FPSAC 2003

On a natural correspondence between bases and reorientations,

related to the Tutte polynomial and linear programming,

in graphs, hyperplane arrangements, and oriented matroids.

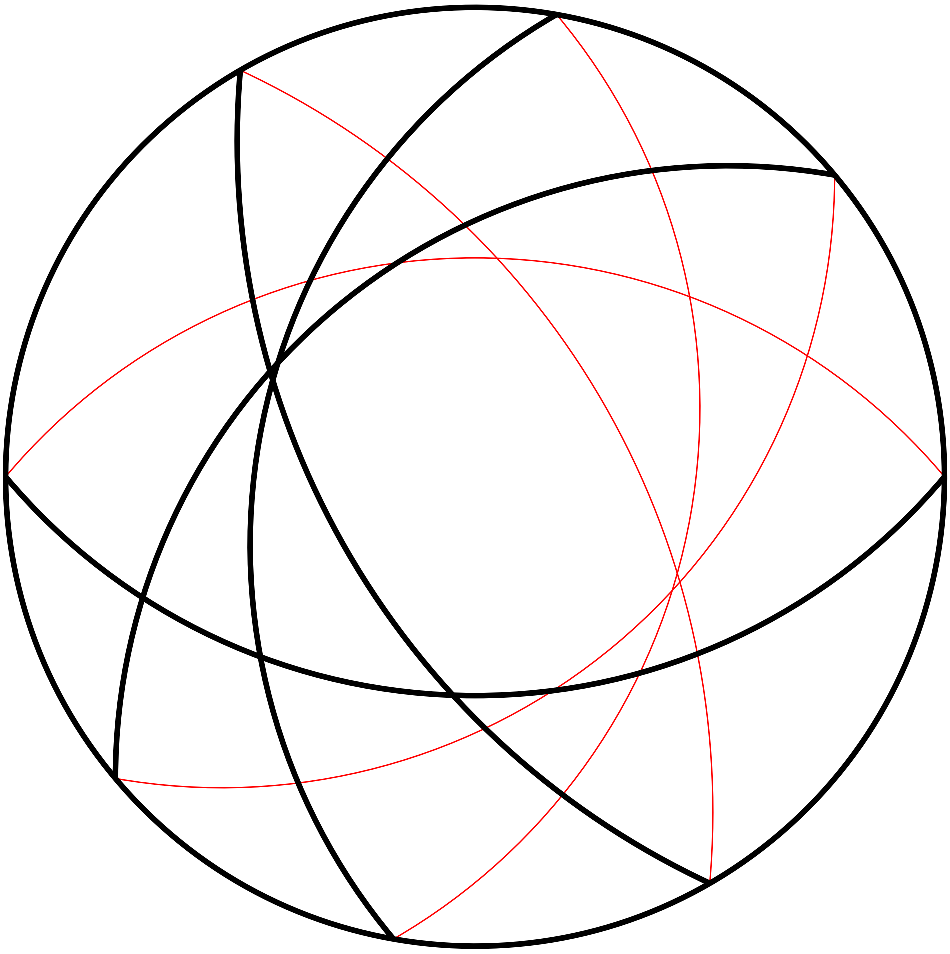
Emeric Gioan

joint work with Michel Las Vergnas

Hyperplane arrangements, graphs, matroids, and oriented matroids

Arrangement of pseudospheres

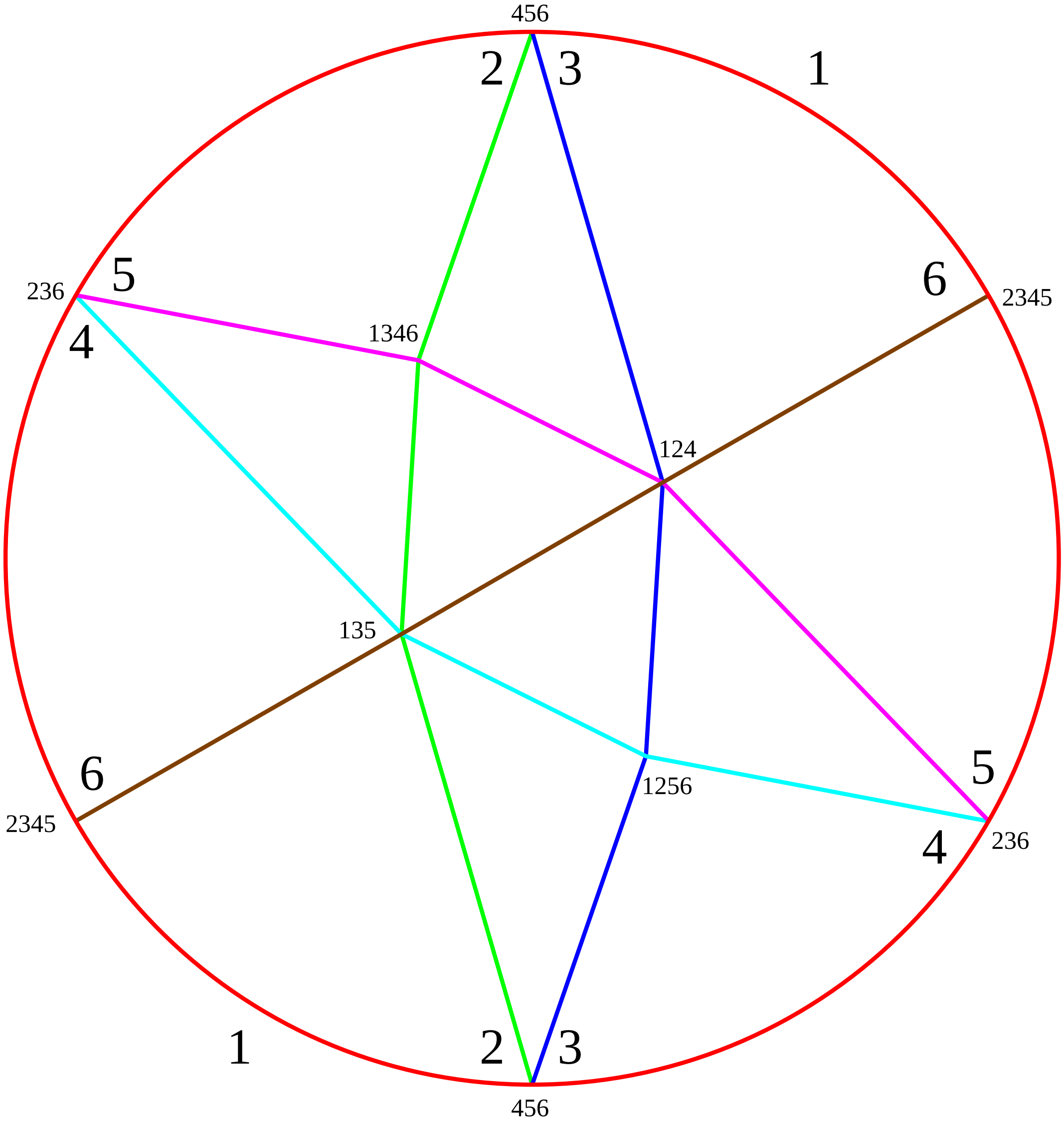
Example. Intersection of an hyperplane arrangement with a central sphere.



Matroid

bases: pseudosimplices, bases for linear algebra, spanning trees in graphs

example: 124, 125, 126, 134, 135, 136, 146, 156, 234, 235, 236, 245, 256, 345, 346, 456



cocircuits: subsets not containing a given vertex

Oriented matroid

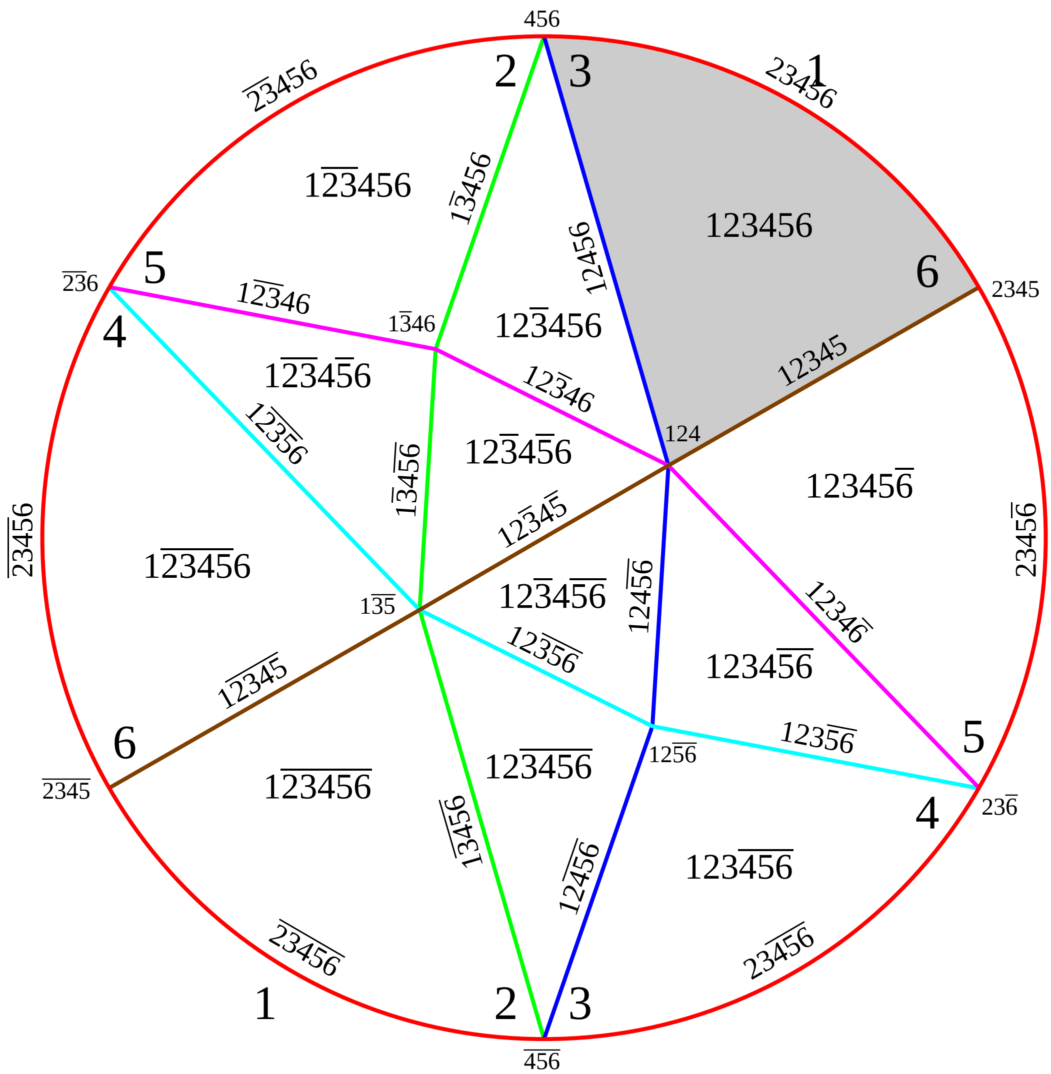
signature: choice of a halfspace + for each $e \in E$

reorientation of $A \subseteq E$: change signature on A

signatures $\leftrightarrow$reorientations

acyclic reorientation: when the intersection of halfspaces + is a region

regions $\leftrightarrow$acyclic reorientations



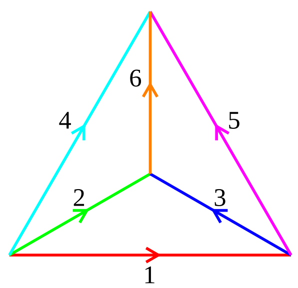
Graph

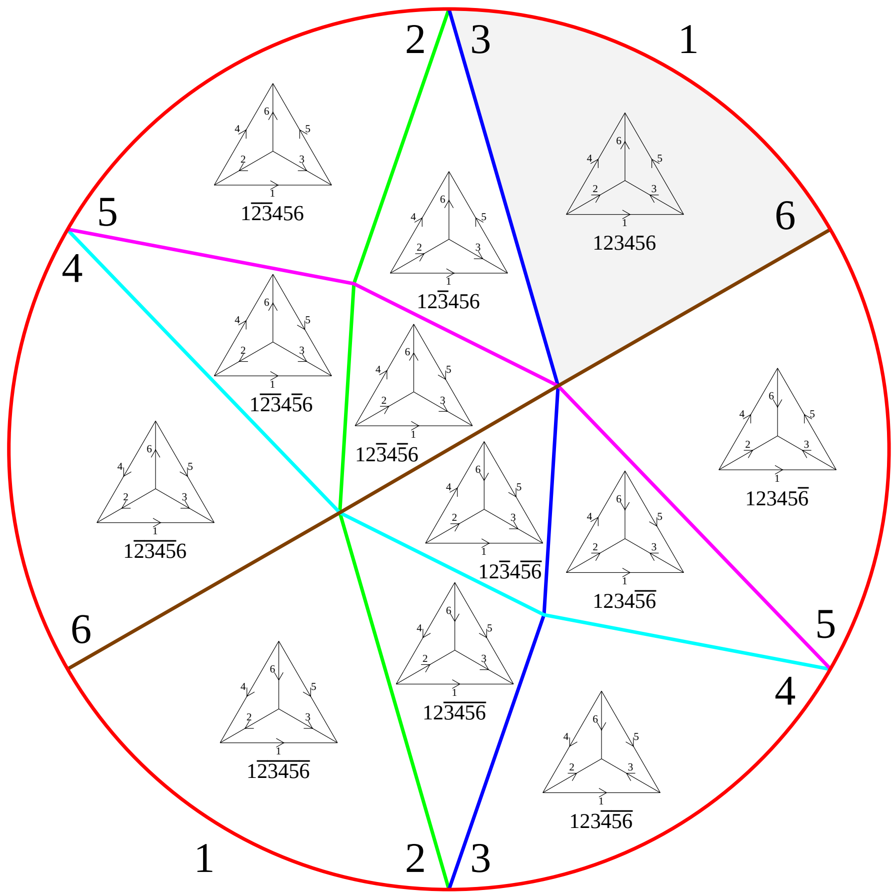
graph $G = (V, E)$

associated hyperplane arrangement:

$$\{\ x_i - x_j = 0 \text{ for } i, j \in V \text{ and } (i, j) \in E\ \}$$

edges $\leftrightarrow$ hyperplanes
acyclic orientations $\leftrightarrow$ regions
cocircuits = cocycles $\leftrightarrow$ vertices (faces of dimension 0)
circuits = elementary cycles $\leftrightarrow$ minimal dependant sets
bases = spanning trees $\leftrightarrow$ simplices



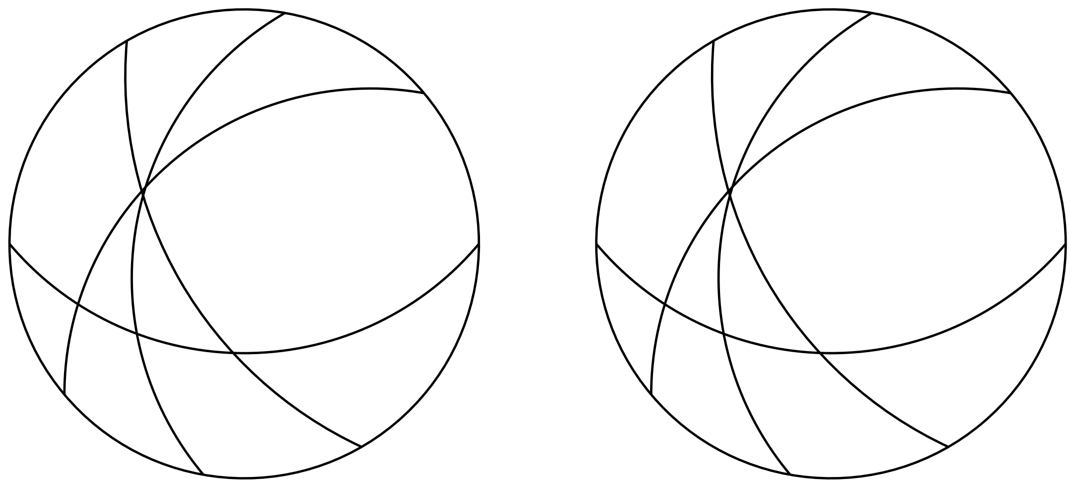


A problem on the Tutte polynomial

A curious property

The number of regions that do not touch a given hyperplane, resp. the number of acyclic orientations in a graph with unique source and sink adjacent on a given edge,

does not depend on
the chosen hyperplane, resp. edge.



This number is $\beta(M)$, coefficient of x (or y) in the Tutte polynomial $t(M; x, y)$ of the matroid M.

The Tutte polynomial

- generating function of rank and cardinality :

$$t(M;x,y) = \sum_{A\subseteq E} (x-1)^{r-r(A)} (y-1)^{|A|-r(A)}$$

- generalisation of the chromatic polynomial of graphs to two dual variables (1950')

$$t(M;x,y) = t(M^*;y,x)$$

- numerous significative evaluations

$$t(M;1,1) = \#\text{ bases of } M$$

- various apparitions (knots, physical models...)

- famous inductive definition by deletion/contraction

Acyclic orientations

• Th. A. Stanley (1973) : $\chi(G;-1) = t(G;2,0)$
$= \#$ acyclic orientations of the graph G

• Th. B. Zaslavski (1975), Las Vergnas (1975) : (Th. B $\Rightarrow$ Th. A)

$$t(M;2,0) = \# \text{ regions of the arrangement } M$$

• Th. C. Greene Zaslavski (1983), Las Vergnas (1977) : the coefficient of x (or y), $\beta(M) = b_{1,0} = b_{0,1}$, is the number of regions not touching any given element (on one side).

• Th. D. Las Vergnas (1984) :
for a *total order* on M oriented matroid

$$t(M;x,y) = \sum_{A\subseteq E} (\frac{x}{2})^{o^*(-_A M)} (\frac{y}{2})^{o(-_A M)}$$

a region M satisfies $o(M)=0$, so Th. D. $\Rightarrow$ Th. B.
a region M does not touch the smallest element if $o(M)=0$ and $o^*(M)=1$, so Th. D. $\Rightarrow$ Th. C.

The basis state model

(activities of bases, Tutte 1954)

M matroid on a linearly ordered set E
B basis of M

$e \in E \setminus B$ is *externally active* with respect to B if e is the smallest element of the (unique) circuit $C(B;e)$ contained in $B \cup e$.

$b \in B$ is *internally active* with respect to B if b is the smallest element of the (unique) cocircuit $C^*(B;e)$ contained in $(E \setminus B) \cup e$.

$\epsilon_M(B) = \#$ externally active elements w.r.t. B
$\iota_M(B) = \#$ internally active elements w.r.t. B

$$t(M;x,y) = \sum_{B \text{ base of } M} x^{\iota_M(B)} y^{\epsilon_M(B)}$$

$$t(M;x,y) = \sum_{i,j} b_{i,j} x^i y^j$$

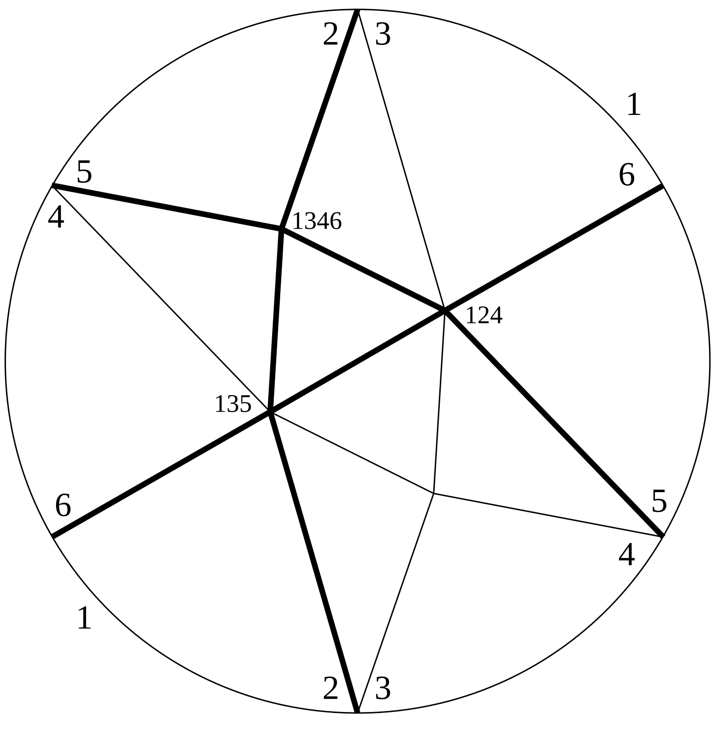
where $b_{i,j} = \#$ bases with *activities* (i,j).

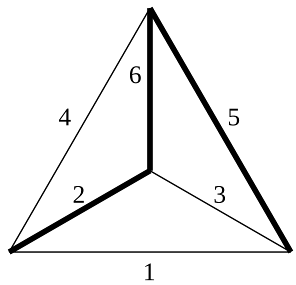
Example. Base 256 of K_4.
- fundamental cocircuits:

$$\begin{array}{lcllllll}
C^*(256;2) & = & 1 & 2 & & 4 & & \\
C^*(256;5) & = & 1 & & 3 & & 5 & \\
C^*(256;6) & = & 1 & & 3 & 4 & & 6
\end{array}$$

$Int(256) = \emptyset$, $\iota(256) = 0$,

- fundamental circuits: $C(256;1) = 1256$,
$C(256;3) = 356$, $C(256;4) = 246$.
$Ext(256) = 13$, $\varepsilon(256) = 2$.





fundamental tableau:

lines $= C^*(B;b)$ for $b \in B$

rows $= C(B;e)$, for $e \in\ E \setminus\ B$

256	1	2	3	4	5	6
1	x					
2	x	x		x		
3			x			
4				x		
5	x		x		x	
6	x		x	x		x

An orientation state model

(Las Vergnas 1984)

M oriented matroid on a <u>linearly ordered</u> set E

$o(M) = \#$ minimal elements of positive circuits of M

$o^*(M) = \#$ minimal elements of positive cocircuits of M

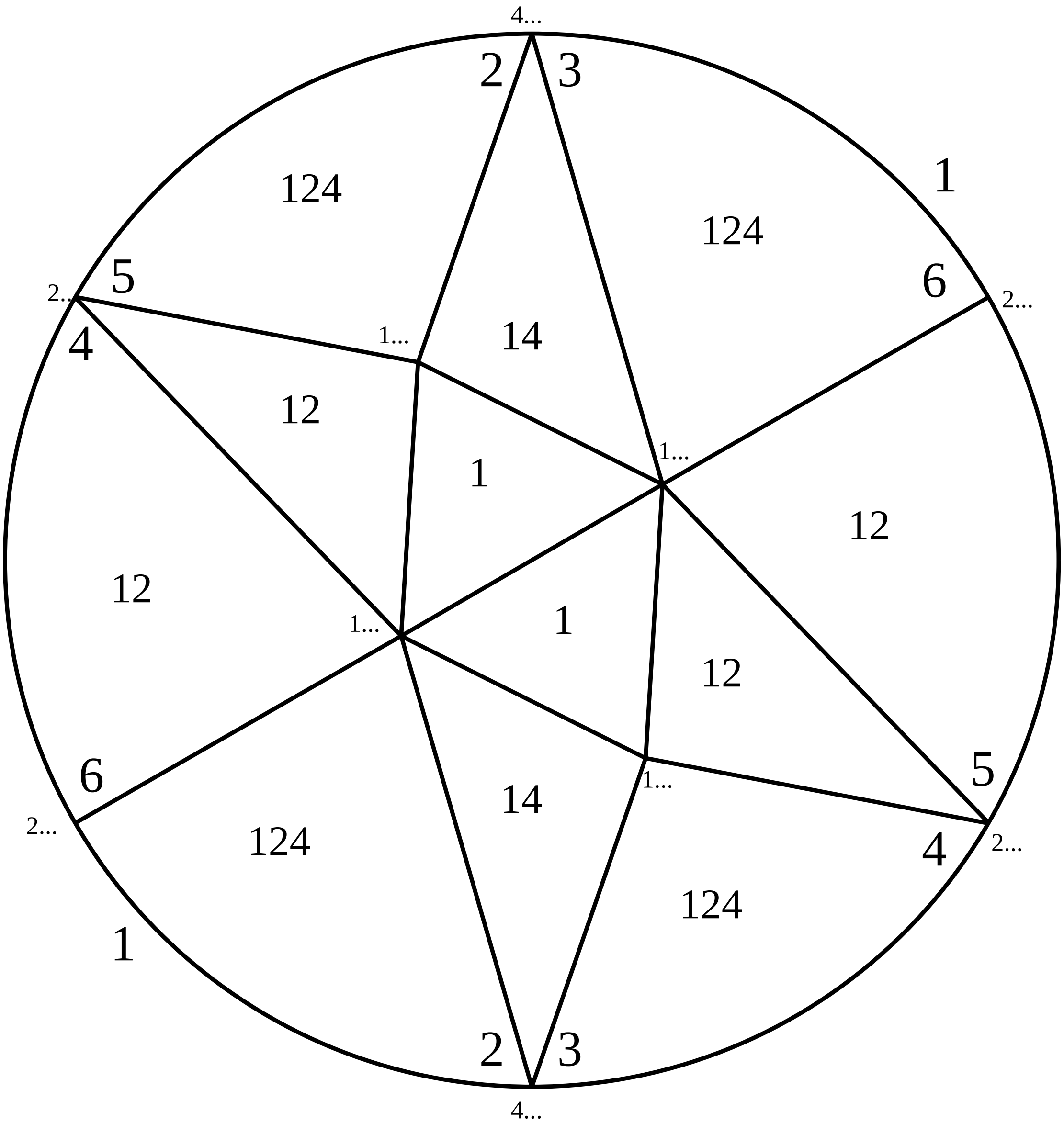
$$t(M;x,y) = \sum_{A\subseteq E} o_{i,j}\left(\frac{x}{2}\right)^{o^*(-_A M)}\left(\frac{y}{2}\right)^{o(-_A M)}$$

$$t(M;x,y) = \sum_{i,j} o_{i,j}\left(\frac{x}{2}\right)^{i}\left(\frac{y}{2}\right)^{j}$$

where $o_{i,j} = \#$ reorientations with *activities* (i,j).

$o(M) = 0$ if and only if M is acyclic (region).
$o^*(M) = 0$ if and only if M is totally cyclic
(strongly connected for a connected graph).

Remark. Activities situate regions with respect to the minimal base for the lexicographic order.



regions with dual activity 1 are *bounded* regions, when the smallest element is *infinity*.

The problem

M oriented matroid on a linearly ordered set E

$$\boxed{o_{i,j} = 2^{i+j} b_{i,j}}$$

Construct and study a

Natural activity preserving correspondence
between bases and reorientations

compatible with the above equality

- find a natural bijection between $(1,0)-$bases and pairs of opposite bounded regions, for

$$o_{1,0} = 2b_{1,0}$$

- use a decomposition of activities to extend this bijection from $(1,0)$ activities to (i,j) activities.

The canonical active correspondence

First decomposition of activities

from (i,j) to $(i,0)$ and $(0,j)$ activities

Theorem (Etienne, Las Vergnas 1998)

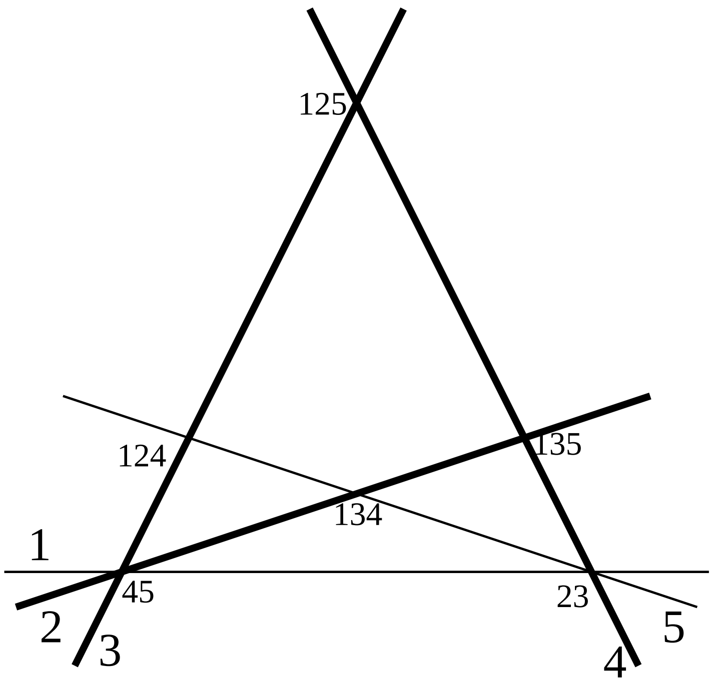
$$t(M;x,y) = \sum_{\substack{F \text{ flat of } M \\ E\setminus F \text{ flat of } M^*}} t(M/F;x,0)\, t(M(F);0,y)$$

• Activities of reorientations
F = union of positive circuits of M
F^* = union of positive cocircuits of M

Proposition. $E = F + F^*$ ('Farkás lemma')
M/F is acyclic, i.e. $o(M/F) = 0$,
and $M(F)$ is totally cyclic, i.e. $o^*(M(F)) = 0$.

Geometrical interpretation : F^* corresponds to the intersection of halfspaces $+$.

- Activities of bases



Base $234 = 23 \cup 4$

Base 23 de $M(123)$:
internal act. 0

Base 4 de $M/123$:
external act. 0

234	1	2	3	4	5
1	x				
2	x	x			x
3	x		x		x
4				x	x
5					x

23	1	2	3
1	x		
2	x	x	
3	x		x

4	4	5
4	x	x
5		x

Second decomposition of activities

from $(i, 0)$ to $(1, 0)$ activities

decomposing sequences of a matroid:

$$\emptyset = F'_\varepsilon \subset ... \subset F'_0 = F = F''_0 \subset ... \subset F''_\iota = E$$

Such a sequence is asociated to any basis or re-orientation.

Theorem (Gioan, Las Vergnas 2002)

$$t(M; x, y) = \sum_{\substack{\emptyset = F'_\varepsilon \subset \dots \subset F'_0 = F \\ F = F''_0 \subset \dots \subset F''_\iota = E \\ \text{decomposing}}} \Big(\prod_{1 \leq k \leq \iota} \beta(M(F'_k)/F'_{k-1}) \Big) \Big(\prod_{1 \leq k \leq \varepsilon} \beta(M(F''_{k-1})/F''_k) \Big) \; x^\iota y^\varepsilon$$

- Activities of reorientations

$a_1 < ... < a_\ell$ the dual-orientation active elements of M

for $0 \leq i \leq \ell - 1$,
F_i = union of positive cocircuits with smallest element $a \geq a_{i+1}$

$$\emptyset = F_\ell \subset F_{\varepsilon-1} \subset ... \subset F_1 \subset F_0 = E$$

$M_i = M(E \setminus F_i)/(E \setminus F_{i-1})$

$Min(F_{i-1} \setminus F_i) = a_i$.

Proposition.
For $1 \leq i \leq \ell$, $o^*(M_i) = 1$, $o(M_i) = 0$

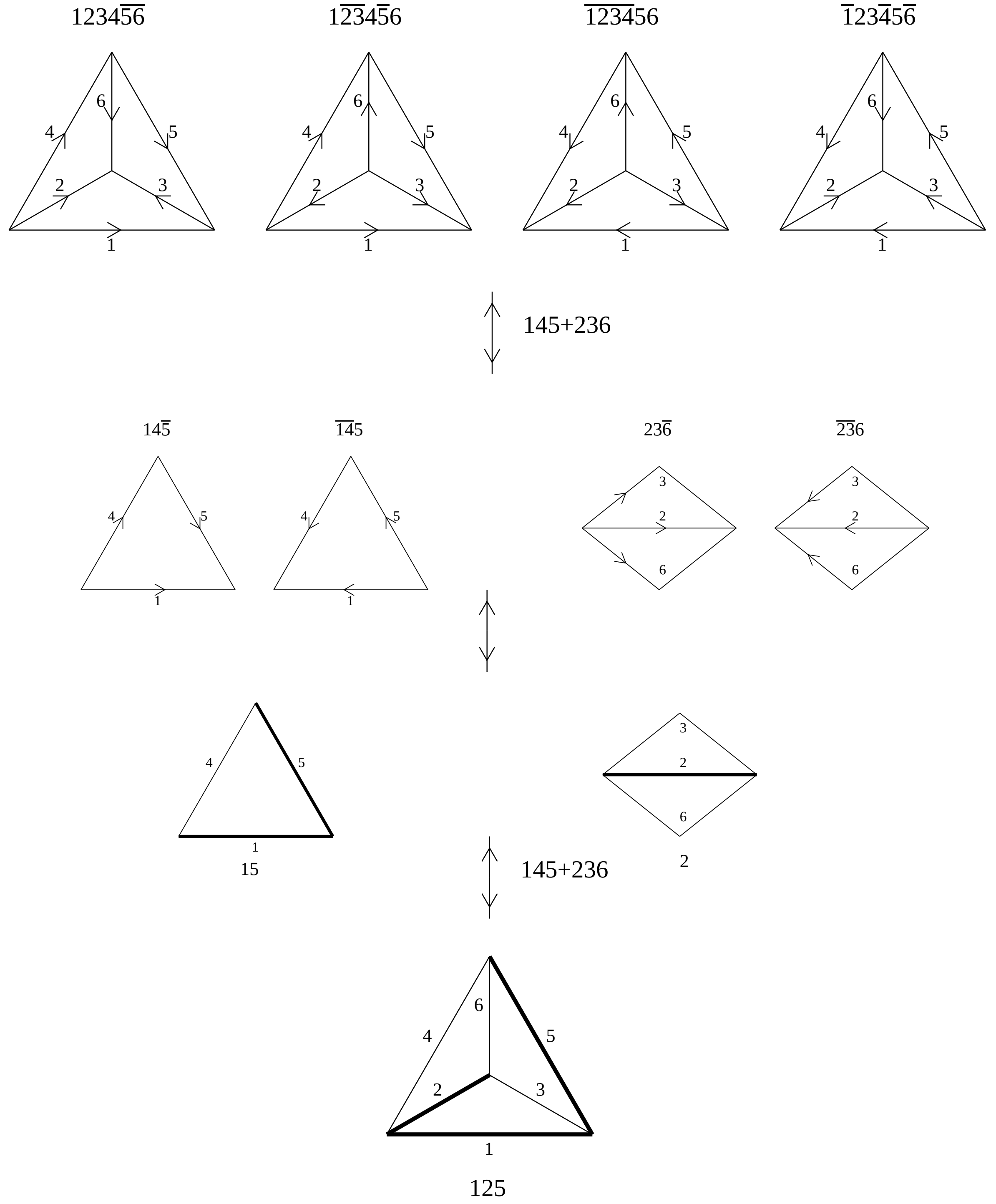
activity class of M: the set of 2^ℓ reorientations obtained by reorienting the ℓ subsets, they all have same decomposing sequence

- Activities of bases : similar but more technical

Extension theorem

Example. Base 125 of K_4

two minors $M(145)$ and $M/145$.



Fundamental bijection for $(1,0)$ activities

- (1,0)-active reorientations: bounded regions
- (1,0)-active bases.

- the smallest element of a line (except the first) belongs to a previous line.

- the smallest element of a row belongs to a previous row.

135	1	2	3	4	5	6
1	x	x				x
2		x				
3		x	x	x		x
4				x		
5		x		x	x	x
6						x

- From (1,0)-bases to bounded regions.

the reorientation $\phi(B)$ is chosen so that, in the fundamental tableau of B in $\phi(B)$:

- the minimal element of each row is +
- the minimal element of each line is -

(except the first)

135	1	2	3	4	5	6
1	+	+				+
2		-				
3		-	+	+		x
4				-		
5		-		x	+	x
6						-

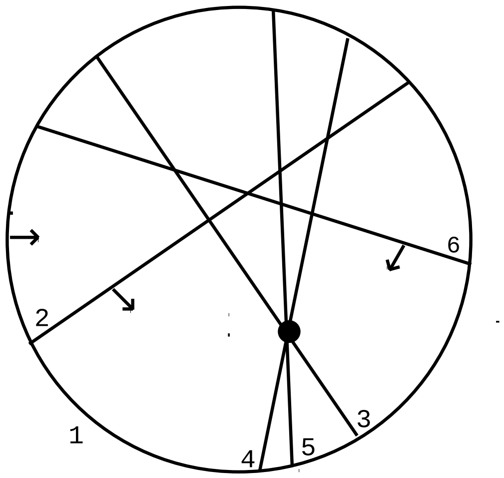
<u>two dual algorithms</u>: consider lines, or rows, step by step, each unsigned element is signed opposite with the smallest element (except first line).

Proposition. *We get a bounded region.*

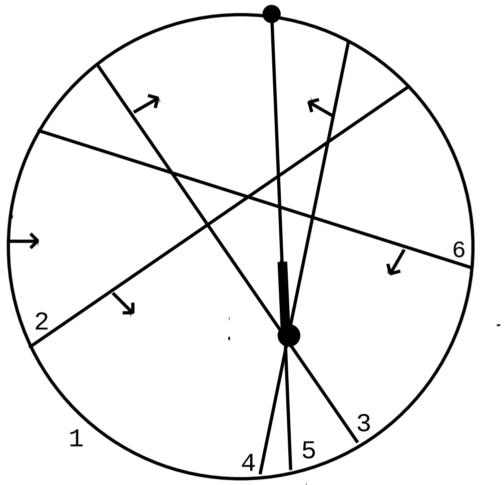
Examples.

Base 135

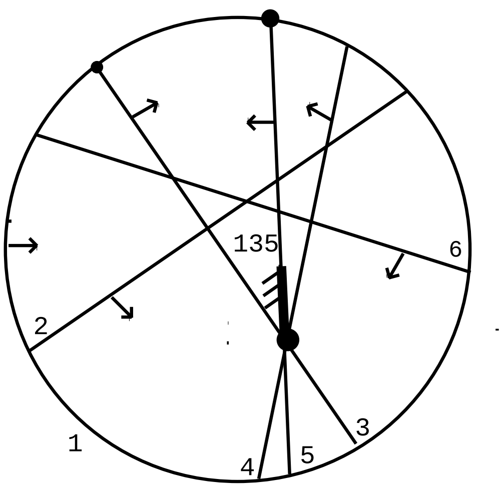
Step 1

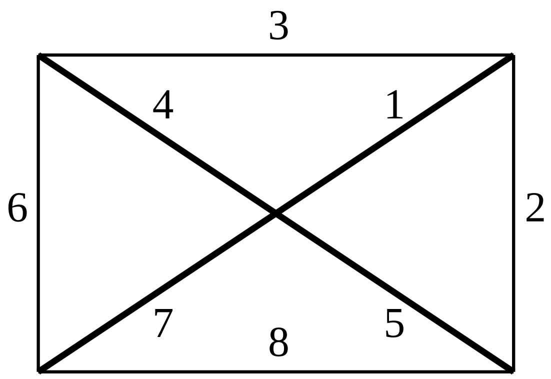


Step 2



Step 3





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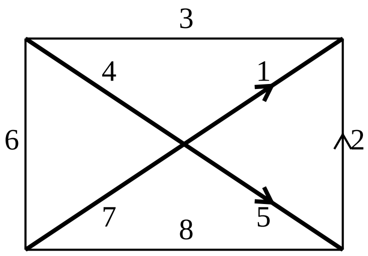
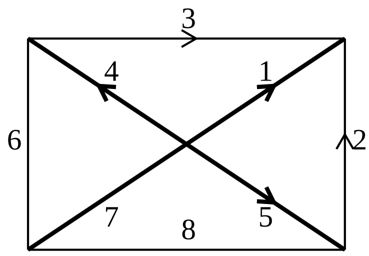
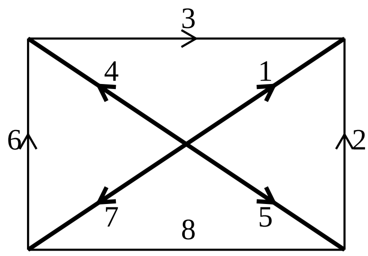


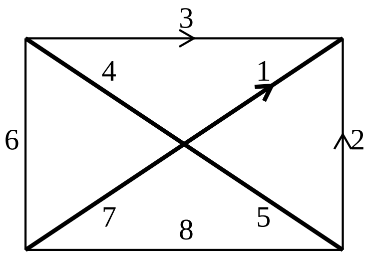
(1.1)



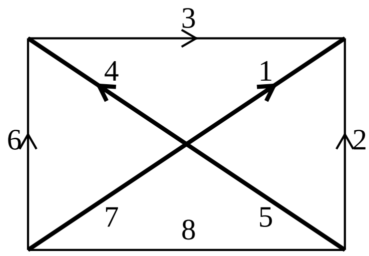
(1.2)



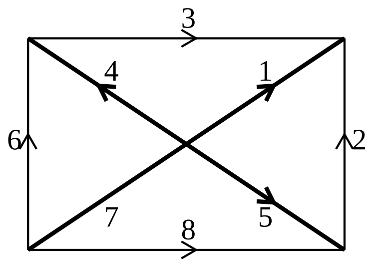
(1.3)



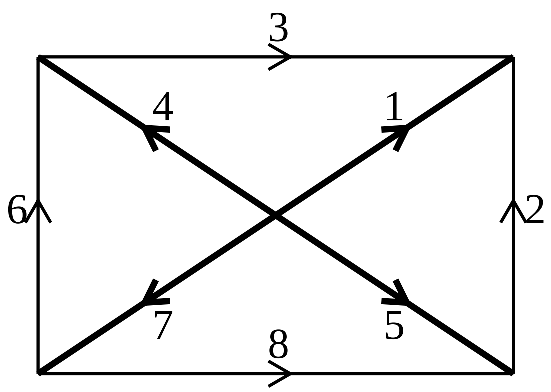
(2.1)



(2.2)



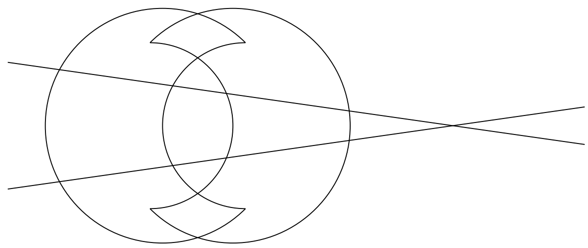
(2.3)



(1.4) = (2.4)

Main theorem. (Gioan, Las Vergnas 2002)
This application defines a bijection.

Laconic proof : impossible figure



• From bounded regions to $(1, 0)$-bases.

$\phi^{-1}(M)$ is the *optimal base* of the bounded region M, it is the <u>unique</u> (1,0)-basis whose fundamental tableau satisfies the required sign properties.

- inductive algorithm by deletion/contraction of the greatest element

- extensions of linear programming

EXTENSIONS OF LINEAR PROGRAMMING

In usual linear programming, the first cocircuit (first line) is *optimal* if it is positive and the first fundamental circuit (first row) is negative (except for the minimal elements)

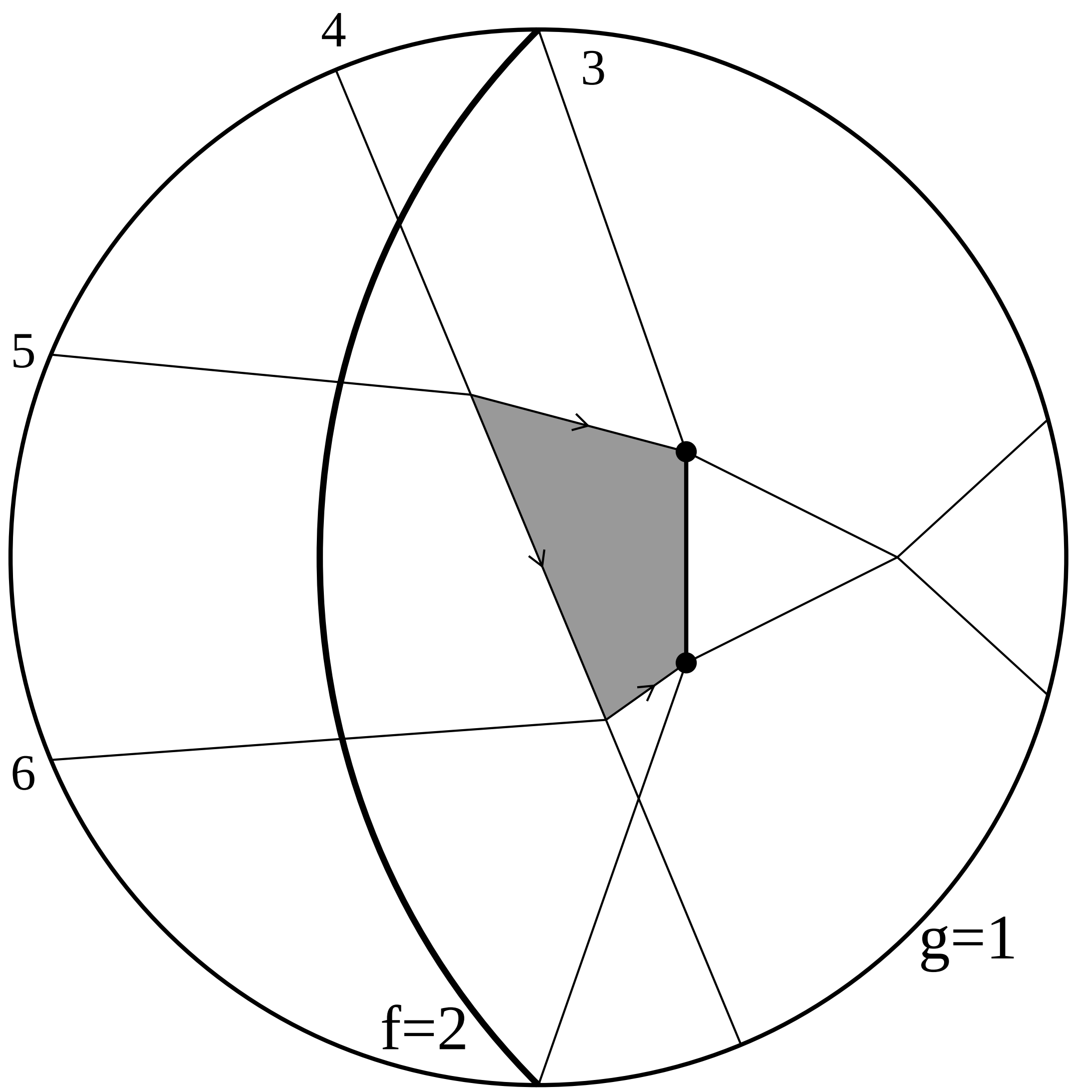
Here, we consider ALL lines, and ALL rows: we optimize a sequence of nested faces (all lines), with respect to a sequence of objective functions (all rows)...

135	1	2	3	4	5	6
1	+	+				+
2		-				
3		-	+	+		x
4				-		
5		-		x	+	x
6						-

... and we get a bijection.

First example of oriented matroid program

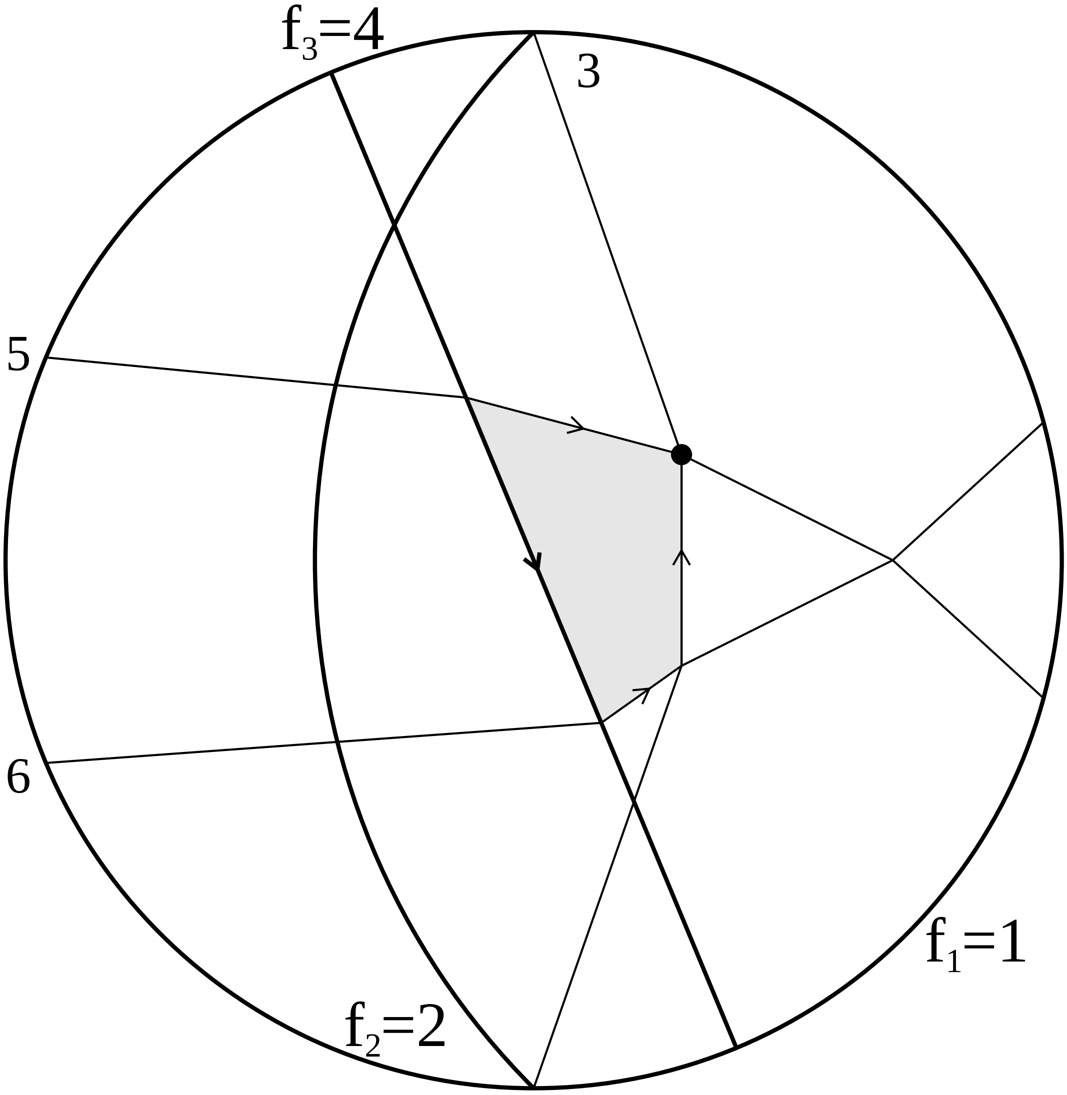
We optimize with respect to one objective function (here f) and get two optimal vertices



First extension of oriented matroid program

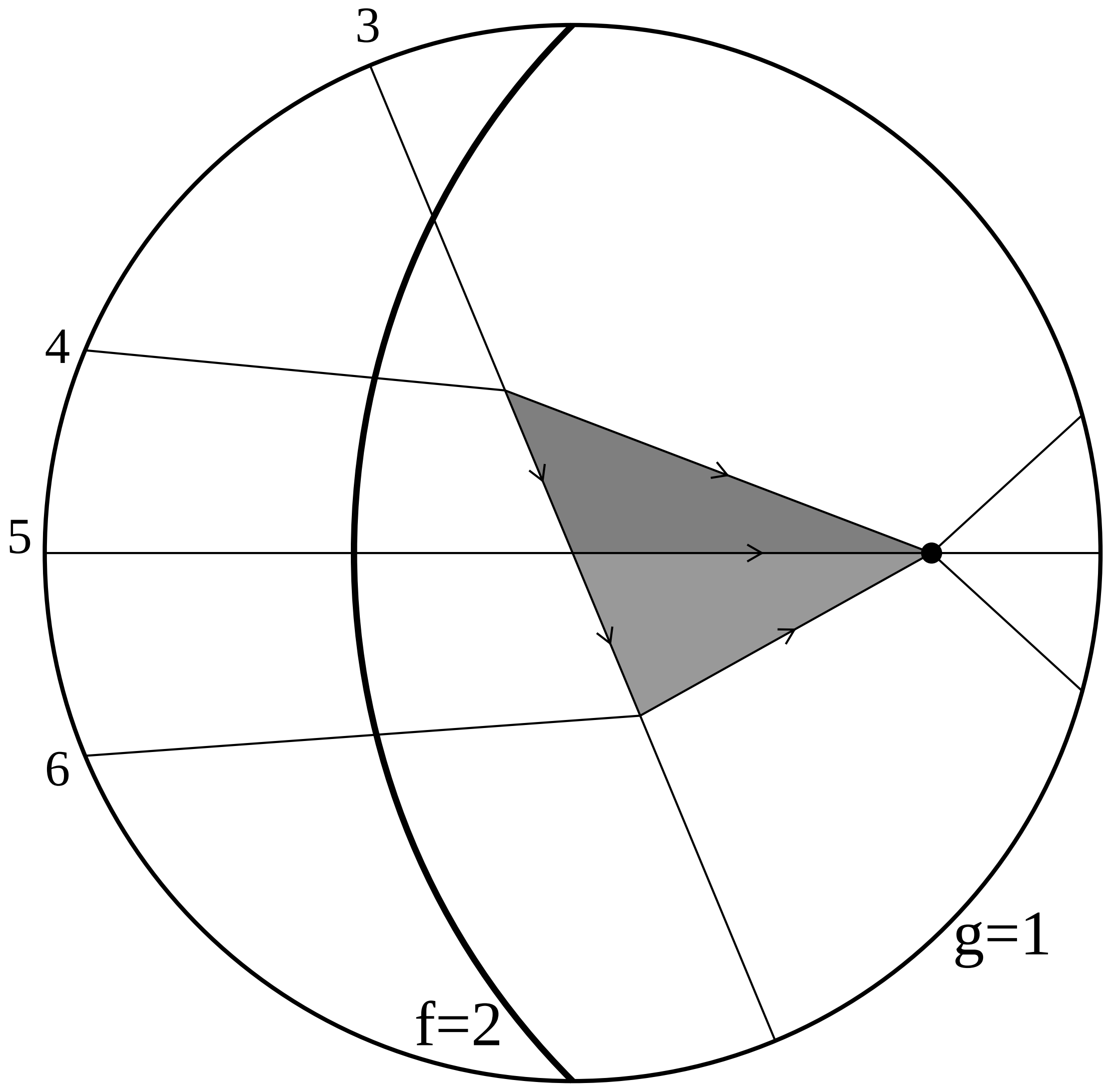
multiobjective programming

We optimize with respect to $r-1$ objective functions (here f_2, then f_3), and get a unique optimal vertex.



Second example of oriented matroid program

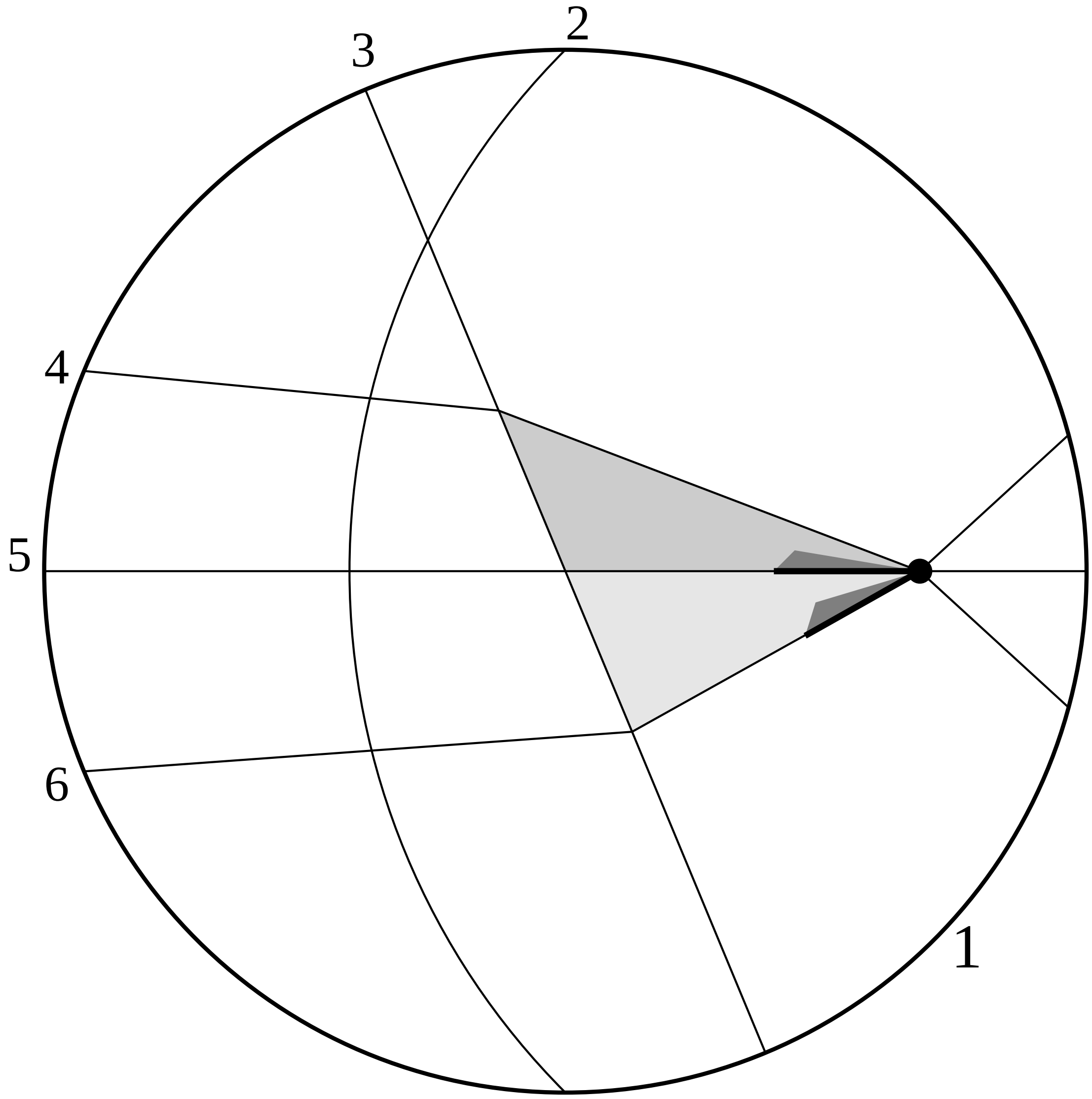
We get the same optimal vertex for two distinct regions.



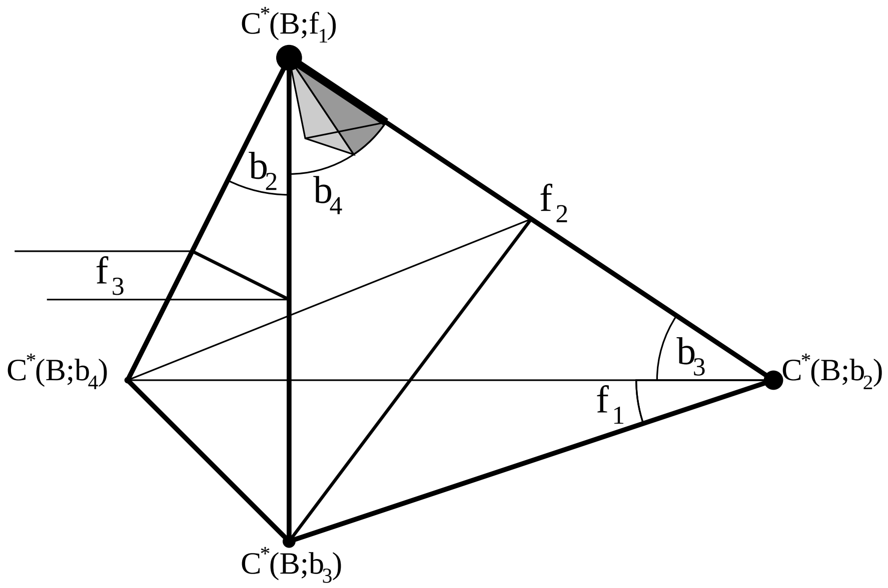
Second extension of oriented matroid program

flag programming

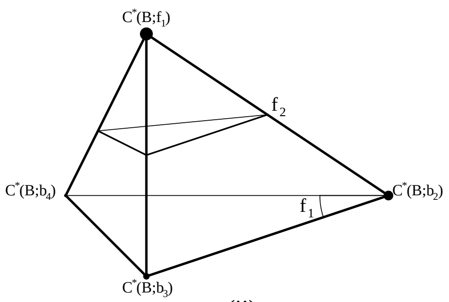
The ordered $(1,0)-$base associated with a region defines an optimal sequence of nested faces, there is a distinct solution in the two regions.



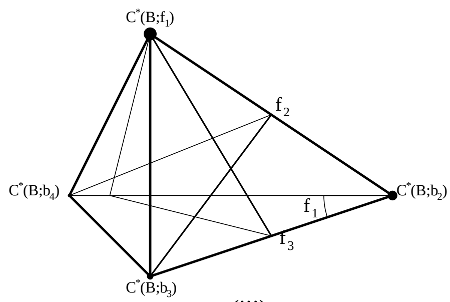
Some situations in rank 4



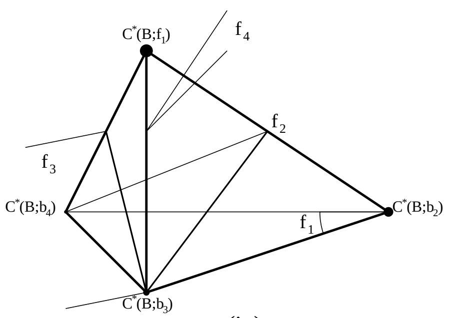
(i)



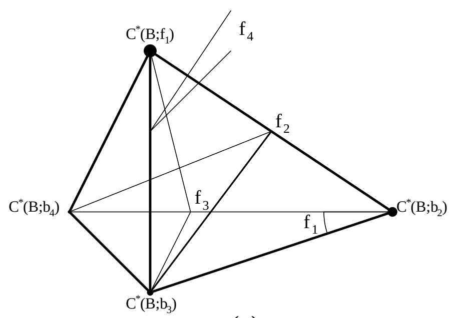
(ii)



(iii)

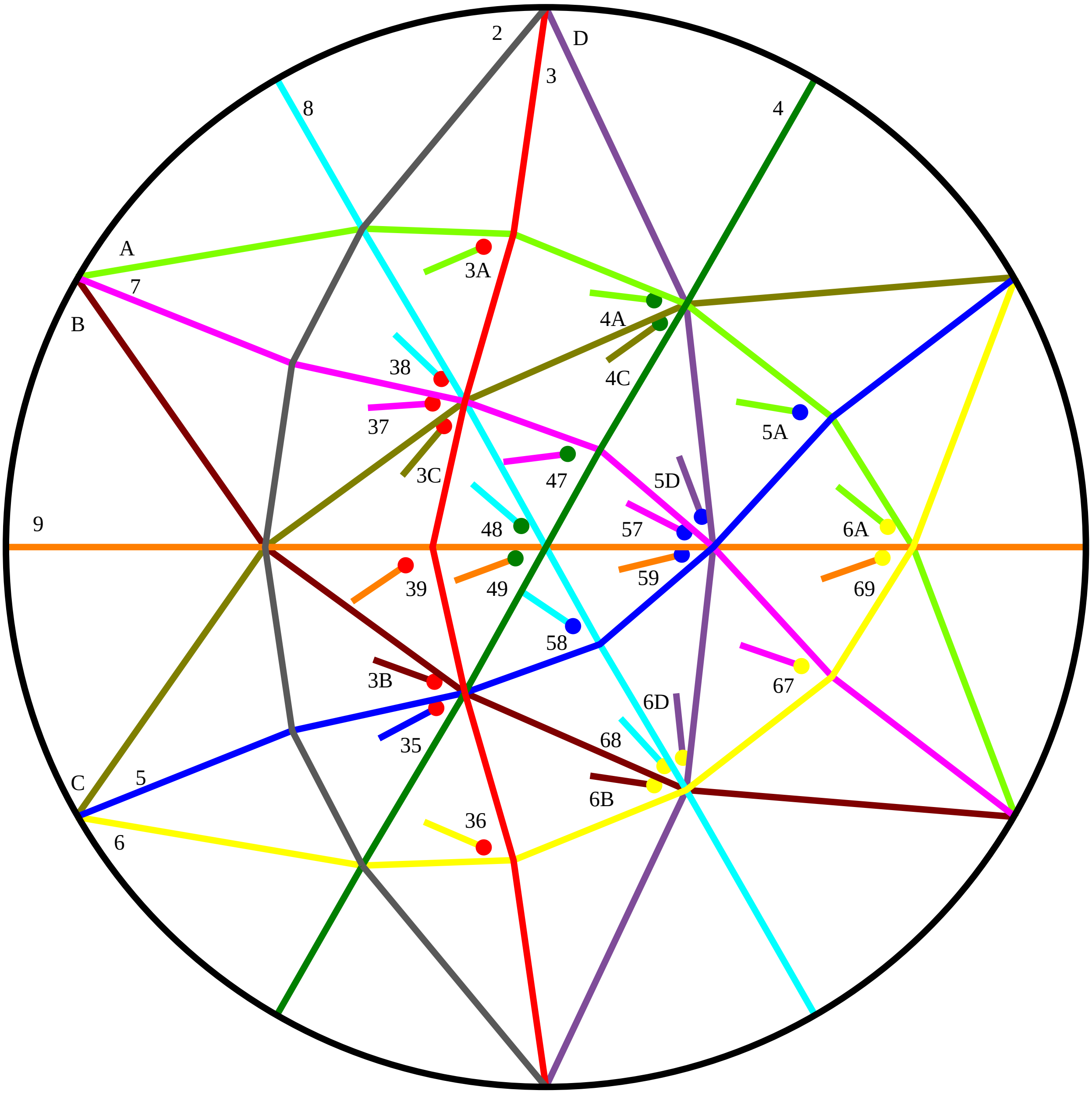


(iv)



(v)

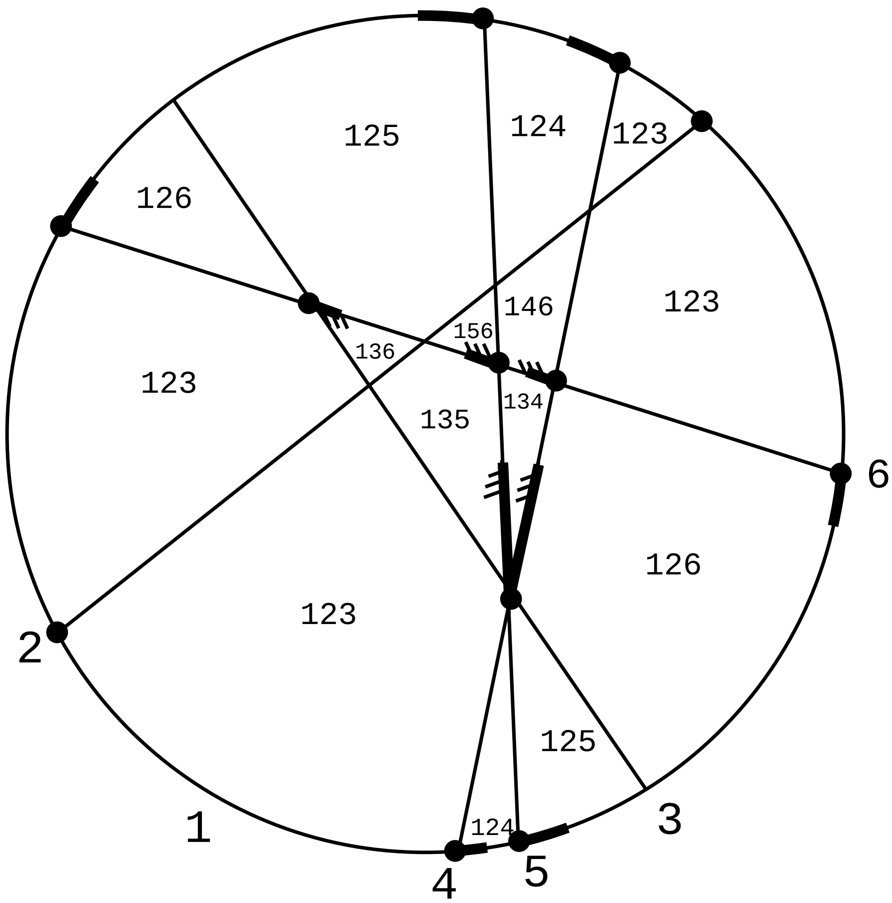
Canonical (attr)active correspondence



phenomenon of attraction with respect to the linear ordering

Active bijections

M matroid, G graph



$(1,0)$ bases $\leftrightarrow$ bounded regions of M,

bipolar orientations of G

$(i,0)$ bases $\leftrightarrow$ activity classes of regions of M, acyclic orientations with unique given sink of G

$(i,0)$ bases $\leftrightarrow$ subsets with no broken circuit $\mathcal{NBC}$

(i,j) bases $\leftrightarrow$ activity classes of (re)orientations