Paths partition with prescribed beginnings in digraphs: a Chvátal-Erdős condition approach.

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Abstract

A digraph D verifies the Chvátal-Erdős conditions if $\alpha(D) \leq \kappa(D)$, where $\alpha(D)$ is the stability of D and $\kappa(D)$ is its vertex-connectivity. Related to the Gallai-Milgram Theorem ([5]), we raise in this context the following conjecture. For every set of $\alpha = \alpha(D)$ vertices $\{x_1, \ldots, x_\alpha\}$, there exists a vertex-partition of D into directed paths $\{P_1, \ldots, P_\alpha\}$ such that P_i begins at x_i for all i. The case $\alpha(D) = 2$ of the conjecture is proved.

1 Introduction.

All topics of the paper deal with digraphs. Considered paths and circuits are directed ones. In our digraphs, circuits of length 2 are allowed, but not loops. We denote by κ the vertex-connectivity, and by α the stability of a digraph. All partitions or coverings of digraphs mentioned in the paper are understood as vertex partitions or coverings.

The classical Gallai-Milgram Theorem (see [5]) states that every digraph admits a partition into α paths. In this paper, we are mainly concerned by finding conditions to prescribe the beginnings of paths in such a partition. This problem is motivated, in a remote way, by coverings of digraphs into circuits (for instance, see [1] or Conjecture 2).

The following definitions are given for a digraph D with vertex set V and arc set E. For a path P of D, we denote by $b(P)$ and $e(P)$ respectively its beginning and its end. The internal vertices of P are the vertices of $P \setminus \{b(P), e(P)\}$ (possibly empty). For two vertices x and y of D, an (x, y)-path is a path with beginning x and end y. By extension, an (X, Y)-path P is an (x, y)-path for

some $x \in X$ and $y \in Y$ such that the set of internal vertices of P and $X \cup Y$ are disjoint.
For a path P and a vertex x of P, xP (resp. Px) denote the maximal sub-path of P which starts (resp. ends) at x. Moreover, if y is a vertex of xP, xPy denotes the maximal sub-path of xP which ends in y (i.e. the sub-path of P which starts at x and ends at y). We denote the concatenation of two paths by . ($P.Q$ is only used when there exists an arc from the end of P to the beginning of Q).
Finally, for an arc $xy \in E$ we also denote by xy the path of length 1 from x to y.

A digraph D verifies the Chvátal-Erdős conditions if we have $\alpha(D) \leq \kappa(D)$. These were named from the following sufficient condition for a (non oriented) graph to have a hamilton cycle, given by Chvátal and Erdős in 1972.

Theorem 1 ([4]) *For a graph G, if $\alpha(G) \leq \kappa(G)$, then G has a hamilton cycle.*

For digraphs the condition $\alpha \leq \kappa$ (κ is here the 'strong' vertex-connectivity) does not imply the existence of hamilton circuit. Infinite families of examples for $\alpha = 2$ and $\alpha = 3$ are given in [7]. However, according to the previous result for graphs, it could seem possible to ask for partitions into paths or circuits in digraphs which satisfy Chvátal-Erdős conditions. Several results and conjectures are stated in a survey of B. Jackson and O. Ordaz (see [7]). We present two new conjectures in this area.

The well-known result of Gallai-Milgram ([5]) asserts that every digraph D admits a path partition into at most $\alpha(D)$ paths. If D satisfies Chvátal-Erdős conditions, we would like to choose the beginnings of these paths.

Conjecture 1 *Let D be a digraph with $\alpha(D) \leq \kappa(D)$. For every set of $\alpha = \alpha(D)$ vertices $\{x_1, \ldots, x_\alpha\}$, there exists $\{P_1, \ldots, P_\alpha\}$ a path partition of D such that P_i begins at x_i for all i*

Note that the existence of a hamilton circuit in D implies the result of Conjecture 1. In particular, Conjecture 1 is true for graphs and for digraphs with stability 1 (according to Camion's theorem [3], strong digraphs with stability 1 have a hamilton circuit). The result of Conjecture 1 is also true if D contains a hamilton path with a prescribed beginning. But, we cannot ask for such a path as a consequence of Chvátal-Erdős condition as seen in Figure 1. This example is derived from those given by B. Jackson and O. Ordaz in [7] to provide digraphs with $\alpha \leq \kappa$ and no hamilton circuit.

The second conjecture deals with partition into circuits. In [6], it is proved that if a digraph satisfies Chvátal-Erdős conditions, then it admits a partition into circuits. In addition, a recent result (see [1]) states that the vertices of every strongly connected digraph can be covered with at most α circuits. So, it could be possible to limit the number of circuits in a circuit partition of a digraph which satisfies Chvátal-Erdős conditions.

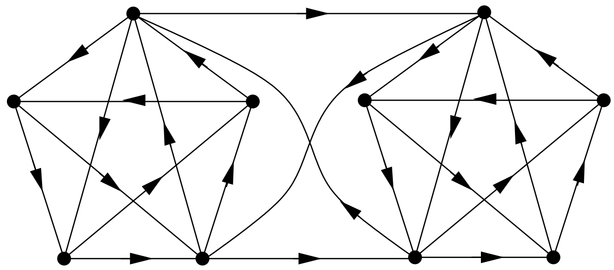

Figure 1: Digraph with $\alpha \leq \kappa = 2$ and no hamilton path starting at x.

Conjecture 2 *Every digraph which satisfies Chvátal-Erdős conditions admits a circuit partition into at most α circuits.*

Note that Conjecture 2 is true for $\alpha = 1$, according to Camion's Theorem ([3]), and this seems the sole known case of resolution.

The two next sections give useful tools, Lemmas 2 and 3, for the proof of Conjecture 2 in the case $\alpha = 2$ which is detailed in Section 4.

2 Vertices reachable from a prescribed vertex.

Let $u_1, \ldots, u_p$ and x be vertices of D. We say that a sequence of paths $(P_1, \ldots, P_p)$ satisfies $[u_1, \ldots, u_p \to x]$ if the u_i are distinct, $b(P_i) = u_i$, $e(P_i) = x$ for every $i = 1, \ldots, p$ and if the paths P_i are pairwise disjoint, except in x. We simply say $[u_1, \ldots, u_p \to x]$ to mean that such a sequence of paths exists. By extension, for $A_1, \ldots, A_p$, p subsets of V, $[A_1, \ldots, A_p \to x]$ means that there exists a vertex u_i in A_i, for $i = 1, \ldots, p$, such that $[u_1, \ldots, u_p \to x]$.
This will not be used here, but it is known that, for a fixed x, the sets of vertices $\{u_1, \ldots, u_k\}$ such that $[u_1, \ldots, u_k \to x]$ form a matroid (for instance, see [9], Chapter 39).
The following lemma is a corollary of Menger's Theorem ([8]).

Lemma 1 *If, for $p \geq 1$, we have $[v_1, \ldots, v_{p-1}, y \to x]$ and $[v_p, \ldots, v_{2p-1} \to y]$, then there exists a sequence of integers $1 \leq i_1 < \ldots < i_p \leq 2p - 1$ such that $[v_{i_1}, \ldots, v_{i_p} \to x]$.*

Proof. First, if $x = y$, we have $[v_p, \ldots, v_{2p-1} \to x]$. So, assume that $x \neq y$ and moreover suppose for the moment that none of the v_i is equal to x. Denote $X = N_D^-(x)$, $D' = D \setminus x$ and $W = \{v_i : 1 \leq i \leq 2p - 1\}$. We prove that every separator from W to X has at least p elements. Indeed, assume that there exists a set S of vertices of D' of size at most $p - 1$ which is a separator from W to X. Denote $\overline{W} = \{z \in V \setminus x :$ there exists a path from W to z in $D' \setminus S\}$ and

$\overline{X} = V \setminus (\overline{W} \cup S)$. Now, y is vertex of D', but if $y \in \overline{W} \cup S$, we contradict $[v_1, \ldots, v_{p-1}, y \to x]$ and if $y \in \overline{X}$ we contradict $[v_p, \ldots, v_{2p-1} \to y]$. So, every separator from W to X has at least p elements and, by Menger's Theorem ([8]), there exists p disjoint paths from W to X in D'. We finally just add x as end of all these paths.
Now, assume that one of the v_i is equal to x. By the hypothesis, at most one of the v_i, for $1 \leq i \leq p-1$, is equal to x and, if this is the case, we suppose $v_{p-1} = x$. Similarly, at most one of the v_i for $p \leq i \leq 2p-1$ is equal to x and again, if this is the case, we suppose $v_{2p-1} = x$. So, none of the v_i for $1 \leq i \leq p-2$ and $p \leq i \leq 2p-2$ is equal to x. By the first case, we have a sequence $1 \leq i_1 < \ldots < i_{p-1} \leq 2p-2$ and $i_j \neq p-1$ such that $[v_{i_1}, \ldots, v_{i_{p-1}} \to x]$. Moreover, according to the construction of the obtained paths, none of this path is the trivial path $\{x\}$. So, we add $\{x\}$ to them to complete the collection of paths. □

In the case $\alpha = 2$, we use the following refinement of Lemma 1.

Lemma 2 *Let u, v, w, x, y be vertices of a digraph D such that $[u, v \to y]$ and $[y, w \to x]$, then we have $[u, v \to x]$ or $[u, w \to x]$.*

Proof. Either Lemma 1 directly provides the result, or we have $[v, w \to x]$. In this case, denote by (P_1, P_2) a couple of paths which satisfies $[v, w \to x]$. As there exists a path from u to x, there exists a path from u to $P_1 \cup P_2$. Denote by P a $(v, P_1 \cup P_2)$-path. If the end of P belongs to P_1, we have $[u, w \to x]$, else the end of P belongs to P_2 and we have $[u, v \to x]$. □

3 Paths exchange.

The following theorem, due to J.A. Bondy (see [2]) provides a useful tool to reduce the number of paths in a path partition of a digraph. Moreover, it gives some control on beginnings and ends of the paths. We will refer later this result as 'paths exchange'.

Theorem 2 ([2]) Paths exchange. *Let D be a digraph and $\{P_1, \ldots, P_k\}$ a path partition of D. If $k > \alpha(D)$, then there exists $\{Q_1, \ldots, Q_{k-1}\}$ a path partition of D such that $\{b(Q_i) : 1 \leq i \leq k-1\} \subset \{b(P_i) : 1 \leq i \leq k\}$ and $\{e(Q_i) : 1 \leq i \leq k-1\} \subset \{e(P_i) : 1 \leq i \leq k\}$*

Finally, the next lemma is an easy corollary of the paths exchange and will be useful in next section.

Lemma 3 *Let D be a digraph with stability 2 and two initial components M_1 and M_2. Then, for all x_1 in M_1 and x_2 in M_2 there exists two disjoint paths P_1 and P_2 which respectively begin in x_1 and x_2 and cover D.*

Proof. First, note that M_1 and M_2 have each stability 1. So by Camion's Theorem (see [3]) there exists a hamilton circuit of M_1 and one of M_2 and then, a hamilton path Q_1 of M_1 which starts at x_1 and a hamilton path Q_2 of M_2 which starts at x_2. Now, apply enough paths exchanges on the set of paths $\mathcal{P} = \{Q_1, Q_2\} \cup \{x : x \in V(D \setminus (M_1 \cup M_2))\}$ in order to obtain two disjoint paths P_1 and P_2 which cover D. The beginnings of P_1 and P_2 belong to $b(\mathcal{P})$, the beginnings of the paths of $\mathcal{P}$. But, in D there is no path from any vertex of $b(\mathcal{P}) \setminus x_1$ to x_1, so x_1 is the beginning of P_1 or P_2. Similarly, x_2 is the beginning of the other path. □

4 The main result.

This section presents a proof of Conjecture 1 for the case $\alpha = 2$.

Theorem 3 *Let D be a digraph with $\alpha(D) \leq 2$ and $\kappa(D) \geq 2$. Then, for any distinct vertices x and y of D, there exists two disjoint paths which respectively start at x and y and cover D.*

Proof. In fact, for a digraph D with stability at most 2 and with at least two vertices, we prove the following stronger statement $(\star)$:

> $(\star)$ *If A and B are non-empty subsets of V such that $[A, B \to x]$ for all x in V, then there exists two disjoint paths P_1 and P_2 which cover D and such that $b(P_1) \in A$ and $b(P_2) \in B$.*

For $\kappa(D) \geq 2$ and fixed x and y, the condition of $(\star)$ holds with $A = \{x\}$ and $B = \{y\}$ and the conclusion of $(\star)$ gives the result of the theorem.
So, assume that $(\star)$ is not true and consider among all the counter-examples of $(\star)$ with minimum number of vertices, one with $|A| + |B|$ minimum. Denote by D this extremal digraph. Note that D has at least 3 vertices, this will be useful to apply induction.
We prove several facts on D, A and B to obtain a contradiction.

Fact 1: The digraph D has a sole initial component. We denote it by M.

Indeed, if not, D has two initial components. Denote it by M_1 and M_2. By hypothesis, M_1 has to contain a vertex a of A and M_2 has to contain a vertex b of B. By Lemma 3, we obtain two disjoint paths which respectively start at a and b and cover D. This contradicts the choice of D.

Fact 2: We have $|A| \geq 2$ and $|B| \geq 2$.

By contradiction, assume that $|A| = 1$ and denote $A = \{a\}$. So, we consider $D' = D \setminus a$, $A' = N^+_D(a)$ and $B' = B \setminus a$ (according that $a \in B$ is possible). It is clear that D' has stability at most 2 and that D' has at least two vertices (as D has at least 3 vertices). So, let us check that D', A' and B' satisfy the hypothesis of $(\star)$. As D has at least two vertices, A' and B' are not empty. Moreover, for every vertex x of D', as $[a, B \to x]$ in D, we have $[A', B' \to x]$

in D'. So, $(\star)$ applied to D', A' and B' provides two disjoint paths P and Q which cover D' and with $b(P) \in A'$ and $b(Q) \in B'$. Finally, the paths $a.P$ and Q contradict the choice of D.

Fact 3: The sets A and B are disjoint.

First, let us prove that every vertex of $A \cap B$ has no in-neighbour in D. Otherwise, consider a vertex x of $A \cap B$ and y an in-neighbour of x. By hypothesis, there exists (P, Q) a couple of paths which satisfies $[A, B \to y]$. As $P \setminus y$ and $Q \setminus y$ are disjoint, x cannot belong to both. Assume that $x \notin P$ and consider $A' = A \setminus x$ (which is not empty because $d(P) \in A'$). Now, for every z of D we have $[A', B \to z]$. Indeed, if for a vertex z, $[A, B \to z]$ does not directly give $[A', B \to z]$, we have $[x, B \to z]$. Denote b a vertex of B such that $[x, b \to z]$. By Lemma 2, as $[d(P), x \to x]$, we have $[d(P), x \to z]$ or $[d(P), b \to z]$ and in both case, $[A', B \to z]$. Then, by minimality of A, there exists two disjoint paths which cover D and start respectively in A' and B. As $A' \subset A$, this contradicts the choice of D. Similarly, we obtain the same contradiction if $x \notin Q$, and conclude that x cannot have an in-neighbour.

Now, we can prove that $A \cap B = \emptyset$. If not, consider a vertex x of $A \cap B$. According to the previous remarks, $d^-_D(x) = 0$ and then, by Fact 1, $\{x\}$ is the sole initial component of D. If $D \setminus x$ has two initial components M_1 and M_2, both intersect $N^+_D(x)$, and by hypothesis, both intersect $A \cup B$. So, pick $y \in M_1 \cap (A \cup B)$ and $z \in M_2 \cap N^+_D(x)$. By Lemma 3, there exists two disjoint paths P and Q which cover $D \setminus x$ with $b(P) = y$ and $b(Q) = z$. But now, P and $x.Q$ are two disjoint paths which cover D and respectively start in $A \cup B$ and $A \cap B$. This contradicts the choice of D.
Then, $D \setminus x$ has a sole initial component, M. By hypothesis, M contains a vertex of $A \cup B$. Assume that there exists a vertex a in $M \cap A$ and consider $D' = D \setminus x$, $A' = A \setminus x$ and $B' = (B \setminus x) \cup N^+_D(x)$. Note that $A' \neq \emptyset$, $B' \neq \emptyset$ and that $|D'| \geq 2$. Let us see that $[A', B' \to z]$ for every z in D'. For z a vertex of D', consider (P, Q) which realize $[A, B \to z]$ in D. Either x does not belong to $P \cup Q$ and then $[A', B' \to z]$ in D' is clear, or x belongs to $P \cup Q$. In this case, as x is the initial component of D, we have either $x = d(Q)$, and through $Q \cap N^+_D(x) \neq \emptyset$, $[A', B' \to z]$ in D' is clear again, or $x = d(P)$. Finally, if $x = d(P)$, denote by x' the successor of x along P ($x' \in B'$). As $a \in M$, there exists a path from a to x'. From $[a, x' \to x']$ and $[x', d(Q) \to z]$, we derive through the Lemma 2 that $[x', a \to z]$ or $[x', d(Q) \to z]$ and, in both cases, we have $[A', B' \to z]$. So, we apply $(\star)$ to A', B' and D' and obtain two disjoint paths P' and Q' which cover D' with $d(P') \in A'$ and $d(Q') \in B'$. If $d(Q') \in N^+_D(x)$, then the paths P' and $x.Q'$ cover D, start respectively in A and B and so, contradict the choice of D. So, $d(Q') \in (B' \setminus N^+_D(x)) \subset B$ and $d(P') \in A' \subset A$ and we apply a paths exchange on P', Q' and the path x. As $\{x\}$ is the initial component of D, we obtain two disjoint paths which cover D whose the beginning of one of them is x. The beginning of the other path is $d(Q')$ or $d(P')$. We finally provide two disjoint paths which cover D and start respectively in $A \cap B$ and $A \cup B$. This contradicts the choice of D.

Fact 4: The sole initial component of D contains $A \cup B$.

By contradiction, assume that there exists $a \in M \setminus A$ and consider $A' = A \setminus a$ which is non empty (as $M \cap A \neq \emptyset$). Denote by X the set $\{x \in V : [A', B \to x]\}$. If $X = V$, then, by minimality of A, we cover D by two disjoint paths which start respectively in A' and B, and as $A' \subset A$, this contradicts the choice of D. So, Y, the complementary of X in V is not empty. By hypothesis, there exists a path from a to every vertex of Y. So, in particular, $Y \cap M = \emptyset$ and if M_Y denotes an initial component of Y, there exists a vertex u in X which is the in-neighbour of a vertex of M_Y. Let us see that u is a separator in D from X to M_Y. Indeed, if not, there exists two vertices $x \in X \setminus u$ and $y \in M_Y$ such that xy is an arc of D. The existence of a path in M_Y from $N^+_{M_Y}(u)$ to y assures that $[x, u \to y]$ in D. But this contradicts the fact that $y \notin X$. Indeed, there exists $a_1, a_2 \in A'$ and $b_1, b_2 \in B$ such that $[a_1, b_1 \to x]$ and $[a_2, b_2 \to u]$. By Lemma 2, $[a_1, b_1 \to x]$ and $[x, u \to y]$ give $[a_1, u \to y]$ ($[a_1, b_1 \to y]$ is impossible through $y \notin X$). And by Lemma 2 again, $[a_2, b_2 \to u]$ and $[a_1, u \to y]$ give $[a_2, b_2 \to y]$ or $[a_1, b_2 \to y]$, both contradicting $y \notin X$.
So, u is a (X, M_Y)-separator in D. Then, consider P a $((A \cup B) \cap M, u)$-path and assume that $d(P) \in A$. Note that $M \setminus P \neq \emptyset$, because P does not intersect B and M must contain a vertex of B (by Fact 3, $A \cap B = \emptyset$ and M contains at least two distinct vertices respectively from A and B). Therefore, there exists M' an initial component of $M \setminus P$ which intersects B. As there is no path in D from a to M, there is no arc from M_Y to M', and as $u \notin M_Y$, there is no arc from M' to M_Y. Finally, M' and M_Y are the two initial components of $D \setminus P$, and by Lemma 3, we provide two disjoint paths Q and Q' which cover $D \setminus P$ and respectively start in $M_Y \cap N^+_D(u)$ and $M' \cap B$. So, the paths Q' and $P' = P.Q'$ are disjoint, cover D and respectively start in B and A. This contradicts the choice of D.
We conclude similarly in the case $d(P) \in B$.

Fact 5: There exists no three distinct vertices $a, a' \in A$ and $b \in B$ such that $[a, b \to a']$ (and similarly, there exists no three distinct vertices $b, b' \in B$ and $a \in A$ such that $[a, b \to b']$).

If not, assume that we have three distinct vertices $a, a' \in A$ and $b \in B$ such that $[a, b \to a']$ and once again, we reduce A. We consider $A' = A \setminus a'$ and claim that $[A', B \to x]$ for every $x \in V$. Indeed, if for a vertex x we have $[a', b' \to x]$ for some $b' \in B$, using Lemma 2 and that $[a, b \to a']$, we obtain that $[a, b \to x]$ or $[a, b' \to x]$. As $a \neq a'$, in both case, we have $[A', B \to x]$. Finally, through $(\star)$, we provide two disjoint paths which cover D and respectively start in $A' \subset A$ and B. This contradicts the choice of D.

Fact 6: There exists no three distinct vertices $a, a', a'' \in A$ such that $[a, a' \to a'']$ (and similarly, there exists no three distinct vertices $b, b', b'' \in B$ such that $[b, b' \to b'']$).

Indeed, as $[a, b \to a]$ for some $b \in B$, using Lemma 2 and that $[a, a' \to a'']$, we have $[a, b \to a'']$ or $[a', b \to a'']$. This is impossible by Fact 5 (note that a, a', a'' and b are disjoint because $A \cap B = \emptyset$ by Fact 3).

Fact 7: There exists no three distinct vertices $a, a' \in A$ and $b \in B$ such that $[a, a' \to b]$ (and similarly, there exists no three distinct vertices $b, b' \in B$ and $a \in A$ such that $[b, b' \to a]$).

Indeed, if not, by the Fact 2, there exists a vertex $b' \in B$ distinct of b, and by the Fact 4, b' is in the initial component of D. So, there exists a path from b' to a and we have $[a, b' \to a]$. By Lemma 2, and through $[a, a' \to b]$, we have $[a, b' \to b]$ or $[a', b' \to b]$, what contradicts in both case Fact 5.

Fact 8: If there exists two disjoint arcs xx' and yy' with $x, x', y, y' \in A \cup B$, then there exists two disjoint paths with length at least 1, which cover d and whose first arcs are respectively xx' and yy'. In particular, by choice of D, we have $x, y \in A$ or $x, y \in B$.

Indeed, assume that there exists two disjoint arcs of D, xx' and yy' with $X = \{x, x', y, y'\} \subset A \cup B$. By Facts 5,6 and 7, x' and y' are not link by a path in $D \setminus \{x, y\}$. In particular, $\{x', y'\}$ is a stable set of D. Now, we prove that every vertex $z \in V \setminus X$ is the end of a $(\{x', y'\})$-path in $D \setminus \{x, y\}$. Indeed, fixed $z \in V \setminus X$, as $\alpha(D) \leq 2$, either there exists an arc from x' or y' to z and we are done, or there exists an arc from z to x' or y', for instance, say that $zx' \in E$. By hypothesis, we have $[a, b \to z]$ for some $a \in A$ and $b \in B$. So, consider P and Q minimal such that (P, Q) realizes $[A \cup B, A \cup B \to z]$. If x' does not belong to $P \cup Q$, one this two paths does not contain x, say P. Then, P does not contain x and x' and starts in $A \cup B$, but now, $P.x'$ and xx' contradict one of the Facts 5,6 or 7. So, x' belongs to $P \cup Q$. By minimality of the paths P and Q, x and y do not belong to $P \cup Q$ and we have a path from x' to z in $D \setminus \{x, y\}$.
Finally, x' and y' are respectively in two distinct initial components of $D \setminus \{x, y\}$. By Lemma 3, there exists two disjoint paths which cover $D \setminus \{x, y\}$ and respectively start at x' and y' what proves Fact 8.

Fact 9: The sets A and B have exactly two elements each.

By Fact 2, we have just to prove that $|A| \leq 3$ and $|B| \leq 3$. If not, assume that A has at least 3 vertices. First, note that there is no circuit in $D[A]$, because a minimum path from B to such a circuit would provide three distinct vertices $a, a' \in A$ and $b \in B$ such that $[a, b \to a']$ what is impossible by Fact 5. So, pick three distinct vertices $a, a', a'' \in A$. Through the previous remark, we can assume that a dominates a' (as $\{a, a', a''\}$ is not a stable set) and a' does not dominate a''. By Fact 5, a'' does not dominate a', and then $\{a', a''\}$ is a stable set of D. Now, consider b and b' two distinct vertices of B (distinct from

a, a' and a'' by Fact 3). By Fact 8, there is no arc from $\{b, b'\}$ to $\{a, a', a'', b, b'\}$ disjoint from aa'. In particular, $\{b, b'\}$ is a stable set of D and a'' dominates b or b', say b. Then, by Facts 5 and 7, a' dominates b'. To obtain a contradiction, we finally look at the vertices a, b and b'. Indeed, $ab' \in E$ and $ab \in E$ arc forbidden by Fact 7 and $b'a \in E$ and $ba \in E$ are forbidden by Fact 8 (respectively consider arcs $a''b$ and $a'b'$). Then, $\{a, b, b'\}$ should be a stable set. This contradicts $\alpha(D) \leq 2$.
By symmetry, we have $|B| \leq 3$.

Fact 10: There is no disjoint arcs ab and $a'b'$ with $a, a' \in A$ and $b, b' \in B$ (and similarly, there is no disjoint arcs ba and $b'a'$ with $a, a' \in A$ and $b, b' \in B$).

Indeed, if the statement holds, by Fact 8, we provide P and P', two disjoint paths which cover $D \setminus \{a, a'\}$ and respectively start at b and b'. If $aa' \in E$, the paths P and $aa'.P'$ contradict the choice of D. Similarly, $a'a \notin E$ and $\{a, a'\}$ is a stable set of D.
Now, as there exists a path from B to a for instance, there exists a vertex x of $P \cup P'$ which dominates a or a'. Without loss of generality, we can assume that x is a vertex of P and that x is the last vertex along P which dominates a or a'. If x is the last vertex of P, then the paths $P.a$ and $a'.P'$ (if $xa \in E$) or the paths a and $P.a'.P'$ (if $xa' \in E$) contradict the choice of D. So, x is not the last vertex of P and we denote by x_+ the successor of x along P. As $\alpha(D) \leq 2$, there exists an arc between x_+ and $\{a, a'\}$ and by choice of x, this arc ends in x_+. We discuss the different cases.

Case 1: We have $xa \in E$ and $ax_+ \in E$. In this case, we insert a in the path P and the paths $Px.a.x_+P$ and $a'.P'$ contradicts the choice of D.

Case 2: We have $xa' \in E$ and $a'x_+ \in E$. As in Case 1, we insert a' in P to obtain a contradiction.

Case 3: We have $xa' \in E$ and $ax_+ \in E$. The paths $a.x_+P$ and $Px.a'.P'$ contradicts the choice of D.

Case 4: We have $xa \in E$ and $a'x_+ \in E$. This case is not so straightforward as the previous ones. Consider the paths $P_1 = Px.a$, $P_2 = x_+P$ and $P_3 = P'$ and, as $\alpha(D) \leq 2$, apply a paths exchange on them in order to obtain Q_1 and Q_2, two disjoint paths which cover $D \setminus a'$. Now, whatever the beginning of P_1, P_2 or P_3 we lost in the path exchange, we can extend one of the paths Q_1 or Q_2 to obtain two disjoint paths which cover D and respectively start in A and B. This contradicts the choice of D.

Finally, we can conclude that such a digraph D does not exist. By Facts 3 and 9, A and B are disjoint and have exactly two elements each. By Facts 8 and 10, there is no two disjoint arcs in $D[A \cup B]$. So, as $\alpha(D) \leq 2$, it is easy to check that $D[A \cup B]$ contains a set X of 3 vertices pairwise linked. By Facts 5, 6 and 7, every vertex of $A \cup B$ has in-degree at most 1 in $D[A \cup B]$. So, $D[X]$

is a circuit of length 3, but by Fact 4, $A \cup B$ is in a sole component of D and there exist a path from the vertex of $(A \cup B) \setminus X$ to X. However, as previously seen, this last remark contradicts one of the Facts 5, 6 or 7. □

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