-matching from U_1 to C or two 2-matchings from U_1 to C which cover all the vertices of C.

Claim 6 *For every 3-cycle C of $\mathcal{F}$, if there is a 3-cover from U_1 to C, then there is no 2-matching from C to U_2. In particular, $|E(C, U_2)| \leq 3$.*

Proof: Assume that $C = xyz$ and that there is a 2-matching $\{zz', xx'\}$ from C to U_2 and a 3-cover from U_1 to C. If there is a 2-matching from U_1 to $\{z, x\}$, we are done. The remaining case occurs when the 3-cover from U_1 to C is formed by a 2-matching $\{ax, by\}$ to $\{x, y\}$ and a 2-matching $\{cy, dz\}$ to $\{y, z\}$ with $a = d$. In this case, we form the circuits axx' and $byzz'$, which contain two 3-cycles extending C. The bound on $|E(C, U_2)|$ follows from Claim 2. ◇

For a fixed U_1, we say that a 3-cycle C of $\mathcal{F}$ is of type **2-m**, **3-m** or **3-c** if there respectively is a 2-matching, a 3-matching or a 3-cover from U_1 to C. A 3-cover is useful to extend a 3-cycle, using Claim 6, but not very convenient in the general case, because the number of arcs that forces a 3-cover from U_1 to some 3-cycle C of $\mathcal{F}$ is the same than the number of arcs that forces a 3-matching (which is seven). However, to prove the existence of a 3-cover, we have the following statement.

Claim 7 *If there are three vertices a, b, c of U_1 such that $d^+_Y(a) \geq 2p$, $d^+_Y(b) \geq 2p - 1$ and $d^+_Y(c) \geq 2p - 2$, where Y is the set of vertices of a set of p 3-cycles $\mathcal{F}' \subset \mathcal{F}$, then $\mathcal{F}'$ contains a 3-c 3-cycle or all the 3-cycles of $\mathcal{F}'$ are 2-m.*

Proof: We prove it by induction on p. If $p = 1$ then there is a 2-matching from $\{a, b\}$ to the 3-cycle of $\mathcal{F}'$. Thus we may assume that $p \geq 2$. There is $6p - 3$ arcs from $\{a, b, c\}$ to the p 3-cycles of $\mathcal{F}'$. Thus there is a 3-cycle C of $\mathcal{F}'$ such that there are at least four arcs from $\{a, b, c\}$ to C and so there is a 2-matching from $\{a, b, c\}$ to C. If C is 3-c, we are done, otherwise each vertex of $\{a, b, c\}$ sends at most two arcs to C. We apply induction on $\mathcal{F}' \setminus C$. ◇

Now we are ready to prove the remaining cases ($k \leq 6$). As mentioned in the beginning of the paper, Conjecture 1.1 is known to hold for all digraphs when $k \leq 3$, so we only have to deal with the cases $k \in \{4, 5, 6\}$.

We will use several times, without referring explicitly, that a 3-cycle of type respectively 2-m and 3-c or 3-m sends respectively at most 7 and 3 arcs to U_2, by Claim 6 and 2. For each of the three cases below, we will use the three first vertices of U for U_1, that is, $U_1 = \{u_1, u_2, u_3\}$.

2.2.2 Case $k = 4$

For $k = 4$, we have $\delta^+(T) \geq 7$ and three 3-cycles in $\mathcal{F}$. There are:

- at least 21-3=18 arcs from U_1 to W and then at most 9 arcs from W to U_1.
- at least $9 \cdot 7 - \frac{1}{2} 9 \cdot 8 = 27$ arcs from W to U and then, at least 18 arcs from W to U_2.

So it is not possible to have types 3-c, 2-m and 2-m for the three 3-cycles of $\mathcal{F}$, otherwise, they send at most $3 + 7 + 7 = 17$ arcs to U_2. Now we prove that there are at least two 3-cycles of type 3-c. As u_1 sends seven arcs to W, one of the 3-cycle, say C_1 receives 3 arcs. If u_2 or u_3 sends one arc to C_1, then C_1 is of type 3-c, if not, then C_2 and C_3 are of type 3-c. So, at least one of the three 3-cycle is of type 3-c, we assume that it is C_1. Note that u_1, u_2 and u_3 send respectively at least 4,3 and 2 arcs to $C_2 \cup C_3$. Using Claim 7, we find a second 3-cycle which is of type 3-c. We assume that this second one is C_2. Now, we have:

- there is no 2-matching from U_1 to C_3, then C_3 receives at most 3 arcs from U_1, and then $C_1 \cup C_2$ receive at least 15 arcs from U_1, what means that there is a 3-matching from U_1 to C_1 for instance.
- $C_1 \cup C_2$ sends at least $6 \cdot 7 - \frac{1}{2} 6 \cdot 5 = 27$ arcs to $U \cup C_3$, at most 3 to U_1 and 6 to U_2, what means that there all the arcs from $C_1 \cup C_2$ to C_3
- C_3 sends at least $18 - 3 - 3 = 12$ arcs to U_2, then, by Claim 5, there is a 3-matching from C_3 to U_2.

Finally, using 3-matchings from U_1 to C_1, from C_1 to C_3 and from C_3 to U_2 and Claim 3, we can extend C_1, C_2 and C_3.

2.2.3 Case $k = 5$

For $k = 5$, we have $\delta^+(T) \geq 9$ and four 3-cycles in $\mathcal{F}$. There is:

- at least 24 arcs from U_1 to W and then at most 12 arcs from W to U_1.
- at least $12 \cdot 9 - \frac{1}{2} 12 \cdot 11 = 42$ arcs from W to U and then, at least 30 arcs from W to U_2.

So, it is not possible to have types 2-m, 2-m, 2-m and 2-m for the four 3-cycles of $\mathcal{F}$, otherwise, they send at most $7+7+7+7 = 28$ arcs to U_2. There are no three type 3-c among the four 3-cycles of $\mathcal{F}$. Otherwise, assume that C_1, C_2 and C_3 are of type 3-c, then, C_4 can not be of type 2-m, and there are at most 3 arcs from U_1 to C_4 and at least 21 arcs from U_1 to $C_1 \cup C_2 \cup C_3$. Then, $C_1 \cup C_2 \cup C_3$ sends at most 3 arcs to U_1, at most 9 arcs to U_2 and at most 27 arcs to C_4. However, there is at least $9 \cdot 9 - \frac{1}{2} 9 \cdot 8 = 45$ arcs going out of $C_1 \cup C_2 \cup C_3$, what gives a contradiction.
Using Claim 7 twice, we find two 3-cycles, C_1 and C_2 for instance, in $\mathcal{F}$ that are of type 3-c. Now, u_1, u_2 and u_3 respectively send at least 3, 2 and 1 arc to C_3 and C_4 and it is easy to find a 2-matching from U_1 to C_3 or C_4.
Now, we assume that C_1 and C_2 have a 3-cover from U_1 and that C_3 have a 2-matching from U_1. We obtain:

- C_4 receives at most three arcs from U_1 (otherwise C_4 would be a fourth 3-cycle of type 2-m).
- U_1 sends at least 21 arcs to $C_1 \cup C_2 \cup C_3$, then there is a 3-matching from U_1 to one of these 3-cycle, say C_1 and there is at most 6 arcs from $C_1 \cup C_2 \cup C_3$ to U_1.
- there is at most $3 + 3 + 7 = 13$ arcs from $C_1 \cup C_2 \cup C_3$ to U_2, and then as there is at least $9 \cdot 9 - \frac{1}{2} 9 \cdot 8 = 45$ arcs going out of $C_1 \cup C_2 \cup C_3$, there is $45 - 6 - 13 = 26$ arcs from $C_1 \cup C_2 \cup C_3$ to C_4. In particular, there is a 3-matching from C_1 to C_4.
- there are at most 13 arcs from $C_1 \cup C_2 \cup C_3$ to U_2, so, there are at least 17 arcs from C_4 to U_2 and then a 3-matching from C_4 to U_2.

Finally, we extend C_1 and C_4 using 3-matchings from U_1 to C_1, from C_1 to C_4 and from C_4 to U_2.

2.2.4 Case $k = 6$

For $k = 6$, we have $\delta^+(T) \geq 11$ and five 3-cycles in $\mathcal{F}$. There is:

- at least 30 arcs from U_1 to W and then at most 15 arcs from W to U_1.
- at least $15 \cdot 11 - \frac{1}{2} 15 \cdot 14 = 60$ arcs from W to U and then, at least 45 arcs from W to U_2.

Finding five 3-cycles of type 2-m in $\mathcal{F}$ is not possible then, because we would have at most $7 \cdot 5 = 35$ arcs from W to U_2. We will see that there are either at least three 3-cycles which are of type 3-c or there are two 3-cycles of type 3-c and two 3-cycles of type 2-m. Using Claim 7 twice, we find two 3-cycles which are of type 3-c, say C_1 and C_2. There remains at least 5, 4 and 3 arcs from respectively u_1, u_2 and u_3 to $C_3 \cup C_4 \cup C_5$. One of the 3-cycles C_3, C_4 or C_5, say C_3, receives at least 4 arcs from $\{u_1, u_2, u_3\}$ and then is of type 2-m. If C_3 is of type 3-c, we are done, otherwise, it receives at most 2 arcs from each of u_1, u_2, u_3, and u_1, u_2 and u_3 respectively send at least 3, 2 and 1 arcs to $C_4 \cup C_5$. We then find another 3-cycle of type 2-m.

First, we consider the case where there are two 3-cycles of type 3-c, C_1 and C_2 and two 3-cycles of type 2-m, C_3 and C_4. Then, we have:

- C_5 receives at most 3 arcs from U_1 (otherwise there is a fifth 3-cycle of type 2-m).
- U_1 sends at least 27 arcs to $C_1 \cup C_2 \cup C_3 \cup C_4$, thus there is at most 9 arcs from $C_1 \cup C_2 \cup C_3 \cup C_4$ to U_1.
- there are at most $3 + 3 + 7 + 7 = 20$ arcs from $C_1 \cup C_2 \cup C_3 \cup C_4$ to U_2.

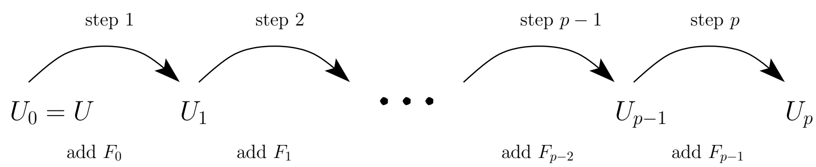

Figure 2: The p steps in the procedure to define free vertices.

- as there are at least $11 \cdot 12 - \frac{1}{2}12 \cdot 11 = 66$ arcs going out of $C_1 \cup C_2 \cup C_3 \cup C_4$, there are at least $66 - 9 - 20 = 37$ arcs from $C_1 \cup C_2 \cup C_3 \cup C_4$ to C_5, which is not possible.

Now, we treat the case where there are three 3-cycles of type 3-c in $\mathcal{F}$, C_1, C_2 and C_3. Then, we obtain:

- C_4 and C_5 receive each at most 3 arcs from U_1 (otherwise we are in one of the previous situations).
- U_1 sends at least 24 arcs to $C_1 \cup C_2 \cup C_3$. Thus there is a 3-matching from U_1 to two of these 3-cycles, say C_1 and C_2 and there are at most 3 arcs from $C_1 \cup C_2 \cup C_3$ to U_1.
- there are at most $3 + 3 + 3 = 9$ arcs from $C_1 \cup C_2 \cup C_3$ to U_2, and then as there are at least $9 \cdot 11 - \frac{1}{2}9 \cdot 8 = 63$ arcs going out of $C_1 \cup C_2 \cup C_3$, there are $63 - 3 - 9 = 51$ arcs from $C_1 \cup C_2 \cup C_3$ to $C_4 \cup C_5$. In particular, there is, a 3-matching from any of 3-cycle of $\{C_1, C_2, C_3\}$ to any of the 3-cycle of $\{C_4, C_5\}$, excepted possibly for one pair, say C_2 to C_4, to be in the worst case.
- there are at least 45-9=36 arcs from $C_4 \cup C_5$ to U_2, so, there are at least 18 arcs from one of the 3-cycle C_4 or C_5 to U_2, say from C_4, and then there is a 3-matching from C_4 to U_2.

Finally, we extend C_1 and C_4 using 3-matchings from U_1 to C_1, from C_1 to C_4 and from C_4 to U_2.

3 Proof of Theorem 1.3: An asymptotic better constant

In this part, we will asymptotically ameliorate the result of Theorem 1.2 by proving Theorem 1.3.

Let α be a real number with $\alpha > 1.5$, and T be a tournament with $\delta^+(T) \geq \alpha k$. We assume that $\alpha < 2$, otherwise Theorem 1.2 gives

We consider a family $\mathcal{F}$ of less than k disjoint 3-cycles in T. We will see that if k is great enough, then we can extend $\mathcal{F}$. As usual, we denote by W the set of vertices of all the 3-cycles of $\mathcal{F}$, and by U the other vertices that form an acyclic part (otherwise, we directly extend $\mathcal{F}$). As $\delta^+(T) \geq \alpha k$, remark that T has at least $2\alpha k$ vertices and then, as $|W| \leq 3k-3$, the size of U is at least $(2\alpha - 3)k$. The main idea of the proof is to obtain (almost) a partition of W into two parts X_1 and X_2 such that, as previously, X_1 receives many arcs from U and X_2 sends many arcs to U, with the requirement that the 3-cycles of $\mathcal{F}$ behave well with respect to the partition. The 3-cycles (or parts of the 3-cycles) of X_1 will act as in-3-cycles and the 3-cycles of X_2 as out-3-cycles. If we assume that $\mathcal{F}$ is maximum, a contradiction will result by computing the number of arcs leaving X_1.

We chose a positive real number ϵ such that $\epsilon < (\alpha - 1.5)/4$. This value corresponds to the room that we have to ignore some vertices, which we will do several times during the proof. Then we fix an integer p with $(3-\alpha)/p < \epsilon/3$, and we will repeat p times the procedure described below to define **free vertices**. We define three families of sets:

- $(F_i)_{0 \leq i \leq p-1}$ the free vertices produced at step i,
- $(U_i)_{0 \leq i \leq p}$ the free vertices produced since the beginning (they will form an acyclic part), and
- $(W_i)_{0 \leq i \leq p}$ the remaining vertices, see Figure 2.

We initialize by setting $U_0 = U$, and $W_0 = W$. For $0 \leq i \leq p-1$, a vertex x of W_i (resp. an arc xy of W_i) is **good at step** i if there exists at least 3^{p+1} disjoint pairs of vertices $\{y, z\}$ (resp. distinct vertices z) of U_i such that $\{x, y, z\}$ induces a 3-cycle. In other words, an element (vertex or arc) is good if it is contained in at least 3^{p+1} 3-cycles which are disjoint on U_i. When we find good elements, we will split the 3-cycles they are involved in into the good vertices (or vertices belonging to a good arc), that we will keep in W_{i+1}, and the others, called later free vertices and that we put with the transitive part U_{i+1}. For a 3-cycle C of $\mathcal{F}$, the vertices of C which we keep in W_{i+1} form the **remainder** of C. The remainder of C can contain one or two vertices. We use the name a **1-remainder** for a remainder of a 3-cycle with one vertex and a **2-remainder** for a remainder with 2 vertices.

Then, for $i = 0, \ldots, p-1$, we initialise $F_i = \emptyset$ and perform the step i of the procedure below, that is, we apply the first of the following rules as long as possible and then we consider the second rule, apply it as long as possible and proceed similarly for the third and fourth rule. When it is no more possible to apply the fourth rule, the step i is over, and we deal with the step $i+1$.

Rule 3.1 *If a 3-cycle or a 2-remainder C belonging to W_i contains a vertex x which is good at step i, then we add $V(C) \setminus \{x\}$ to F_i.*

Rule 3.2 *If C, C' and C'' are 3-cycles or 2-remainders belonging to W_i and $T < V(C) \cup V(C') \cup V(C'') >$ contains three disjoint arcs, say xy, $x'y'$ and $x''y''$, which are good at step i, then we add $V(C) \cup V(C') \cup V(C'') \setminus \{x, x', x'', y, y', y''\}$ to F_i.*

Rule 3.3 *If C and C' are 3-cycles or 2-remainders belonging to W_i and $T < V(C) \cup V(C') >$ contains two disjoint arcs, say xy and $x'y'$, which are good at step i, then we add $V(C) \cup V(C') \setminus \{x, x', y, y'\}$ to F_i.*

Rule 3.4 *If a 3-cycle C of $\mathcal{F}$ belonging to W_i contains a good arc xy at step i, then we add $V(C) \setminus \{x, y\}$ to F_i.*

Now, we fix the sets U_{i+1} to $U_i \cup F_i$ and W_{i+1} to $W_i \setminus F_i$. Furthermore, we call U_i the **free vertices at step** i. The next claim shows that these vertices are 'free to form a 3-cycle'.

Claim 8 *If the final set of free vertices, U_p, contains a 3-cycle, then, we can extend the family $\mathcal{F}$.*

Proof: Assume that U_p contains a 3-cycle xyz, we will build a family $\mathcal{F}'$ of 3-cycles with $|\mathcal{F}'| = |\mathcal{F}| + 1$. The family $\mathcal{F}'$ initially contains xyz and all the 3-cycles of $\mathcal{F}$ that still exist in W_p. We will inductively complete $\mathcal{F}'$ with 3-cycles formed from remainings of 3-cycles of $\mathcal{F}$ that are in W_p by going step by step backward from the step p to the initial configuration. A vertex of $U_p \setminus U_0 = \cup_{i=0}^{p-1} F_i$ is called **busy** if it is currently contained in a 3-cycle of $\mathcal{F}'$. At the end of step p, only x, y, z are possibly busy (and only if they do not belong to U_0), and, for $i = 1, \ldots, p$ we will prove the following (where stage i corresponds to the ith level of undoing the steps performed above, starting with stage 1 where we undo step p):

At stage i, every remainder created at step $p-i+1$ is contained in a 3-cycle of $\mathcal{F}'$ or in a 2-remainder previously created and U_{p-i} contains at most 3^{i+1} busy vertices. $(\star)$

Let us see what happens when $i = 1$. If, $\{x, y, z\} \cap F_{p-1} = \emptyset$, then using the vertices of F_{p-1} and the corresponding remainders we undo step $p-1$ to re-create original 3-cycles, which we add to $\mathcal{F}'$ or 2-remainders previously created (if Rule 3.1 has been used on a 2-remainder at step $p-1$). So, in this case, the only possible busy vertices of U_{p-1} are x, y and z and the property $(\star)$ holds for $i = 1$. Otherwise, consider a busy vertex in $\{x, y, z\}$ which is contained in F_{p-1}. It became free through the application of one of the Rules 3.1, 3.2, 3.3 or 3.4. In each of these cases, it has been separated from good elements (vertex or arc(s)), and these good elements can be re-completed into 3-cycles by adding at most three vertices (two for Rule 3.1, three for Rule 3.2, two for Rule 3.3 and one for Rule 3.4). Each of these good elements can be completed into at least 3^{p+1} disjoint (on U_{p-1}) 3-cycles. Hence, it is always possible to complete them disjointly with vertices of U_{p-1}. In the worst

case, 3 vertices were busy in the beginning (x, y and z) and each of the corresponding good element needs 3 vertices in U_{p-1} to be completed, producing 9 busy elements in U_{p-1}. Finally, the vertices of F_{p-1} that are not busy are used to re-create 3-cycles or 2-remainders destroyed at step p.
For $i = 2, \dots, p-1$, we apply exactly same arguments to pass from stage i to stage $i+1$, provided that at each stage i at most $3^{i+1} \leq 3^{p+1}$ busy vertices are present in U_{p-i}. For the last stage, that is to undo step 1, everything is similar, except that, by definition, U_0 contains no busy vertices and hence the corresponding vertices can be directly taken to form the last 3-cycles of $\mathcal{F}'$.
Finally, $\mathcal{F}'$ contains one 3-cycle for each remainder in W_p and xyz, so $|\mathcal{F}'| = |\mathcal{F}| + 1$. $\diamond$

An immediate consequence of Claim 8 is that the size of set W_p can not be less than $\alpha \cdot k$, because the first vertex of U_p has its out-neighbour-hood contained in W_p. So, the number of free vertices added to $U_0 = U$, that is $\cup_{i=0}^{p-1} F_i$, is at most $(3-\alpha)k$, and thus there is a step i_0+1 with $0 \leq i_0 \leq p-1$, with $|F_{i_0}| < (3-\alpha)k/p < \epsilon k/3$. We stop just before this step i_0+1, and denote by $\mathcal{R}$ the set of 3-cycles or 2-remainders with at least one vertex in F_{i_0}. So, the size of $R = V(\mathcal{R})$, is at most ϵk. We symbolically remove the small set R and go on working on the other 3-cycles and remainders. Remark that, now, in $W_{i_0} \setminus R$ there are no more free elements.

For any $q \leq p$, we say that a set of vertices (or abusively a sub-digraph) S of W_q is **insertable in U_q up to l vertices**, if there exists a partition of U_q into three sets Z_1, Z_2 and Z such that: there is no arc from Z_1 to Z_2, $|Z| \leq l$ and there is no arc from Z_1 to S and no arc from S to Z_2 .

Claim 9 *Every vertex $x \in W_{i_0} \setminus R$ belonging to a 3-cycle of $\mathcal{F}$ or a 2-remainder is insertable in U_{i_0} up to 3^{p+1} vertices. Furthermore, every 3-cycle of $\mathcal{F}$ contained in $W_{i_0} \setminus R$ is insertable in U_{i_0} up to $5 \cdot 3^{p+1}$ vertices.*

Proof: Consider C a 3-cycle of $\mathcal{F}$ or a 2-remainder which is contained in $W_{i_0} \setminus R$ and let x be a vertex of C. As U_{i_0} is an acyclic tournament by Claim 8, we denote by $\{u_1, u_2, \dots, u_r\}$ its vertices in such way that U_{i_0} contains no arc $u_i u_j$ with $i < j$. Among all the $r+1$ cuts of type $(Z_1 = \{u_1, \dots, u_i\}, Z_2 = \{u_{i+1}, \dots, u_r\})$, we choose one for which $d^+(x, Z_2) + d^+(Z_1, x)$ is minimum[1] and abusively denote it by (Z_1, Z_2) with $Z_1 = \{u_1, \dots, u_i\}$. If $d^+(Z_1, x) = l$ then it is possible to build l 3-cycles containing x and some vertices of Z_1 which are all disjoint on Z_1. Indeed, we denote by $(u_{in(j)})_{1 \leq j \leq l}$ (resp. $(u_{out(j)})_{1 \leq j \leq i-l}$) the in-neighbours of x in Z_1 (resp. the out-neighbours of x in Z_1) sorted according to the order $(u_i, u_{i-1}, \dots, u_1)$. Then, assume that for some j, $x u_{out(j)} u_{in(j)}$ is not a 3-cycle (because $u_{out(j)}$ is after $u_{in(j)}$, or because $u_{out(j)}$ does not exist), it means that x has more in-neighbours than out-neighbours in the set $\{u_i, u_{i-1}, \dots u_{in(j)}\}$, which contradicts the choice of the partition (Z_1, Z_2). So, it is possible to form all the 3-cycles $(x u_{out(j)} u_{in(j)})_{1 \leq j \leq l}$. Similarly, the same statement holds with Z_2, and globally it is possible to provide $d^+(x, Z_2) + d^+(Z_1, x)$ 3-cycles containing x and all disjoint on U_{i_0}. Then, as $x \in W_{i_0} \setminus R$ we have $d^+(x, Z_2) + d^+(Z_1, x) \leq 3^{p+1}$ and hence x is insertable in U_{i_0} up to 3^{p+1} vertices.

For the second part of the claim, consider a 3-cycle $C = xyz$ which is contained in $W_{i_0} \setminus R$. By the first part of the claim, we know that there exist three sets of vertices Z_x, Z_y and Z_z in U_{i_0} of size at most 3^{p+1} such that $(U_{i_0} \setminus \{Z_x \cup Z_y \cup Z_z\}) - \{xy, yz, zx\}$ forms an acyclic digraph. We consider an acyclic ordering of this digraph. If one of three arcs xy, yz or zx, say xy, is backward in this ordering and 'jumps' across more than 3^{p+1} vertices of U_{i_0}, then the arc xy is good and C should have been put in $\mathcal{R}$. So, as C can have one or two backward arcs with respect to this order, it is possible to remove from $U_{i_0} \setminus \{Z_x \cup Z_y \cup Z_z\}$ two further sets of vertices of size at most 3^{p+1} to insert C. $\diamond$

Now, using Claim 9, we give to every vertex x of a 2-remainder which is contained in $W_{i_0} \setminus R$ a position $p(x)$ in the ordering of U_{i_0}. More precisely, there exists a set Z of at most 3^{p+1} vertices of U_{i_0} such that there is no arc from $\{u_1, \dots, u_{p(x)}\} \setminus Z$ to x and no arc from x to $\{u_{p(x)+1}, \dots, u_r\} \setminus Z$. If there is several possibilities to choose $p(x)$, we pick one arbitrarily. Similarly, for a 3-cycle $C = xyz$ which lies in $W_{i_0} \setminus R$, we assign to each of its vertices a position $p(x) = p(y) = p(z)$, such that up to $5 \cdot 3^{p+1}$ vertices, C is insertable between $\{u_1, \dots, u_{p(x)}\}$ and $\{u_{p(x)+1}, \dots, u_r\}$.

[1]Here for two disjoint sets of vertices R, S $d^+(R, S)$ denotes the number of arcs from R to S.

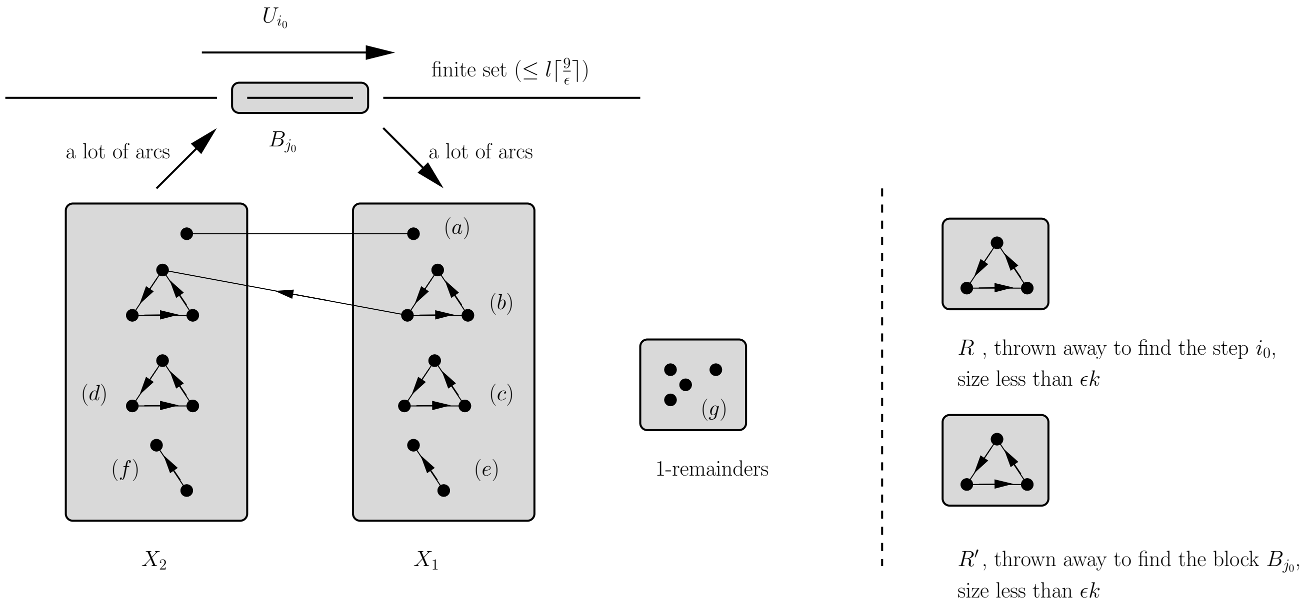

Figure 3: The situation in the proof of Theorem 1.3.

Then, we fix an integer l such that $(\alpha-1.5)l > 2\cdot 3^{p+2}$ and $l > 3\cdot 3^{p+1}$ and we consider a partition of the first vertices of U_{i_0} into $\lceil 9/\epsilon\rceil$ blocks of l vertices, provided that U_{i_0} is large enough. This is insured if U, of size at least $(2\alpha-3)k$, is large enough, that is if $(2\alpha-3)k > l\lceil 9/\epsilon\rceil$, what is possible as l and ϵ only depend on α. Exactly, for $j=1,\dots,\lceil 9/\epsilon\rceil$, the block B_j is $B_j=\{u_{(j-1)l+1},u_{(j-1)l+2},...,u_{jl}\}$. As $W_{i_0}\setminus R$ contains at most $3k$ vertices, there is at most $3k$ different values $p(x)$ for $x\in W_{i_0}\setminus R$. So, one of the $\lceil 9/\epsilon\rceil$ blocks B_j, say B_{j_0}, contains at most $3k/\lceil 9/\epsilon\rceil \leq \epsilon k/3$ values $p(x)$ for x being a vertex of a 2-remainder or of a 3-cycle of $W_{i_0}\setminus R$. We call $\mathcal{R}'$ these 2-remainders and 3-cycles of $W_{i_0}\setminus R$ and denote by R' the set $V(\mathcal{R}')$. Remark that R' has size at most ϵk.

So, we partition the remaining vertices of $W_{i_0}\setminus(R\cup R')$ into two parts: $X_1=\{x\in W_{i_0}: p(x)\leq (j_0-1)l\}$ and $X_2=\{x\in W_{i_0}: p(x)>j_0 l\}$. By the definition of p, a 3-cycle C of $\mathcal{F}$ which lies in $W_{i_0}\setminus(R\cup R')$ satisfies $V(C)\subseteq X_1$ or $V(C)\subseteq X_1$. Whereas the 2-remainders of $W_{i_0}\setminus(R\cup R')$ can intersect both parts of the partition (X_1,x_2) of $W_{i_0}\setminus(R\cup R')$. The situation is depicted in Figure 3.

We have the following property on the partition $((X_1,x_2)$ of $W_{i_0}\setminus(R\cup R')$.

Claim 10 *Every arc from X_1 to X_2 is a good arc.*

Proof: Let xy be an arc from X_1 to X_2. By definition, we know that $p(x)\leq (j_0-1)l$ and that $p(y)>j_0 l$. That means that all the vertices of B_{j_0} dominate x except for at most 3^{p+1} of them, and that all the vertices of B_{j_0} are dominated by y except for at most 3^{p+1} of them. As $|B_{j_0}|>3\cdot 3^{p+1}$, we can find 3^{p+1} vertices of B_{j_0} that are dominated by y and that dominate x, implying that xy is a good arc. $\diamond$

Now, according to their behaviour, we classify the 2-remainders and 3-cycles which are in $W_{i_0}\setminus(R\cup R')$:

- A 2-remainders which have one vertex in X_1 and the other in X_2 is of type (a).
- We consider a maximal collection of disjoint pairs of 3-cycles $\{C,C'\}$ where C is in X_1, C' is in X_2 and there is at least one arc from C to C'. All the 3-cycles involved in this collection are of type (b).
- A 3-remainder included in X_1 is of type (c) if it is not of type (b).
- A 3-remainder included in X_2 is of type (d) if it is not of type (b).

- A 2-remainder included in X_1 is of type (e).
- A 2-remainder included in X_2 is of type (f).

We abusively denote by a (resp. c, d, e and f) the number of remainders of type (a) (resp. (c), (d), (e) and (f)). We denote by b the number of pairs of 3-remainders of type (b). Finally, we denote by g the number of 1-remainders. At this point of the proof, any of this value is an integer between 0 and $k-1$ and X_1 or X_2 could be empty. This will be settled by the computation at the end of the proof. For the moment, we have the following properties.

Claim 11 *We have the following bounds on the number of arcs from X_1 to X_2:*

- *The number of arcs from X_1 to X_2 linking a 3-cycle with another 3-cycle or a 2-remainder is at most 3.*
- *The number of arcs going from X_1 to X_2 and linking a vertex of an element of type (a) and a vertex of an element of type (b) is at most $4ab$.*
- *There is no arc from a 3-cycle of type (c) to a 3-cycle of type (d).*

Proof: To prove the first point, consider C a 3-cycle of X_1 and C' a 3-cycle or a 2-remainder of X_2. If there are more than three arcs from C to C', then we find a 2-matching from C to C'. By Claim 10, this 2-matching is made of good arcs and the Rule 3.3 could apply to find a free vertex in C, contradicting that $W_{i_0} \setminus R$ has no free elements.

For the second point, consider a 2-remainder of type (a) on vertices v_1 and v_2 with $v_1 \in X_1$ and $v_2 \in X_2$ and a pair (C_1, C_2) of 3-cycles of type (b) with $C_1 = x_1y_1z_1$ contained in X_1 and $C_2 = x_2y_2z_2$ contained in X_2 and x_1x_2 being an arc of T. To find a contradiction, assume that the number of arcs from v_1 to C_2 plus the number of arcs from C_1 to v_2 is at least 5. It means that either v_1 dominates C_2 or C_1 dominates v_2, say that v_1 dominates C_2. Now, v_2 is dominated by at least two vertices of C_1, and one of these two is not x_1, say that it is y_1. But, the arcs x_1x_2, y_1v_2 and v_1y_2 are good by Claim 10, and z_1 and z_2 should be free by Rule 3.2, contradicting that $W_{i_0} \setminus R$ has no free elements.

The third point follows from the definition of 3-cycles of type (c) and (d). ◇

Now, we can derive a number of in-equalities from the structure derived so far, in order to obtain a contradiction, knowing that it has not been possible to increase the size of $\mathcal{F}$ above.

The first in-equality comes from the fact that there is a most $k-1$ remainders and 3-cycles in W_{i_0}.

$$a + 2b + c + d + e + f + g < k \tag{2}$$

For the second one, we compute the number of arcs going outside of B_{j_0}, which has size l. There are at least $\alpha kl - l(l-1)/2$ such arcs. The number of arcs from B_{j_0} to $\cup_{j=1}^{j_0-1} B_j$ is at most $l^2\lceil 9/\epsilon \rceil$. There is no arc from B_{j_0} to $U_{i_0} \setminus (\cup_{j=1}^{j_0} B_j)$. The number of arcs from B_{j_0} to X_2 is at most $|X_2|3^{p+1}$, because every vertex of X_2 is insertable into U_{i_0} before B_{j_0} up to 3^{p+1} vertices. As $X_2 \subseteq W$ we can bound this number by $3k3^{p+1}$. Finally, the remaining arcs going outside of B_{j_0} are at most $l(a + 3b + 3c + 2e + g + |R| + |R'|)$, and we obtain:

$$\alpha kl - \frac{l(l-1)}{2} \leq \lceil \frac{9}{\epsilon} \rceil l^2 + k3^{p+2} + l(a + 3b + 3c + 2e + g + 2\epsilon k,)$$

which we rewrite as

$$(\alpha - 1.5)kl - \frac{l(l-1)}{2} - \lceil \frac{9}{\epsilon} \rceil l^2 - k3^{p+2} - 2l\epsilon k \leq l(a + 3b + 3c + 2e + g - 1.5k)$$

And finally arrange in:

$$\left(\frac{(\alpha - 1.5)l}{2} - 3^{p+2} \right) k + \left((\frac{(\alpha - 1.5)}{2} - 2\epsilon)k - \frac{l-1}{2} - \lceil \frac{9}{\epsilon} \rceil l \right) l \leq l(a + 3b + 3c + 2e + g - 1.5k)$$

By choice of l, the first term is positive, and as $\epsilon < (\alpha - 1.5)/4$, if k is large enough, the second term is strictly positive too, implying that:

$$a + 3(b + c) + 2e + g > 1.5k \tag{3}$$

For the last in-equality, we compute the number of arcs going outside of X_1. As previously, we first show that, if k is large enough, we have $\alpha k|X_1| - d^+(X_1, U_{i_0}) - d^+(X_1, R \cup R') - 1.5k|X_1|$ is positive. Indeed, this term is greater than $(\alpha - 1.5)k|X_1| - \lceil \frac{9}{\epsilon} \rceil l|X_1| - 2\epsilon k|X_1|$ which is

$$|X_1|\left(((\alpha - 1.5) - 2\epsilon)k - \lceil \frac{9}{\epsilon} \rceil l\right)$$

and this is positive if k is large enough. Now, we have to take in account the arcs inside X_1 and those from X_1 to X_2 and to the 1-remainders. By the calculation above we still have at least $1.5k|X_1|$ arcs incident with vertices of X_1 to account for. Using Claim 11 we obtain

$$\begin{aligned} & 1.5k(a + 3(b + c) + 2e) - \frac{1}{2}(a + 3(b + c) + 2e)^2 < \\ & a(a + 4b + 3d + 2f + g) + b(3b + 3d + 3f + 3g) + c(3a + 3b + 3f + 3g) + e(2a + 3b + 3d + 4f + 2g) = \\ & a(a + 4b + 3c + 3d + 2e + 2f + g) + b(3b + 3c + 3d + 3e + 3f + 3g) + c(3f + 3g) + e(3d + 4f + 2g) \end{aligned} \tag{4}$$

Considering (2), (3) and (4), an equation solver leads to a contradiction. We just indicate how to manage the computation 'by hand'. Suppose that there exists a solution $X = (a, b, c, d, e, f, g)$ to these three in-equalities, we will show that then $X' = (a + b + f + g, 0, b + c + d + e, 0, 0, 0, 0)$ is also a solution to these equations. It is easy to check that X' is a solution to (2) and (3). For (4), we denote by $\phi(a, b, c, d, e, f, g)$ the value

$$2a(a + 4b + 3c + 3d + 2e + 2f + g) + 2b(3b + 3c + 3d + 3e + 3f + 3g) + 2c(3f + 3g) + 2e(3d + 4f + 2g)$$
$$+(a + 3(b + c) + 2e)^2 - 3k(a + 3(b + c) + 2e)$$

Then, we compute $\phi(a + b + f + g, 0, b + c + d + e, 0, 0, 0, 0) - \phi(a, b, c, d, e, f, g)$ and obtain:

$$2a(2b + 3d + 2e + f + 2g) + 3b(3b + 2c + 8d + 4e + 4f + 4g) + 6c(3d + e + f + g) + 3d(3d + 4e + 4f + 4g)$$
$$+e(5e + 4f + 8g+) + 3f(f + 2g) + 3g^2 - 3k(b + 3d + e + f + g)$$

Using the fact that X is a solution of (3), we have $-3k > -2(a + 3(b + c) + 2e + g)$ and so $\phi(a + b + f + g, 0, b + c + d + e, 0, 0, 0, 0) - \phi(a, b, c, d, e, f, g)$ is greater than

$$2a(b + e + g) + b(3b + 6d + 2e + 6f + 4g) + 3d(3d + 4f + 2g) + e(e + 2g) + f(3f + 4g) + g^2$$

which is positive. So, $\phi(a + b + f + g, 0, b + c + d + e, 0, 0, 0, 0)$ is strictly positive and X' is a solution of (4).
Now, there is a solution to the in-equations (2), (3) and (4) of type $(a', 0, c', 0, 0, 0, 0)$, what is impossible: (2) gives $a' + c' < 1$ and (4) gives $3(a' + 3c')(a' + c' - 1) > 0$.

This concludes the proof of Theorem 1.3. As a last remark, note that k_α is larger than a polynomial function in l, which is larger than an exponential in p, itself larger than a linear function in the inverse of $\alpha - 1.5$. So, k_α is an exponential function in the inverse of $\alpha - 1.5$.

4 Some Remarks

It is perhaps worth pointing out that the following obvious idea does not lead to a proof of Conjecture 1.1 for tournaments: find a 3-cycle C which is not dominated by any vertex of $V(T) - V(C)$, remove C and apply induction. This approach does not work because of the following[2].

Proposition 4.1 *For infinitely many $k \geq 3$ there exists a tournament T with $\delta^+(T) = 2k - 1$ such that every 3-cycle C is dominated by at least one vertex of minimum out-degree.*

Proof: Consider the quadratic residue tournament T on 11 vertices $V(T) = \{1, 2, \ldots, 11\}$ and arcs $A(T) = \{i \to i + p \mod 11 \ : \ i \in V(T), p \in \{1, 3, 4, 5, 9\}\}$. The possible types of 3-cycles in T are $i \to i+1 \to i+10 \to i, i \to i+1 \to i+6 \to i, i \to i+3 \to i+6 \to i, i \to i+3 \to i+7 \to i$, where indices are taken modulo 11. These are dominated by the vertices $i-3, i-3, i+2, i+2$ respectively. By substituting an arbitrary tournament for each vertex of T, we can obtain a tournament with the property that every 3-cycle is dominated by some vertex of minimum out-degree. $\diamond$

On the other hand, removing a 2-cycle from a digraph D with $\delta^+(D) \geq 2k - 1$ clearly results in a new digraph D' with $\delta^+(D') \geq 2(k-1) - 1$ and hence, when trying to prove Conjecture 1.1, we may always assume that the digraph in question has no 2-cycles. In particular the following follows directly from Theorem 1.2[3].

Corollary 4.2 *Every semicomplete digraph D with $\delta^+(D) \geq 2k - 1$ contains k disjoint cycles.* $\diamond$

A **chordal bipartite digraph** is a bipartite digraph with no induced cycle of length greater than 4. Note that in particular semicomplete bipartite digraphs [3, page 35] are chordal bipartite. It is easy to see that Conjecture 1.1 holds for chordal bipartite digraphs.

Proposition 4.3 *Every chordal bipartite digraph D with $\delta^+(D) \geq 2k - 1$ contains k disjoint cycles.*

Proof: This follows from the fact that such a digraph contains a directed cycle C of length 2 or 4 as long as $k \geq 1$. As D is bipartite, no vertex dominates more than half of the vertices on C and so we have $\delta^+(D - C) \geq 2(k-1) - 1$ and the result follows by induction on k. $\diamond$

An **extension** of a digraph $D = (V, A)$ is any digraph which can be obtained by substituting an independent set I_v for each vertex $v \in V$. More precisely we replace each vertex v of V by an independent set I_v and then add all arcs from I_u to I_v precisely if $uv \in A$.

Proposition 4.4 *Let $D = T[I_{n_1}, I_{n_2}, \ldots, I_{n_{|V(T)|}}]$ be an extension of a tournament T such that I_{n_i} is an independent set on n_i vertices for $i \in \{1, 2, \ldots, |V(T)|\}$. If $\delta^+(D) \geq 2k - 1$, then D contains k disjoint 3-cycles.*

Proof: Let T' be the tournament that we obtain from D by replacing each I_{n_i} by a transitive tournament on n_i vertices. Then $\delta^+(T') \geq 2k - 1$ and hence, by theorem 1.2, T' contains k disjoint 3-cycles $C_1, C_2, \ldots, C_k$. By the definition of an extension and the fact that we replaced independent sets by acyclic digraphs, no C_i can contain more than one vertex from any I_{n_i}, implying that $C_1, C_2, \ldots, C_k$ are also cycles in D. $\diamond$

References

[1] N. Alon. Disjoint directed cycles. *J. Combin. Theory Ser. B*, 68(2):167–178, 1996.

[2] See also [2, Section 9.1].

[3] A digraph is semicomplete if there is at least one arc between any pair of vertices.

[2] N. Alon and J.H. Spencer. *The Probabilistic Method*. John Wiley & Sons, New York, 2nd edition, 2000.

[3] J. Bang-Jensen and G. Gutin. *Digraphs: Theory, Algorithms and Applications, 2nd Edition*. Springer-Verlag, London, 2009.

[4] J.-C. Bermond and C. Thomassen. Cycles in digraphs—a survey. *J. Graph Theory*, 5(1):1–43, 1981.

[5] S. Bessy, N. Lichiardopol, and J.-S. Sereni. Two proofs of the bermond-thomassen conjecture for tournaments with bounded minimum in-degree. *Discrete Mathematics*, 310:557–560, 2010.

[6] N. Lichiardopol. Vertex-disjoint directed cycles of prescribed length in tournaments with given minimum out-degree and in-degree. *Discrete Mathematics*, 310:2567–2570, 2010.

[7] N. Lichiardopol, A. Pór, and J.-S. Sereni. A step towards the Bermond-Thomassen conjecture about disjoint cycles in digraphs. *Siam J. Discrete Math.*, 23:979–992, 2009.

[8] C. Thomassen. Disjoint cycles in digraphs. *Combinatorica*, 3(3-4):393–396, 1983.