Disjoint 3-cycles in tournaments: a proof of the Bermond-Thomassen conjecture for tournaments

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Abstract

We prove that every tournament with minimum out-degree at least $2k - 1$ contains $k$ disjoint 3-cycles. This provides additional support for the conjecture by Bermond and Thomassen that every digraph $D$ of minimum out-degree $2k - 1$ contains $k$ vertex disjoint cycles. We also prove that for every $\epsilon > 0$, when $k$ is large enough, every tournament with minimum out-degree at least $(1.5 + \epsilon)k$ contains $k$ disjoint cycles. The linear factor 1.5 is best possible as shown by the regular tournaments.

Keywords: Disjoint cycles, tournaments.

1 Introduction

Notation not given below is consistent with [3]. Paths and cycles are always directed unless otherwise specified. In a digraph $D = (V, A)$, a $k$-cycle is a cycle of length $k$, and for $k \geq 3$, we denote by $x_1x_2\ldots x_k$ the $k$-cycle on $\{x_1, \ldots, x_k\}$ with arc set $\{x_1x_2, x_2x_3, \ldots, x_{k-1}x_k, x_kx_1\}$. The minimum length of a cycle in $D$ is called the girth of $D$. The underlying graph of a digraph $D$, denoted $UG(D)$, is obtained from $D$ by suppressing the orientation of each arc and deleting multiple edges.

For a set $X \subseteq V$, we use the notation $D(X)$ to denote the subdigraph of $D$ induced by the vertices in $X$. For two disjoint sets $X$ and $Y$ of vertices of $D$, we say that $X$ dominates $Y$ if $xy$ is an arc of $D$ for every $x \in X$ and every $y \in Y$. In the digraph $D$, if $X$ and $Y$ are two disjoint subsets of vertices of $D$ or subdigraphs of $D$, we say that there is a $k$-matching from $X$ to $Y$ if the set of arcs from $X$ to $Y$ contains a matching (in $UG(D)$) of size at least $k$. A tournament is an orientation of a complete graph, that is a digraph $D$ such that for every pair $\{x, y\}$ of distinct vertices of $D$ either $xy \in A(D)$ or $yx \in A(D)$, but not both. Finally, an out-neighbour (resp. in-neighbour) of a vertex $x$ of $D$ is a vertex $y$ with $xy \in A(D)$ (resp. $yx \in A(D)$). The out-degree (resp. in-degree) $d^+_D(x)$ (resp. $d^-_D(x)$) of a vertex $x \in V$ is the number of out-neighbours (resp. in-neighbours) of $x$. We denote by $\delta^+(D)$ the minimum out-degree of a vertex in $D$.

The following conjecture, due to J.C. Bermond and C. Thomassen, gives a relationship between $\delta^+$ and the maximum number of vertex disjoint cycles in a digraph.

Conjecture 1.1 (Bermond and Thomassen, 1981) [4] If $\delta^+(D) \geq 2k - 1$ then $D$ contains $k$ vertex disjoint cycles.

Remark that the complete digraph (with all the possible arcs) shows that this statement is best possible. The conjecture is trivial for $k = 1$ and it has been verified for general digraphs when $k = 2$
in [8] and for $k = 3$ in [7]. N. Alon proved in [1] that a lower bound of $64k$ on the minimum outdegree gives $k$ disjoint cycles.

It was shown in [5] that every tournament with both minimum out-degree and minimum in-degree at least $2k - 1$ has $k$ disjoint cycles each of which have length 3. Very recently Lichiardopol [6] obtained a generalization of this result to the existence of $k$ disjoint cycles of prescribed length $q$ in a tournament with sufficiently high minimum degree.

In this paper we will prove Conjecture 1.1 for tournaments. Recall that by Moon’s Theorem [3, Theorem 1.5.1], a tournament has $k$ disjoint cycles if and only if it has $k$ disjoint 3-cycles.

**Theorem 1.2** Every tournament $T$ with $\delta^+(T) \geq 2k - 1$ has $k$ disjoint cycles each of which have length 3.

We also show how to improve this result for tournaments with large minimum out-degree.

**Theorem 1.3** For every value $\alpha > 1.5$, there exists a constant $k_\alpha$, such that for every $k \geq k_\alpha$, every tournament $T$ with $\delta^+(T) \geq \alpha k$ has $k$ disjoint 3-cycles.

Remark that the constant 1.5 is best possible in the previous statement. Indeed, a family of sharp examples is provided by the rotative tournaments $TR_{2p+1}$ on $2p+1$ vertices $\{x_1, \ldots, x_{2p+1}\}$ with arc set $\{x_ix_j : j - i \mod 2p+1 \in \{1, \ldots, p\}\}$. For $2p+1 = 0 \mod 3$, we denote $2p+1/3$ by $k$. Then, we have $\delta^+(TR_{2p+1}) = \lfloor 1.5k \rfloor$ and $TR_{2p+1}$ admits a partition into $k$ vertex disjoint 3-cycles and no more.

Theorem 1.3 does not give any result both for small values of $k$ and for tournaments with $\delta^+ \geq 1.5k$, even asymptotically. We conjecture that we could still have $k$ disjoint 3-cycles in these cases. Furthermore, in the light of the sharp examples to Conjecture 1.1 and Theorem 1.3, we extend these questions to digraphs with no short cycles. Namely, we conjecture the following.

**Conjecture 1.4** For every integer $g > 1$, every digraph $D$ with girth at least $g$ and with $\delta^+(D) \geq \frac{g}{g-1}k$ contains $k$ disjoint cycles.

Once again, the constant $\frac{g}{g-1}$ is best possible. Indeed, for every integers $p$ and $g$, we define the digraph $D_{g,p}$ on $n = p(g - 1) + 1$ vertices with vertex set $\{x_1, \ldots, x_n\}$ and arc set $\{x_ix_j : j - i \mod n \in \{1, \ldots, p\}\}$. The digraph $D_{g,p}$ has girth $g$ and we have $\delta^+(D_{g,p}) = p = \lfloor \frac{g}{g-1}k \rfloor$. Moreover, for $n = 0 \mod g$, the digraph $D_{g,p}$ admits a partition into $k$ vertex disjoint 3-cycles and no more.

Even a proof of Conjecture 1.4 for large values of $k$ or $g$ (or both) would be of interest by itself. On the other hand, for $g = 3$, the first case of our conjecture which differs from Conjecture 1.1 and which is not already known corresponds to the following question: does every digraph $D$ without 2-cycles and $\delta^+(D) \geq 6$ admit four vertex disjoint cycles?

In Section 2 and Section 3, we respectively prove Theorem 1.2 and Theorem 1.3. Before starting these, we precise notations that will be used in both next sections. Let $T$ be a tournament and $F$ a maximal collection of 3-cycles of $T$. The 3-cycles of $F$ are denoted by $C_1, \ldots, C_p$ and their ground set $V(C_1) \cup \cdots \cup V(C_p)$ is denoted by $W$. The remaining part of $T$, $T \setminus W$ is denoted by $U$. By the choice of $F$, $U$ induces an acyclic tournament on $T$, and we denote its vertices by $\{u_p, u_{p-1}, \ldots, u_2, u_1\}$, such that the arc $u_iu_j$ exists if and only if $i > j$.

## 2 Proof of Conjecture 1.1 for tournaments

In this section, we prove Conjecture 1.1 for tournaments. In fact, we strengthen a little bit the statement and prove the following:

**Theorem 2.1** For every tournament $T$ with $\delta^+(T) \geq 2k - 1$ and every collection $F = \{C_1, \ldots, C_{k-1}\}$ of $k - 1$ disjoint 3-cycles of $T$, there exists a collection of $k$ disjoint 3-cycles of $T$ which intersects $T - V(C_1) \cup \cdots \cup V(C_{k-1})$ on at most $\frac{k}{2}$ vertices.
This result implies Theorem 1.2. Indeed, for a tournament $T$ with $\delta^+(T) \geq 2k_0 - 1$, we apply Theorem 2.1 $k_0$ times, with $k = 1$ to obtain a family $F$ of one 3-cycle, and then with this family $F$ and $k = 2$ to obtain a new family $F$ of two 3-cycles, and so on.

To prove Theorem 2.1, we consider a counter-example $T$ and a family $F$ of $k - 1$ disjoint 3-cycles with $k$ minimum. The chosen family $F$ is then maximal. So, from now on we use the notation stated in the first Section.

We will say that $i$-3-cycles of $F$, with $i = 1$ or $i = 2$ can be extended if we can make $i + 1$ 3-cycles using the vertices of the initial $i$-3-cycles and at most four vertices of $U$. If there one or two 3-cycles in $F$ can be extended, we say that we could extend $F$. If this happens, it would contradict the choice of $T$ and $F$. The following definition will be very useful in all this section. For an arc $xy$ with $x, y \in W$, we say that a vertex $z$ of $U$ is a breaker of the arc $xy$ if $xyz$ forms a 3-cycle. By extension, a vertex $z$ of $U$ is a breaker of a 3-cycle $C_i$ of $F$ if it is a breaker of one of the arcs of $C_i$.

The following claim is fundamental, and we will use it later several times without explicit mention.

Claim 1 Every 3-cycle $C$ of $F$ has breakers for at most two of its three arcs, and every arc of $C$ has at most three breakers. As a consequence, $C$ has at most six breakers.

Proof: Consider a 3-cycle $C_1 = xyz$ of $F$. Assume that $C_1$ has a breaker for each of its arcs. We denote by $v_e$ a breaker of the arc $e$, for $e \in \{xy, yz, zx\}$. If $v_{yz}$ dominates $v_{xz}$ then we form the 3-cycles $xyv_{xy}$ and $yzv_{yz}v_{xy}$, which intersect $U$ on three vertices and we extend $F$. So, by symmetry, we obtain that $v_{xx}v_{yz}v_{xy}$ forms a 3-cycle. This contradicts that $T(U)$ is acyclic.

Now, if an arc $xy$ of $C_1$ has four breakers $v_1, v_2, v_3, v_4$ in $U$, then in $T \setminus \{x, y\}$ every vertex has out-degree at least $2(k - 1) - 1$, and $F \setminus C_1$ forms a collection of $k - 2$ 3-cycles. So, by the choice of $T$, there exists a collection $F'$ of $k - 1$ 3-cycles of $T \setminus \{x, y\}$ which intersect $U \cup z$ in at most four vertices. Then $F'$ does not contain one of the vertices $v_1, v_2, v_3, v_4, z$. If $z \notin V(F')$, we complete $F'$ with the 3-cycle $xyz$, and obtain a collection of $k$ 3-cycles which has the same intersection with $U$ than $F'$. If $z \in V(F')$, then one of the $v_i$, say $v_1$ does not belong to $V(F')$ and $F'$ intersect $U$ on at most three vertices. Then, we complete $F'$ with the 3-cycle $xyzv_1$, and obtain a collection of $k$ 3-cycles which intersect $U$ on at most four vertices.

Observe that if a 3-cycle $xyz$ of $F$ has a breaker for two of its arcs, then these breakers are disjoint. Indeed, if $x'$ and $y'$ are respectively breaker of $xy$ and $yz$ then $yx'$ and $y'y$ are arcs of $T$. As $T$ has no 2-cycle, $x'$ and $y'$ have to be distinct.

Informally, Claim 1 gives that every 3-cycle $C$ of $F$ can be extended or can be inserted in the transitive tournament $T(U)$, that is, there exists a partition $(U_2, U_1)$ of $U$ such that there is no arc from $U_1$ to $U_2$, there is few arcs from $U_1$ to $C$ and few from $C$ to $U_2$ (otherwise, too roughly many breakers appear). This will be settled at Claim 2. The condition on the minimum out-degree of $T$ will then allow one or two 3-cycles of $F$ to be extended. Fixing precisely the computation will show, in the following subsection, that $k$ cannot be too large ($k \leq 6$). Then, we treat the small cases in the last subsection.

2.1 A bound on $k$

For any partition $(U_1, U_2)$ of $U$ with no arc from $U_1$ to $U_2$, we have the following.

Claim 2 For every 3-cycle $C = xyz$ of $F$, we have:

1. If $C$ receives at least four arcs from $U_1$ then there exists a $2$-matching from $U_1$ to $C$.

2. If $C$ receives at least eight arcs from $U_1$ then either there exists a $3$-matching from $U_1$ to $C$ or, up to permutation on $x, y, z$, $yz$ has three breakers, $xy$ has at least two breakers and $x$ has in-degree at least five in $U_1$. Furthermore, $x$ is dominated by $U_2$ and both $y$ and $z$ have each at most one out-neighbour in $U_2$. 

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3. Consequently, if \( C \) receives at least eight arcs from \( U_1 \) then, there is no 2-matching from \( C \) to \( U_2 \) and, in particular, \( C \) sends at most three arcs to \( U_2 \).

Symmetrically, the same statements hold if we exchange the role of \( U_1 \) and \( U_2 \), and the bounds on in- and out-neighbours for every vertex.

**Proof:** 1. Assume that there is no 2-matching from \( U_1 \) to \( C \) then one vertex \( x \) of \( U_1 \cup C \) belongs to all the arcs from \( U_1 \) to \( C \). It is clear that \( x \in C \). Hence if \( y \) is the successor of \( x \) in \( C \), then four in-neighbours of \( x \) in \( U_1 \) form four breakers for the arc \( xy \), which is not possible.

2. If there is no 3-matching from \( U_1 \) to \( C \), then two vertices \( \{x, y\} \) in \( U_1 \cup C \) belongs to all arcs from \( U_1 \) to \( C \). If \( x \in U_1 \) and \( y \in C \), then there exists at least four in-neighbours of \( y \) different from \( x \) which form four breakers for the arc \( yz \), where \( z \) is the successor of \( y \) in \( C \), which is forbidden. As the case \( \{x, y\} \subseteq U_1 \) is not possible, we have \( \{x, y\} \subseteq C \). Assume that \( x \) dominates \( y \) and call \( z \) the third vertex of \( C \). If \( d_{U_1}^{-}(y) \leq 2 \), then \( d_{U_1}^{-}(x) \geq 6 \) and \( xy \) has four breakers, which is not possible. If \( d_{U_1}^{y}(y) \geq 4 \), \( yz \) has four breakers. So \( d_{U_1}^{y}(y) = 3 \) and \( d_{U_1}^{y}(x) \geq 5 \) which means that \( yz \) has three breakers and \( x \) has at least two in-neighbours in \( U_1 \) which are not in-neighbours of \( y \), and so, are breakers of \( xy \). If \( x \) has an out-neighbour \( x' \) in \( U_2 \), we extend \( C \) using the 3-cycles \( xx'x_1 \) and \( yy_1y \), where \( x_1 \) and \( y_1 \) are breakers of respectively \( xy \) and \( yz \). So \( U_2 \) must hold (that is, there is no arc from \( x \) to \( U_2 \)). Now, if \( y \) has two out-neighbours in \( U_2 \), they form two more breakers for \( xy \), and \( xy \) would have two breakers. Finally, if \( z \) has two out-neighbours in \( U_2 \), one of these is in-neighbour of \( y \) and would form a new breaker for \( yz \), which had already three.

3. Assume that \( C \) receives at least eight arcs from \( U_1 \) and that there is a 2-matching from \( C \) to \( U_2 \). If there exists a 3-matching from \( U_1 \) to \( C \), then we can extend \( C \) using at most four vertices of \( U \). If not, then we are in the case described in the point 2, and \( C \) has at least five breakers in \( U_1 \), three for \( yz \) and at least two for \( xy \). We can conclude except if the 2-matching from \( C \) to \( U_2 \) starts from \( y \) and \( z \). We denote it by \( \{yy', zz'\} \). If \( z'y \) is an arc of \( T \), then \( yz \) would have four breakers. Then \( yz' \) is an arc of \( T \), but then, as \( U_2 \) dominates \( x \), the vertices \( y' \) and \( z' \) would be two breakers of \( xy \), which already has two.

The two following claims are useful to extend two 3-cycles of \( F \) in order to form three new 3-cycles.

**Claim 3** There are no two 3-cycles \( C \) and \( C' \) of \( F \) with a 3-matching from \( U_1 \) to \( C \), a 3-matching from \( C \) to \( U_2 \) and a 3-matching from \( C' \) to \( U_2 \).

**Proof:** If this happens, we respectively denote these matchings by \( \{x_1x, y_1y, z_1z\} \), \( \{xx', yy', zz'\} \) and \( \{x'x_2, y'yy_2, z'z_2\} \), where \( V(C) = \{x, y, z\} \), \( V(C') = \{x', y', z'\} \), \( x_1, y_1, z_1 \in U_1 \) and \( x_2, y_2, z_2 \in U_2 \). If all three of \( xx_2, yy_2, zz_2 \) are arcs of \( T \), then we can extend \( C \) and \( C' \) by \( xx'x_1, yy'y_2 \) and \( zz'z_2 \). So, we can assume that \( xx_2 \) is an arc of \( T \). If one of the arcs \( yy_2 \) or \( zz_2 \) exists then, we can extend \( C \). So, \( xx_2, yy_2 \) and \( zz_2 \) are arcs of \( T \) and we extend \( C \) and \( C' \) using the 3-cycles \( xx_2x_1, yy'y_2 \) and \( zz'z_2 \).

**Claim 4** There are no two 3-cycles \( C, C' \) such that \( |E(U_1, C)| \geq 8 \), \( |E(C, C')| \geq 7 \) and \( |E(C', U_2)| \geq 8 \).

**Proof:** Assume that \( C \) and \( C' \) satisfy the hypothesis of the claim. We denote \( V(C) = \{x, y, z\} \) and \( V(C') = \{x', y', z'\} \). As \( |E(C, C')| \geq 7 \) there is a 3-matching between \( C \) and \( C' \). By the Claim 3, one cannot both find a 3-matching from \( U_1 \) to \( C \) and a 3-matching from \( C' \) to \( U_2 \). By symmetry, two cases arise:

**Case 1:** there are no 3-matching from \( U_1 \) to \( C \) and from \( C' \) to \( U_2 \). We fix the orientations of \( C \) and \( C' \). \( C = xyz \) and \( C' = x'y'z' \). By Claim 2, up to permutation, we can assume that \( yz \) has three breakers in \( U_1 \) and \( xy \) at least two, and that \( x'y' \) has three breakers in \( U_2 \) and \( y'z' \) at least two. Furthermore we know, by Claim 2 that \( U_2 \) dominates \( z \) has at most one out-neighbour in \( U_2 \), \( z' \) dominates \( U_1 \) and \( x' \) has at most one in-neighbour in \( U_1 \). We denote then by \( x_1 \) a breaker of
Claim 2, up to permutation, we can assume that \( k \) arcs each to \( U \) receive at least 8 arcs each from \( U \). In the right part, we bound the number of arcs leaving \( U \).

Case 2: there is no 3-matching from \( U_1 \) to \( C \) and there is a 3-matching from \( C' \) to \( U_2 \). We fix the orientation of \( C, \) \( C = x\overline{yz} \), but we do not fix the orientation of \( C' \). We just assume that \( \{xx', yy', zz'\} \) is a 3-matching between \( C \) and \( C' \). We denote by \( \{x'x_2, y'y_2, z'z_2\} \) a 3-matching from \( C' \) to \( U_2 \). By Claim 2, up to permutation, we can assume that \( yz \) has three breakers in \( U_1 \), we denote by \( y_1 \) one of them, and that \( x \) has at least two. Furthermore we know, that \( d_{U_1}(x) \geq 5 \), that \( U_2 \) dominates \( x \) and that \( y \) and \( z \) have at most one out-neighbour in \( U_2 \). The situation is depicted in Figure 1.

To obtain a contradiction, we follow the next implications:

- \( zz_2 \) is an arc of \( T \), otherwise we form the three circuits \( zz'_2, xx'_2 \) and \( yy'_2y_1 \), which contain three 3-cycles intersecting \( U \) on at most four vertices.
- \( yz_2 \) is an arc of \( T \), otherwise \( z_2 \) is a fourth breaker of \( yz \).
- \( xx \) and \( yz \) dominate \( y \) and \( z \). Indeed, the only out-neighbour of \( y \) and \( z \) in \( U_2 \) is \( z_2 \).
- \( \{y', y_2, z, z'\} \) form an acyclic tournament. Indeed if \( \{y', y_2, z, z'\} \) contains a circuit, we pick this circuit, \( xx'x_2 \) and \( y_2y_1 \) to extend \( C \) and \( C' \). In particular, the orientation of \( C' \) is \( x'y'z' \) and \( y'z \in A(T) \).
- \( xy' \) is an arc of \( T \). Otherwise, \( y'z \) and \( y'x \) are the only arcs from \( C' \) to \( U \) and we form the 3-cycles \( xx'_2, xx'_2 \) and \( yy'_2y_2 \) to extend \( C \) and \( C' \).
- \( z'x_2 \) is an arc of \( T \). Otherwise, we form the 3-cycles \( z'x'_2, xx'y_2 \) and \( yzy_1 \).

Finally, we extend \( C \) and \( C' \) using the 3-cycles \( zz'_2x_2, xy'_2y_2 \) and \( yzy_1 \).

Now, we will show that \( k \leq 6 \). For this, we consider the partition \( (U_2, U_1) \) of \( U \) with \( |U_1| = 5 \) (as \( W \) contains \( 3k - 3 \) vertices, and \( T \) has at least \( 4k - 1 \) vertices, \( U \) contains at least \( k + 2 \) vertices, and provided that \( k \geq 3 \), it is possible to consider such a \( U_1 \)). So, we denote by \( T \) the set of 3-cycles which receive at least 8 arcs each from \( U_1 \) (the in 3-cycles), by \( T \) the set of 3-cycles which send at least 8 arcs each to \( U_2 \) (the out 3-cycles) and by \( R \) the remaining 3-cycles of \( F \setminus (T \cup O) \). Furthermore, \( i, o \) and \( r \) respectively denote the size of \( I, O \) and \( R \) (with \( i + o + r = k - 1 \) as \( I \cap O = \emptyset \) by Claim 2).

First, we bound below and above the number of arcs leaving \( U_1 \), and obtain:

\[
5(2k - 1) - 10 \leq 15i + 7(k - 1 - i - o) + 3o
\]

In the right part, we bound the number of arcs from \( U_1 \) to \( I \), to \( R \) and to \( O \) (using Claim 2). Finally, we have:
\[3k + 4o - 8 \leq 8i\] (1)

Now, we bound below and above the number of arcs leaving \( F \setminus O \) and obtain

\[3(k - 1 - o)(2k - 1) - \frac{1}{2} 3(k - 1 - o)(3(k - 1 - o) - 1) \leq 9ro + 6io + 7r + 3i + (15(i + r) - (10k - 15 - 3o))\]

In the right part, we bound the number of arcs from \( R \) to \( O \), from \( I \) to \( O \) (using Claim 4), from \( R \) to \( U_2 \), from \( I \) to \( U_2 \) (using Claim 2) and from \( I \cup R \) to \( U_1 \). For the last bound, we know that at least 5\((2k - 1) - 10 = 10k - 15\) arcs leave \( U_1 \) and that at most 3\(o\) of these arcs go to \( O \). So, at least 10\(k - 15 - 3o\) arcs go from \( U_1 \) to \( I \cup R \) on the 15\((i + r)\) possible arcs between these two parts. Now, we replace \( r \) by \( k - 1 - i - o \) and obtain:

\[9o^2 - 12ko + 6io + 41o + 3k^2 - 21k + 8i + 8 \leq 0\]

We bound \( i \) from below using (1) to get (after adjusting to get integral coefficients):

\[16o^2 - 13ko + 52o + 4k^2 - 24k \leq 0\]

This inequality admits solution for \( o \) only if

\[(52 - 13k)^2 - 4 \cdot 16 \cdot (4k^2 - 24k) = -87k^2 + 184k + 2704\]

is positive, that is, if \( k \leq 6 \).

### 2.2 Small cases

Below we handle the cases \( k \leq 6 \). The partition \((U_1, U_2)\) is no more fixed by \(|U_1| = 5\), we will specify its size later.

#### 2.2.1 Some remarks

We need some more general statements to solve the cases \( k \leq 6 \). For the following Claim 5 and Claim 6, symmetric statements hold if we exchange the roles of \( U_1 \) and \( U_2 \), and the bounds on in- and out-neighbours for every vertex.

**Claim 5** If \( |E(U_1, C)| \geq 10 \), then there exists a 3-matching from \( U_1 \) to \( C \).

**Proof:** Otherwise, two vertices, \( \{x, y\} \), belong to all arcs from \( U_1 \) to \( C \). As \( \{x, y\} \subset U_1 \) is not possible (otherwise only at most 6 arcs go from \( U_1 \) to \( C \), either \( x \in U_1 \) and \( y \in C \) or \( \{x, y\} \subset C \). In the first case, \( y \) has at least seven in-neighbours in \( U_1 \) distinct of \( x \), and if \( z \) is the out-neighbour of \( y \) in \( C \), these seven vertices would be breakers of \( yz \), contradicting Claim 1. So, we have \( \{x, y\} \subset C \).

We assume that \( x \) dominates \( y \) and that the orientation of \( C \) is \( C = xyz \). Then \( y \) has at most three in-neighbours in \( U_1 \), otherwise \( yz \) would have four breakers, and \( x \) has at most three in-neighbours in \( U_1 \) which are not also in-neighbours of \( y \), otherwise \( xy \) would have four breakers. But then there are at most nine arcs from \( U_1 \) to \( C \), contradicting the hypothesis. \( \diamond \)

As for Claim 2, it is possible to obtain the same result by exchanging \( U_1 \) and \( U_2 \) and the role of in- and out-neighbours for every vertex.

We say that a 3-cycle \( C \) has a 3-cover from \( U_1 \) if there is a 3-matching from \( U_1 \) to \( C \) or two 2-matchings from \( U_1 \) to \( C \) which cover all the vertices of \( C \).

**Claim 6** For every 3-cycle \( C \) of \( F \), if there is a 3-cover from \( U_1 \) to \( C \), then there is no 2-matching from \( C \) to \( U_2 \). In particular, \( |E(C, U_2)| \leq 3 \).
Proof: Assume that $C = xyz$ and that there is a 2-matching $\{zz',xx'\}$ from $C$ to $U_2$ and a 3-cover from $U_1$ to $C$. If there is a 2-matching from $U_1$ to $\{z,x\}$, we are done. The remaining case occurs when the 3-cover from $U_1$ to $C$ is formed by a 2-matching $\{ax, by\}$ to $\{x, y\}$ and a 2-matching $\{cy, dz\}$ to $\{y, z\}$ with $a = d$. In this case, we form the circuits $axx'$ and $byzz'$, which contain two 3-cycles extending $C$. The bound on $|E(C, U_2)|$ follows from Claim 2.

For a fixed $U_1$, we say that a 3-cycle $C$ of $F$ is of type 2-m, 3-m or 3-c if there respectively is a 2-matching, a 3-matching or a 3-cover from $U_1$ to $C$. A 3-cover is useful to extend a 3-cycle, using Claim 6, but not very convenient in the general case, because the number of arcs that forces a 3-cover from $U_1$ to some 3-cycle $C$ of $F$ is the same than the number of arcs that forces a 3-matching (which is seven). However, to prove the existence of a 3-cover, we have the following statement.

Claim 7 If there are three vertices $a, b, c$ of $U_1$ such that $d^+_Y(a) \geq 2p$, $d^+_Y(b) \geq 2p - 1$ and $d^+_Y(c) \geq 2p - 2$, where $Y$ is the set of vertices of a set of $p$ 3-cycles $F' \subset F$, then $F'$ contains a 3-c 3-cycle or all the 3-cycles of $F'$ are 2-m.

Proof: We prove it by induction on $p$. If $p = 1$ then there is a 2-matching from $\{a, b\}$ to the 3-cycle of $F'$. Thus we may assume that $p \geq 2$. There is $6p - 3$ arcs from $\{a, b, c\}$ to the $p$ 3-cycles of $F'$. Thus there is a 3-cycle $C$ of $F'$ such that there are at least four arcs from $\{a, b, c\}$ to $C$ and so there is a 2-matching from $\{a, b, c\}$ to $C$. If $C$ is 3-c, we are done, otherwise each vertex of $\{a, b, c\}$ sends at most two arcs to $C$. We apply induction on $F' \setminus C$.

Now we are ready to prove the remaining cases ($k \leq 6$). As mentioned in the beginning of the paper, Conjecture 1.1 is known to hold for all digraphs when $k \leq 3$, so we only have to deal with the cases $k \in \{4, 5, 6\}$.

We will use several times, without referring explicitly, that a 3-cycle of type respectively 2-m and 3-c or 3-m sends respectively at most 7 and 3 arcs to $U_2$, by Claim 6 and 2. For each of the three cases below, we will use the three first vertices of $U$ for $U_1$, that is, $U_1 = \{u_1, u_2, u_3\}$.

2.2.2 Case $k = 4$

For $k = 4$, we have $\delta^+(T) \geq 7$ and three 3-cycles in $F$. There are:

- at least $21-3=18$ arcs from $U_1$ to $W$ and then at most 9 arcs from $W$ to $U_1$.

- at least $9 \cdot 7 - \frac{1}{2} \cdot 9 \cdot 8 = 27$ arcs from $W$ to $U$ and then, at least 18 arcs from $W$ to $U_2$.

So it is not possible to have types 3-c, 2-m and 2-m for the three 3-cycles of $F$, otherwise, they send at most $3 + 7 + 7 = 17$ arcs to $U_2$. Now we prove that there are at least two 3-cycles of type 3-c. As $u_1$ sends seven arcs to $W$, one of the 3-cycle, say $C_1$ receives 3 arcs. If $u_2$ or $u_3$ sends one arc to $C_1$, then $C_1$ is of type 3-c, if not, then $C_2$ and $C_3$ are of type 3-c. So, at least one of the three 3-cycle is of type 3-c, we assume that it is $C_1$. Note that $u_1, u_2$ and $u_3$ send respectively at least 4, 3 and 2 arcs to $C_2 \cup C_3$. Using Claim 7, we find a second 3-cycle which is of type 3-c. We assume that this second one is $C_2$. Now, we have:

- there is no 2-matching from $U_1$ to $C_3$, then $C_3$ receives at most 3 arcs from $U_1$, and then $C_1 \cup C_2$ receive at least 15 arcs from $U_1$, what means that there is a 3-matching from $U_1$ to $C_1$ for instance.

- $C_1 \cup C_2$ sends at least $6 \cdot 7 - \frac{1}{2} \cdot 6 \cdot 5 = 27$ arcs to $U \cup C_3$, at most 3 to $U_1$ and 6 to $U_2$, what means that there all the arcs from $C_1 \cup C_2$ to $C_3$

- $C_3$ sends at least $18 - 3 - 3 = 12$ arcs to $U_2$, then, by Claim 5, there is a 3-matching from $C_3$ to $U_2$.

Finally, using 3-matchings from $U_1$ to $C_1$, from $C_1$ to $C_3$ and from $C_3$ to $U_2$ and Claim 3, we can extend $C_1, C_2$ and $C_3$. 

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2.2.3 Case $k = 5$

For $k = 5$, we have $\delta^+(T) \geq 9$ and four 3-cycles in $\mathcal{F}$. There is:

- at least 24 arcs from $U_1$ to $W$ and then at most 12 arcs from $W$ to $U_1$.
- at least $12 \cdot 9 - \frac{1}{2} \cdot 12 \cdot 11 = 42$ arcs from $W$ to $U$ and then, at least 30 arcs from $W$ to $U_2$.

So, it is not possible to have types 2-m, 2-m and 2-m for the four 3-cycles of $\mathcal{F}$, otherwise, they send at most $7 + 7 + 7 + 7 = 28$ arcs to $U_2$. There are no three type 3-c among the four 3-cycles of $\mathcal{F}$. Otherwise, assume that $C_1$, $C_2$ and $C_3$ are of type 3-c, then, $C_4$ can not be of type 2-m, and there are at least 3 arcs from $U_1$ to $C_4$ and at most 21 arcs from $U_1$ to $C_1 \cup C_2 \cup C_3$. Then, $C_1 \cup C_2 \cup C_3$ sends at most 3 arcs to $U_1$, at most 9 arcs to $U_2$, and at most 27 arcs to $C_4$. However, there is at least $9 \cdot 9 - \frac{1}{2} \cdot 9 \cdot 8 = 45$ arcs going out of $C_1 \cup C_2 \cup C_3$, what gives a contradiction.

Using Claim 7 twice, we find two 3-cycles, $C_1$ and $C_2$ for instance, in $\mathcal{F}$ that are of type 3-c. Now, $u_1$, $u_2$ and $u_3$ respectively send at least 3, 2 and 1 arc to $C_3$ and $C_4$ and it is easy to find a 2-matching from $U_1$ to $C_3$ or $C_4$.

Now, we assume that $C_1$ and $C_2$ have a 3-cover from $U_1$ and that $C_3$ have a 2-matching from $U_1$. We obtain:

- $C_4$ receives at least 3 arcs from $U_1$ (otherwise $C_4$ would be a fourth 3-cycle of type 2-m).
- $U_1$ sends at least 21 arcs to $C_1 \cup C_2 \cup C_3$, then there is a 3-matching from $U_1$ to one of these 3-cycle, say $C_1$ and there is at most 6 arcs from $C_1 \cup C_2 \cup C_3$ to $U_1$.
- there is at least $3 + 3 + 7 = 13$ arcs from $C_1 \cup C_2 \cup C_3$ to $U_2$, and then as there is at least $9 \cdot 9 - \frac{1}{2} \cdot 9 \cdot 8 = 45$ arcs going out of $C_1 \cup C_2 \cup C_3$, there is $45 - 6 - 13 = 26$ arcs from $C_1 \cup C_2 \cup C_3$ to $C_4$. In particular, there is a 3-matching from $C_1$ to $C_4$.
- there are at most 13 arcs from $C_1 \cup C_2 \cup C_3$ to $U_2$, so, there are at least 17 arcs from $C_4$ to $U_2$ and then a 3-matching from $C_4$ to $U_2$.

Finally, we extend $C_1$ and $C_4$ using 3-matchings from $U_1$ to $C_1$, from $C_1$ to $C_4$ and from $C_4$ to $U_2$.

2.2.4 Case $k = 6$

For $k = 6$, we have $\delta^+(T) \geq 11$ and five 3-cycles in $\mathcal{F}$. There is:

- at least 30 arcs from $U_1$ to $W$ and then at most 15 arcs from $W$ to $U_1$.
- at least $15 \cdot 9 - \frac{1}{2} \cdot 15 \cdot 14 = 60$ arcs from $W$ to $U$ and then, at least 45 arcs from $W$ to $U_2$.

Finding five 3-cycles of type 2-m in $\mathcal{F}$ is not possible then, because we would have at most $7 \cdot 5 = 35$ arcs from $W$ to $U_2$. We will see that there are either at least three 3-cycles which are of type 3-c or there are two 3-cycles of type 3-c and two 3-cycles of type 2-m. Using Claim 7 twice, we find two 3-cycles which are of type 3-c, say $C_1$ and $C_2$. There remains at least 5, 4 and 3 arcs from respectively $u_1$, $u_2$ and $u_3$ to $C_3 \cup C_4 \cup C_5$. One of the 3-cycles $C_3$, $C_4$, or $C_5$, say $C_3$, receives at least 4 arcs from $\{u_1, u_2, u_3\}$ and then is of type 2-m. If $C_3$ is of type 3-c, we are done, otherwise, it receives at most 2 arcs from each of $u_1$, $u_2$, and $u_3$ respectively send at least 3, 2 and 1 arcs to $C_4 \cup C_5$. We then find another 3-cycle of type 2-m.

First, we consider the case where there are two 3-cycles of type 3-c, $C_1$ and $C_2$ and two 3-cycles of type 2-m, $C_3$ and $C_4$. Then, we have:

- $C_5$ receives at most 3 arcs from $U_1$ (otherwise there is a fifth 3-cycle of type 2-m).
- $U_1$ sends at least 27 arcs to $C_1 \cup C_2 \cup C_3 \cup C_4$, thus there is at most 9 arcs from $C_1 \cup C_2 \cup C_3 \cup C_4$ to $U_1$.
- there are at most $3 + 3 + 7 + 7 = 20$ arcs from $C_1 \cup C_2 \cup C_3 \cup C_4$ to $U_2$. 


Finally, we extend

Now, we treat the case where there are three 3-cycles of type 3-c in \( F \), \( C_1 \), \( C_2 \) and \( C_3 \). Then, we obtain:

- \( C_4 \) and \( C_5 \) receive each at most 3 arcs from \( U_1 \) (otherwise we are in one of the previous situations).
- \( U_1 \) sends at least 24 arcs to \( C_1 \cup C_2 \cup C_3 \). Thus there is a 3-matching from \( U_1 \) to two of these 3-cycles, say \( C_1 \) and \( C_2 \) and there are at most 3 arcs from \( C_1 \cup C_2 \cup C_3 \) to \( U_1 \).
- there are at most \( 3 + 3 + 3 = 9 \) arcs from \( C_1 \cup C_2 \cup C_3 \) to \( U_2 \), and then as there are at least \( 9 \cdot 11 - \frac{1}{2} \cdot 12 \cdot 11 = 66 \) arcs going out of \( C_1 \cup C_2 \cup C_3 \cup C_4 \), there are at most \( 66 - 9 - 20 = 37 \) arcs from \( C_1 \cup C_2 \cup C_3 \cup C_4 \) to \( C_5 \), which is not possible.

Finally, we extend \( C_1 \) and \( C_4 \) using 3-matchings from \( U_1 \) to \( C_1 \), from \( C_1 \) to \( C_4 \) and from \( C_4 \) to \( U_2 \).

3 Proof of Theorem 1.3: An asymptotic better constant

In this part, we will asymptotically ameliorate the result of Theorem 1.2 by proving Theorem 1.3.

Let \( \alpha \) be a real number with \( \alpha > 1.5 \), and \( T \) be a tournament with \( \delta^+(T) \geq \alpha k \). We assume that \( \alpha < 2 \), otherwise Theorem 1.2 gives

\[
\delta^+(T) \geq \alpha k
\]

We consider a family \( F \) of less than \( k \) disjoint 3-cycles in \( T \). We will see that if \( k \) is great enough, then we can extend \( F \). As usual, we denote by \( W \) the set of vertices of all the 3-cycles of \( F \), and by \( U \) the other vertices that form an acyclic part (otherwise, we directly extend \( F \)). As \( \delta^+(T) \geq \alpha k \), remark that \( T \) has at least \( 2\alpha k \) vertices and then, as \( |W| \leq 3k - 3 \), the size of \( U \) is at least \((2\alpha - 3)k\).

The main idea of the proof is to obtain (almost) a partition of \( W \) into two parts \( X_1 \) and \( X_2 \) such that, as previously, \( X_1 \) receives many arcs from \( U \) and \( X_2 \) sends many arcs to \( U \), with the requirement that the 3-cycles of \( F \) behave well with respect to the partition. The 3-cycles (or parts of the 3-cycles) of \( X_1 \) will act as in-3-cycles and the 3-cycles of \( X_2 \) as out-3-cycles. If we assume that \( F \) is maximum, a contradiction will result by computing the number of arcs leaving \( X_1 \).

We chose a positive real number \( \epsilon \) such that \( \epsilon < (\alpha - 1.5)/4 \). This value corresponds to the room that we have to ignore some vertices, which we will do several times during the proof. Then we fix an integer \( p \) with \( (3 - \alpha)/p < \epsilon/3 \), and we will repeat \( p \) times the procedure described below to define free vertices. We define three families of sets:

- \( (F_i)_{0 \leq i \leq p - 1} \) the free vertices produced at step \( i \),
- \( (U_i)_{0 \leq i \leq p} \) the free vertices produced since the beginning (they will form an acyclic part), and
- \( (W_i)_{0 \leq i \leq p} \) the remaining vertices, see Figure 2.
We initialize by setting $U_0 = U$, and $W_0 = W$. For $0 \leq i \leq p - 1$, a vertex $x$ of $W_i$ (resp. an arc $xy$ of $W_i$) is \textbf{good at step} $i$ if there exists at least $3^{i+1}$ disjoint pairs of vertices $\{y, z\}$ (resp. distinct vertices $z$) of $U_i$ such that $\{x, y, z\}$ induces a 3-cycle. In other words, an element (vertex or arc) is good if it is contained in at least $3^{i+1}$ 3-cycles which are disjoint on $U_i$. When we find good elements, we will split the 3-cycles they are involved in into the good vertices (or vertices belonging to a good arc), that we will keep in $W_{i+1}$, and the others, called later free vertices and that we put with the transitive part $U_{i+1}$. For a 3-cycle $C$ of $F$, the vertices of $C$ which we keep in $W_{i+1}$ form the \textbf{remainder} of $C$. The remainder of $C$ can contain one or two vertices. We use the name a \textbf{1-remainder} for a remainder of a 3-cycle with one vertex and a \textbf{2-remainder} for a remainder with 2 vertices.

Then, for $i = 0, \ldots, p - 1$, we initialise $F_i = \emptyset$ and perform the step $i$ of the procedure below, that is, we apply the first of the following rules as long as possible and then we consider the second rule, apply it as long as possible and proceed similarly for the third and fourth rule. When it is no more possible to apply the fourth rule, the step $i$ is over, and we deal with the step $i + 1$.

\textbf{Rule 3.1} If a 3-cycle or a 2-remainder $C$ belonging to $W_i$ contains a vertex $x$ which is good at step $i$, then we add $V(C) \setminus \{x\}$ to $F_i$.

\textbf{Rule 3.2} If $C$, $C'$ and $C''$ are 3-cycles or 2-remainders belonging to $W_i$ and $T < V(C) \cup V(C') \cup V(C'')$ contains three disjoint arcs, say $xy$, $x'y'$ and $x''y''$, which are good at step $i$, then we add $V(C) \cup V(C') \cup V(C'') \setminus \{x, x', x'', y, y', y''\}$ to $F_i$.

\textbf{Rule 3.3} If $C$ and $C'$ are 3-cycles or 2-remainders belonging to $W_i$ and $T < V(C) \cup V(C')$ contains two disjoint arcs, say $xy$ and $x'y'$, which are good at step $i$, then we add $V(C) \cup V(C') \setminus \{x, x', y, y'\}$ to $F_i$.

\textbf{Rule 3.4} If a 3-cycle $C$ of $F$ belonging to $W_i$ contains a good arc $xy$ at step $i$, then we add $V(C) \setminus \{x, y\}$ to $F_i$.

Now, we fix the sets $U_{i+1}$ to $U_i \cup F_i$ and $W_{i+1}$ to $W_i \setminus F_i$. Furthermore, we call $U_i$ the \textbf{free vertices at step} $i$. The next claim shows that these vertices are 'free to form a 3-cycle'.

\textbf{Claim 8} If the final set of free vertices, $U_p$, contains a 3-cycle, then, we can extend the family $F$.

\textbf{Proof:} Assume that $U_p$ contains a 3-cycle $xyz$, we will build a family $F'$ of 3-cycles with $|F'| = |F| + 1$. The family $F'$ initially contains $xyz$ and all the 3-cycles of $F$ that still exist in $W_p$. We will inductively complete $F'$ with 3-cycles formed from remainings of 3-cycles of $F$ that are in $W_p$ by going step by step backward from the step $p$ to the initial configuration. A vertex of $U_p \setminus U_0 = \bigcup_{i=0}^{p-1} F_i$ is called \textbf{busy} if it is currently contained in a 3-cycle of $F'$. At the end of step $p$, only $x, y, z$ are possibly busy (and only if they do not belong to $U_0$), and, for $i = 1, \ldots, p$ we will prove the following (where stage $i$ corresponds to the ith level of undoing the steps performed above, starting with stage 1 where we undo step $p$):

\textbf{*} At stage $i$, every remainder created at step $p - i + 1$ is created in a 3-cycle of $F'$ \hspace{1cm} (\star)
\textbf{or in a 2-remainder previously created and $U_{p-i}$ contains at most $3^{i+1}$ busy vertices.}

Let us see what happens when $i = 1$. If, $\{x, y, z\} \cap F_{p-1} = \emptyset$, then using the vertices of $F_{p-1}$ and the corresponding remainders we undo step $p - 1$ to re-create original 3-cycles, which we add to $F'$ or 2-remainers previously created (if Rule 3.1 has been used on a 2-remainder at step $p - 1$). So, in this case, the only possible busy vertices of $U_{p-1}$ are $x, y$ and $z$ and the property (\star) holds for $i = 1$. Otherwise, consider a busy vertex in $\{x, y, z\}$ which is contained in $F_{p-1}$. It became free through the application of one of the Rules 3.1, 3.2, 3.3 or 3.4. In each of these cases, it has been separated from good elements (vertex or arc(s)), and these good elements can be re-completed into 3-cycles by adding at most three vertices (two for Rule 3.1, three for Rule 3.2, two for Rule 3.3 and one for Rule 3.4). Each of these good elements can be completed into at least $3^{p+1}$ disjoint (on $U_{p-1}$) 3-cycles. Hence, it is always possible to complete them disjointly with vertices of $U_{p-1}$. In the worst
case, 3 vertices were busy in the beginning \((x, y \text{ and } z)\) and each of the corresponding good element needs 3 vertices in \(U_{p-1}\) to be completed, producing 9 busy elements in \(U_{p-1}\). Finally, the vertices of \(F_{p-1}\) that are not busy are used to re-create 3-cycles or 2-remainders destroyed at step \(p\).

For \(i = 2, \ldots, p - 1\), we apply exactly same arguments to pass from stage \(i\) to stage \(i + 1\), provided that at each stage \(i\) at most \(3^{i+1} \leq 3^{p+1}\) busy vertices are present in \(U_{p-1}\). For the last stage, that is to undo step 1, everything is similar, except that, by definition, \(U_0\) contains no busy vertices and hence the corresponding vertices can be directly taken to form the last 3-cycles of \(F'\).

Finally, \(F'\) contains one 3-cycle for each remainder in \(W_p\) and \(xyz\), so \(|F'| = |F| + 1\). \(\diamondsuit\)

An immediate consequence of Claim 8 is that the size of set \(W_p\) can not be less than \(\alpha \cdot k\), because the first vertex of \(U_p\) has its out-neighborhood contained in \(W_p\). So, the number of free vertices added to \(U_0 = U\), that is \(\bigcup_{i=0}^{p-1} F_i\), is at most \((3 - \alpha)k\), and thus there is a step \(i_0 + 1\) with \(0 \leq i_0 \leq p - 1\), with \(|F_{i_0}| < (3 - \alpha)k/p < \epsilon k/3\). We stop just before this step \(i_0 + 1\), and denote by \(R\) the set of 3-cycles or 2-remainders with at least one vertex in \(F_{i_0}\). So, the size of \(R = V(R)\), is at most \(\epsilon k\). We symbolically remove the small set \(R\) and go on working on the other 3-cycles and remainders. Remark that, now, in \(W_{i_0} \setminus R\) there are no more free elements.

For any \(q \leq p\), we say that a set of vertices (or abusively a sub-digraph) \(S\) of \(W_q\) is \textbf{insertable in} \(U_q\) \textbf{up to} \(l\) \textbf{vertices}, if there exists a partition of \(U_q\) into three sets \(Z_1, Z_2\) and \(Z\) such that: there is no arc from \(Z_1\) to \(Z_2\), \(|Z| \leq l\) and there is no arc from \(Z_1\) to \(S\) and no arc from \(S\) to \(Z_2\).

**Claim 9** Every vertex \(x \in W_{i_0} \setminus R\) belonging to a 3-cycle of \(F\) or a 2-remainder is insertable in \(U_{i_0}\) up to \(3^{p+1}\) vertices. Furthermore, every 3-cycle of \(F\) contained in \(W_{i_0} \setminus R\) is insertable in \(U_{i_0}\) up to \(5 \cdot 3^{p+1}\) vertices.

**Proof:** Consider \(C\) a 3-cycle of \(F\) or a 2-remainder which is contained in \(W_{i_0} \setminus R\) and let \(x\) be a vertex of \(C\). As \(U_{i_0}\) is an acyclic tournament by Claim 8, we denote by \(\{u_1, u_2, \ldots, u_r\}\) its vertices in such way that \(U_{i_0}\) contains no arc \(u_iu_j\) with \(i < j\). Among all the \(r + 1\) cuts of type \(Z_1 = \{u_1, \ldots, u_i\}, Z_2 = \{u_{i+1}, \ldots, u_r\}\), we choose one for which \(d^+(x, Z_2) + d^+(Z_1, x)\) is minimum\(^1\) and abusively denote it by \((Z_1, Z_2)\) with \(Z_1 = \{u_1, \ldots, u_i\}\). If \(d^+(Z_1, x) = l\) then it is possible to build \(l\) 3-cycles containing \(x\) and some vertices of \(Z_1\) which are all disjoint on \(Z_1\). Indeed, we denote by \((u_{in(j)})_{1 \leq j \leq l}\) (resp. \((u_{out(j)})_{1 \leq j \leq l-1}\) the in-neighbours of \(x\) in \(Z_1\) (resp. the out-neighbours of \(x\) in \(Z_1\)) sorted according to the order \((u_1, u_1-1, \ldots, u_1)\). Then, assume that for some \(j\), \(xu_{out(j)}u_{in(j)}\) is not a 3-cycle (because \(u_{out(j)}\) is after \(u_{in(j)}\), or because \(u_{out(j)}\) does not exist), that means that \(x\) has more in-neighbours than out-neighbours in the set \(\{u_1, u_1-1, \ldots, u_{in(j)}\}\), which contradicts the choice of the partition \((Z_1, Z_2)\). So, it is possible to form all the \(3\) cycles \((xu_{out(j)}u_{in(j)})_{1 \leq j \leq l}\). Similarly, the same statement holds with \(Z_2\) and globally it is possible to provide \(d^+(x, Z_2) + d^+(Z_1, x)\) 3-cycles containing \(x\) and all disjoint on \(U_{i_0}\). Then, as \(x \in W_{i_0} \setminus R\) we have \(d^+(x, Z_2) + d^+(Z_1, x) \leq 3^{p+1}\) and hence \(x\) is insertable in \(U_{i_0}\) up to \(3^{p+1}\) vertices.

For the second part of the claim, consider a 3-cycle \(C = xyz\) which is contained in \(W_{i_0} \setminus R\). By the first part of the claim, we know that there exist three sets of vertices \(Z_x, Z_y, Z_z\) in \(U_{i_0}\) of size at most \(3^{p+1}\) such that \((U_{i_0} \setminus \{Z_x \cup Z_y \cup Z_z\}) \setminus \{xy, yz, zx\}\) forms an acyclic digraph. We consider an acyclic ordering of this digraph. If one of three arcs \(xy, yz\) or \(zx\), say \(xy\), is backward in this ordering and 'jumps' across more than \(3^{p+1}\) vertices of \(U_{i_0}\), then the arc \(xy\) is good and \(C\) should have been put in \(R\). So, as \(C\) can have one or two backward arcs with respect to this order, it is possible to remove from \(U_{i_0} \setminus \{Z_x \cup Z_y \cup Z_z\}\) two further sets of vertices of size at most \(3^{p+1}\) to insert \(C\). \(\diamondsuit\)

Now, using Claim 9, we give to every vertex \(x\) of a 2-remainder which is contained in \(W_{i_0} \setminus R\) a position \(p(x)\) in the ordering of \(U_{i_0}\). More precisely, there exists a set \(Z\) of at most \(3^{p+1}\) vertices of \(U_{i_0}\) such that there is no arc from \(\{u_1, \ldots, u_{p(x)}\} \setminus Z\) to \(x\) and no arc from \(x\) to \(\{u_{p(x)+1}, \ldots, u_r\} \setminus Z\).

If there is several possibilities to choose \(p(x)\), we pick one arbitrarily. Similarly, for a 3-cycle \(C = xyz\) which lies in \(W_{i_0} \setminus R\), we assign to each of its vertices a position \(p(x) = p(y) = p(z)\), such that up to \(5 \cdot 3^{p+1}\) vertices, \(C\) is insertable between \(\{u_1, \ldots, u_{p(x)}\}\) and \(\{u_{p(x)+1}, \ldots, u_r\}\).

\(^1\)Here for two disjoint sets of vertices \(R, S\) \(d^+(R, S)\) denotes the number of arcs from \(R\) to \(S\).
Then, we fix an integer $l$ such that $(\alpha - 1.5)l > 2 \cdot 3^{p+2}$ and $l > 3 \cdot 3^{p+1}$ and we consider a partition of the first vertices of $U_{i_0}$ into $\lfloor 9/\epsilon \rfloor$ blocks of $l$ vertices, provided that $U_{i_0}$ is large enough. This is insured if $U$, of size at least $(2\alpha - 3)k$, is large enough, that is if $(2\alpha - 3)k > l\lfloor 9/\epsilon \rfloor$, what is possible as $l$ and $\epsilon$ only depend on $\alpha$. Exactly, for $j = 1, \ldots, \lfloor 9/\epsilon \rfloor$, the block $B_j$ is $B_j = \{u_{(j-1)l+1}, u_{(j-1)l+2}, \ldots, u_{jl}\}$. As $W_{i_0} \setminus R$ contains at most $3k$ vertices, there is at most $3k$ different values $p(x)$ for $x \in W_{i_0} \setminus R$. So, one of the $\lfloor 9/\epsilon \rfloor$ blocks $B_j$, say $B_{j_0}$, contains at most $3k/\lfloor 9/\epsilon \rfloor \leq \epsilon k$ values $p(x)$ for $x$ being a vertex of a 2-remainder or of a 3-cycle of $W_{i_0} \setminus R$. We call $R'$ these 2-remainers and 3-cycles of $W_{i_0} \setminus R$ and denote by $R'$ the set $V(R')$. Remark that $R'$ has size at most $\epsilon k$.

So, we partition the remaining vertices of $W_{i_0} \setminus (R \cup R')$ into two parts: $X_1 = \{x \in W_{i_0} : p(x) \leq (j_0 - 1)l\}$ and $X_2 = \{x \in W_{i_0} : p(x) > j_0l\}$. By the definition of $p$, a 3-cycle $C$ of $F$ which lies in $W_{i_0} \setminus (R \cup R')$ satisfies $V(C) \subseteq X_1$ or $V(C) \subseteq X_1$. Whereas the 2-remainers of $W_{i_0} \setminus (R \cup R')$ can intersect both parts of the partition $(X_1, x_2)$ of $W_{i_0} \setminus (R \cup R')$. The situation is depicted in Figure 3.

We have the following property on the partition $(X_1, x_2)$ of $W_{i_0} \setminus (R \cup R')$.

**Claim 10** Every arc from $X_1$ to $X_2$ is a good arc.

**Proof:** Let $xy$ be an arc from $X_1$ to $X_2$. By definition, we know that $p(x) \leq (j_0 - 1)l$ and that $p(y) > j_0l$. That means that all the vertices of $B_{j_0}$ dominate $x$ except for at most $3^{p+1}$ of them, and that all the vertices of $B_{j_0}$ are dominated by $y$ except for at most $3^{p+1}$ of them. As $|B_{j_0}| > 3 \cdot 3^{p+1}$, we can find $3^{p+1}$ vertices of $B_{j_0}$ that are dominated by $y$ and that dominate $x$, implying that $xy$ is a good arc. \hfill \Box

Now, according to their behaviour, we classify the 2-remainers and 3-cycles which are in $W_{i_0} \setminus (R \cup R')$:

- A 2-remainder which have one vertex in $X_1$ and the other in $X_2$ is of type $(a)$.
- We consider a maximal collection of disjoint pairs of 3-cycles $(C, C')$ where $C$ is in $X_1$, $C'$ is in $X_2$ and there is at least one arc from $C$ to $C'$. All the 3-cycles involved in this collection are of type $(b)$.
- A 3-remainder included in $X_1$ is of type $(c)$ if it is not of type $(b)$.
- A 3-remainder included in $X_2$ is of type $(d)$ if it is not of type $(b)$.

Figure 3: The situation in the proof of Theorem 1.3.
• A 2-remainder included in \( X_1 \) is of type \((e)\).
• A 2-remainder included in \( X_2 \) is of type \((f)\).

We abusively denote by \( a \) (resp. \( c, d, e \) and \( f \)) the number of remainders of type \((a)\) (resp. \((c), (d), (e)\) and \((f)\)). We denote by \( b \) the number of pairs of 3-remainders of type \((b)\). Finally, we denote by \( g \) the number of 1-remainders. At this point of the proof, any of this value is an integer between 0 and \( k - 1 \) and \( X_1 \) or \( X_2 \) could be empty. This will be settled by the computation at the end of the proof. For the moment, we have the following properties.

**Claim 11** We have the following bounds on the number of arcs from \( X_1 \) to \( X_2 \):

- The number of arcs from \( X_1 \) to \( X_2 \) linking a 3-cycle with another 3-cycle or a 2-remainder is at most 3.
- The number of arcs going from \( X_1 \) to \( X_2 \) and linking a vertex of an element of type \((a)\) and a vertex of an element of type \((b)\) is at most \( 4ab \).
- There is no arc from a 3-cycle of type \((c)\) to a 3-cycle of type \((d)\).

**Proof:** To prove the first point, consider \( C \) a 3-cycle of \( X_1 \) and \( C' \) a 3-cycle or a 2-remainder of \( X_2 \). If there are more than three arcs from \( C \) to \( C' \), then we find a 2-matching from \( C \) to \( C' \). By Claim 10, this 2-matching is made of good arcs and the Rule 3.3 could apply to find a free vertex in \( C \), contradicting that \( W_{i_0} \setminus R \) has no free elements.

For the second point, consider a 2-remainder of type \((a)\) on vertices \( v_1 \) and \( v_2 \) with \( v_1 \in X_1 \) and \( v_2 \in X_2 \) and a pair \((C_1, C_2)\) of 3-cycles of type \((b)\) with \( C_1 = x_1y_1z_1 \) contained in \( X_1 \) and \( C_2 = x_2y_2z_2 \) contained in \( X_2 \) and \( x_1x_2 \) being an arc of \( T \). To find a contradiction, assume that the number of arcs from \( v_1 \) to \( C_2 \) plus the number of arcs from \( C_1 \) to \( v_2 \) is at least 5. It means that either \( v_1 \) dominates \( C_2 \) or \( C_1 \) dominates \( v_2 \), say that \( v_1 \) dominates \( C_2 \). Now, \( v_2 \) is dominated by at least two vertices of \( C_1 \), and one of these two is not \( x_1 \), say that it is \( y_1 \). But, the arcs \( x_1x_2, y_1v_2 \) and \( v_1y_2 \) are good by Claim 10, and \( z_1 \) and \( z_2 \) should be free by Rule 3.2, contradicting that \( W_{i_0} \setminus R \) has no free elements.

The third point follows from the definition of 3-cycles of type \((c)\) and \((d)\). \( \diamond \)

Now, we can derive a number of in-qualities from the structure derived so far, in order to obtain a contradiction, knowing that it has not been possible to increase the size of \( F \) above.

The first in-equality comes from the fact that there is a most \( k - 1 \) remainders and 3-cycles in \( W_{i_0} \).

\[
a + 2b + c + d + e + f + g < k \quad (2)
\]

For the second one, we compute the number of arcs going outside of \( B_{j_0} \), which has size \( l \). There are at least \( akl - l(l - 1)/2 \) such arcs. The number of arcs from \( B_{j_0} \) to \( U_{i_0} \setminus (\cup_{j=1}^{j_0} B_j) \) is at most \( l^2 \lceil 9/\epsilon \rceil \). There is no arc from \( B_{j_0} \) to \( U_{i_0} \setminus (\cup_{j=1}^{j_0} B_j) \). The number of arcs from \( B_{j_0} \) to \( X_2 \) is at most \( |X_2|3^{p+1} \), because every vertex of \( X_2 \) is insertable into \( U_{i_0} \) before \( B_{j_0} \) up to \( 3^{p+1} \) vertices. As \( X_2 \subseteq W \) we can bound this number by \( 3k3^{p+1} \). Finally, the remaining arcs going outside of \( B_{j_0} \) are at most \( l(a + 3b + 3c + 2e + g + |R| + |R'|) \), and we obtain:

\[
\alpha kl - \frac{l(l - 1)}{2} \leq \left\lfloor \frac{9}{\epsilon} \right\rfloor l^2 + 3k3^{p+2} + l(a + 3b + 3c + 2e + g + 2\epsilon k,)
\]

which we rewrite as

\[
(\alpha - 1.5)kl - \frac{l(l - 1)}{2} - \left\lfloor \frac{9}{\epsilon} \right\rfloor l^2 - k3^{p+2} - 2\epsilon k l \leq l(a + 3b + 3c + 2e + g - 1.5k)
\]

And finally arrange in:

\[
\left( \alpha - 1.5 \right) l - 3^{p+2} \right) k + \left( (\alpha - 1.5) - 2\epsilon - \frac{l - 1}{2} - \left\lfloor \frac{9}{\epsilon} \right\rfloor l \right) l \leq l(a + 3b + 3c + 2e + g - 1.5k)
\]
By choice of $l$, the first term is positive, and as $\epsilon < (\alpha - 1.5)/4$, if $k$ is large enough, the second term is strictly positive too, implying that:

$$a + 3(b + c) + 2e + g > 1.5k$$  \hspace{1cm} (3)

For the last in-equality, we compute the number of arcs going outside of $X_1$. As previously, we first show that, if $k$ is large enough, we have $\alpha(k|X_1| - d^+(X_1, U_{in}) - d^+(X_1, R \cup R') - 1.5k|X_1|$ is positive. Indeed, this term is greater than $(\alpha - 1.5)k|X_1| - \left\lfloor \frac{9}{2} \right\rfloor|X_1| - 2\epsilon k|X_1|$ which is

$$|X_1|\left( (\alpha - 1.5) - 2\epsilon k - \left\lfloor \frac{9}{2} \epsilon \right\rfloor \right)$$

and this is positive if $k$ is large enough. Now, we have to take in account the arcs inside $X_1$ and those from $X_1$ to $X_2$ and to the 1-remainders. By the calculation above we still have at least $1.5k|X_1|$ arcs incident with vertices of $X_1$ to account for. Using Claim 11 we obtain

$$1.5k(a + 3(b + c) + 2e) - \frac{1}{2}(a + 3(b + c) + 2e)^2 <$$

$$a(a + 4b + 3d + 2f + g) + b(3b + 3d + 3f + 3g) + c(3a + 3b + 3f + 3g) + c(2a + 3b + 3d + 4f + 2g) =$$

$$a(a + 4b + 3c + 3d + 2e + 2f + g) + b(3b + 3c + 3d + 3e + 3f + 3g) + c(3f + 3g) + c(3d + 4f + 2g)$$ \hspace{1cm} (4)

Considering (2), (3) and (4), an equation solver leads to a contradiction. We just indicate how to manage the computation 'by hand'. Suppose that there exists a solution $X = (a, b, c, d, e, f, g)$ to these three in-equalities, we will show that then $X' = (a + b + f + g, 0, b + c + d + e, 0, 0, 0, 0)$ is also a solution to these equations. It is easy to check that $X'$ is a solution to (2) and (3). For (4), we denote by $\phi(a, b, c, d, e, f, g)$ the value

$$2a(a + 4b + 3c + 3d + 2e + 2f + g) + b(3b + 3c + 3d + 3e + 3f + 3g) + 2c(3f + 3g) + 2c(3d + 4f + 2g)$$

$$+(a + 3(b + c) + 2e)^2 - 3k(a + 3(b + c) + 2e)$$

Then, we compute $\phi(a + b + f + g, 0, b + c + d + e, 0, 0, 0, 0) - \phi(a, b, c, d, e, f, g)$ and obtain:

$$2a(2b + 3d + 2e + f + g) + 3b(3b + 2c + 8d + 4e + 4f + 4g) + 6c(3d + e + f + g) + 3d(3d + 4e + 4f + 4g)$$

$$+ c(5e + 4f + 8g +) + 3f(3f + 2g) + 3g^2 - 3k(b + 3d + e + f + g)$$

Using the fact that $X$ is a solution of (3), we have $-3k > -2(a + 3(b + c) + 2e + g)$ and so $\phi(a + b + f + g, 0, b + c + d + e, 0, 0, 0, 0) - \phi(a, b, c, d, e, f, g)$ is greater than

$$2a(b + e + g) + b(3b + 6d + 2e + 6f + 4g) + 3d(3d + 4f + 2g) + e(e + 2g) + f(3f + 4g) + g^2$$

which is positive. So, $\phi(a + b + f + g, 0, b + c + d + e, 0, 0, 0, 0)$ is strictly positive and $X'$ is a solution of (4).

Now, there is a solution to the in-equations (2), (3) and (4) of type $(a', 0, c', 0, 0, 0, 0)$, what is impossible: (2) gives $d' + c' < 1$ and (4) gives $3(a' + 3c')(a' + c' - 1) > 0$.

This concludes the proof of Theorem 1.3. As a last remark, note that $k_\alpha$ is larger than a polynomial function in $l$, which is larger than an exponential in $p$, itself larger than a linear function in the inverse of $\alpha - 1.5$. So, $k_\alpha$ is an exponential function in the inverse of $\alpha - 1.5$. 

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4 Some Remarks

It is perhaps worth pointing out that the following obvious idea does not lead to a proof of Conjecture 1.1 for tournaments: find a 3-cycle $C$ which is not dominated by any vertex of $V(T) - V(C)$, remove $C$ and apply induction. This approach does not work because of the following.

Proposition 4.1 For infinitely many $k \geq 3$ there exists a tournament $T$ with $\delta^+(T) = 2k - 1$ such that every 3-cycle $C$ is dominated by at least one vertex of minimum out-degree.

Proof: Consider the quadratic residue tournament $T$ on 11 vertices $V(T) = \{1, 2, \ldots, 11\}$ and arcs $A(T) = \{i \to i + p \mod 11 : i \in V(T), p \in \{1, 3, 4, 5, 9\}\}$. The possible types of 3-cycles in $T$ are $i \to i + 1 \to i + 10 \to i, i \to i + 1 \to i + 6 \to i, i \to i + 3 \to i + 6 \to i, i \to i + 3 \to i + 7 \to i$, where indices are taken modulo 11. These are dominated by the vertices $i - 3, i - 1, i + 2, i + 4$. By substituting an arbitrary tournament for each vertex of $T$, we can obtain a tournament with the property that every 3-cycle is dominated by some vertex of minimum out-degree.

On the other hand, removing a 2-cycle from a digraph $D$ with $\delta^+(D) \geq 2k - 1$ clearly results in a new digraph $D'$ with $\delta^+(D') \geq 2(k - 1) - 1$ and hence, when trying to prove Conjecture 1.1, we may always assume that the digraph in question has no 2-cycles. In particular the following follows directly from Theorem 1.2.

Corollary 4.2 Every semicomplete digraph $D$ with $\delta^+(D) \geq 2k - 1$ contains $k$ disjoint cycles.

A chordal bipartite digraph is a bipartite digraph with no induced cycle of length greater than 4. Note that in particular semicomplete bipartite digraphs [3, page 35] are chordal bipartite. It is easy to see that Conjecture 1.1 holds for chordal bipartite digraphs.

Proposition 4.3 Every chordal bipartite digraph $D$ with $\delta^+(D) \geq 2k - 1$ contains $k$ disjoint cycles.

Proof: This follows from the fact that such a digraph contains a directed cycle $C$ of length 2 or 4 as long as $k \geq 1$. As $D$ is bipartite, no vertex dominates more than half of the vertices on $C$ and so we have $\delta^+(D - C) \geq 2(k - 1) - 1$ and the result follows by induction on $k$.

An extension of a digraph $D = (V, A)$ is any digraph which can be obtained by substituting an independent set $I_v$ for each vertex $v \in V$. More precisely we replace each vertex $v$ of $V$ by an independent set $I_v$ and then add all arcs from $I_v$ to $I_v$ precisely if $uv \in A$.

Proposition 4.4 Let $D = T[I_{n_1}, I_{n_2}, \ldots, I_{n_{|V(T)|}}]$ be an extension of a tournament $T$ such that $I_{n_i}$ is an independent set on $n_i$ vertices for $i \in \{1, 2, \ldots, |V(T)|\}$. If $\delta^+(D) \geq 2k - 1$, then $D$ contains $k$ disjoint 3-cycles.

Proof: Let $T'$ be the tournament that we obtain from $D$ by replacing each $I_{n_i}$ by a transitive tournament on $n_i$ vertices. Then $\delta^+(T') \geq 2k - 1$ and hence, by theorem 1.2, $T'$ contains $k$ disjoint 3-cycles $C_1, C_2, \ldots, C_k$. By the definition of an extension and the fact that we replaced independent sets by acyclic digraphs, no $C_i$ can contain more than one vertex from any $I_{n_i}$, implying that $C_1, C_2, \ldots, C_k$ are also cycles in $D$.

References


2See also [2, Section 9.1].

3A digraph is semicomplete if there is at least one arc between any pair of vertices.


