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EuroComb 07

Fully Optimal Bases and the Active Bijection in Graphs, Hyperplane Arrangements, and Oriented Matroids

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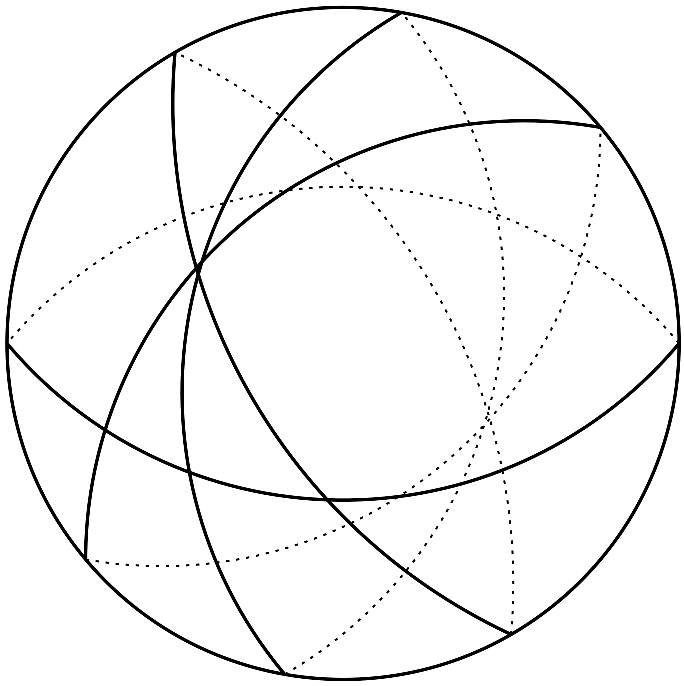
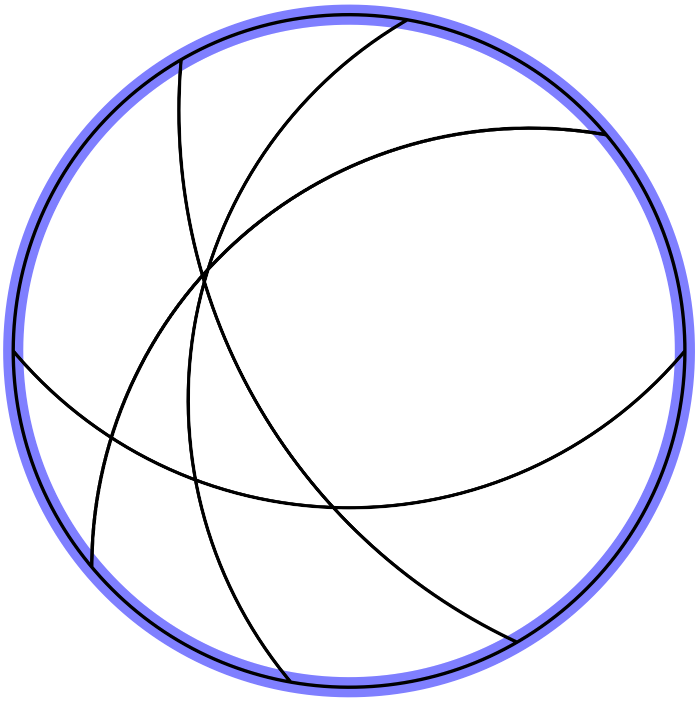
A FAMOUS CURIOUS PROPERTY

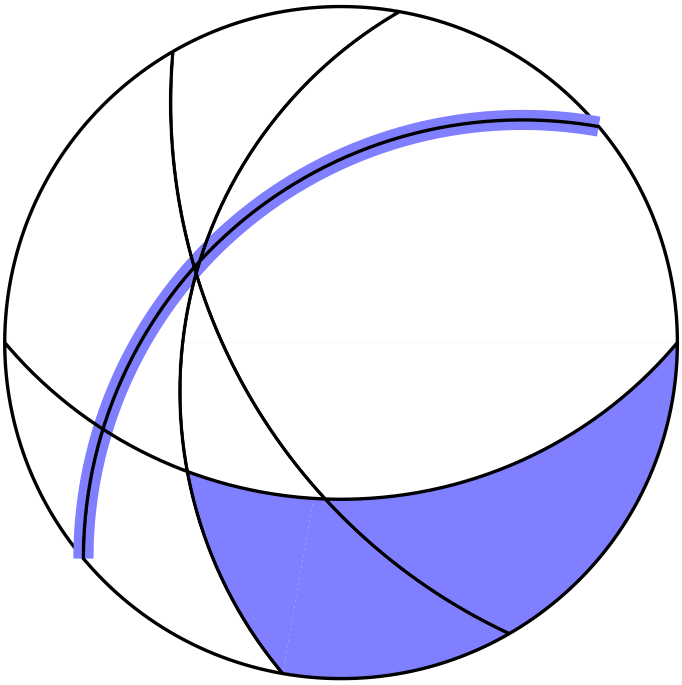
Hyperplane arrangement, represented by its intersection with a central sphere:



two symmetric halfspheres delimited by one the hyperplanes

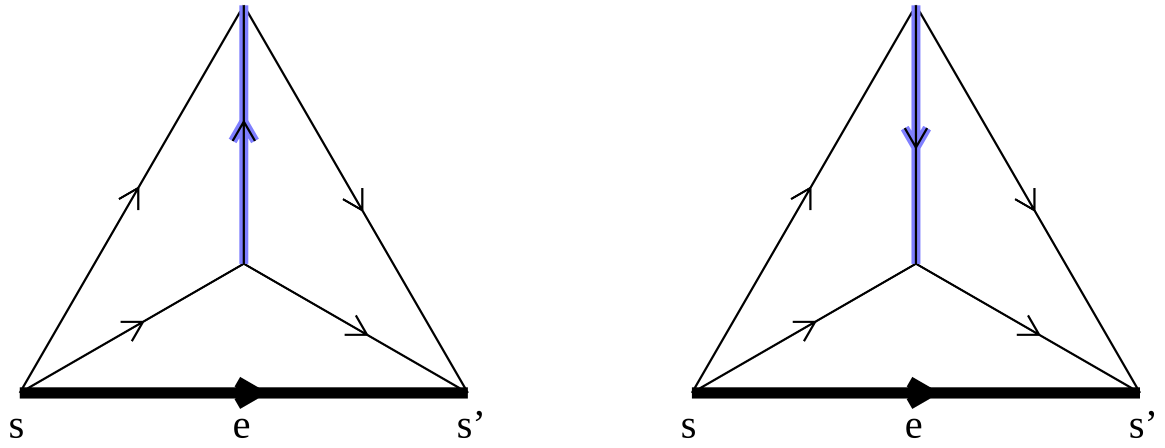


Consider the number of regions that do not touch a given hyperplane (on a given side): *bounded regions* w.r.t. the hyperplane chosen as *hyperplane at infinity*



This number does not depend on the chosen hyperplane!

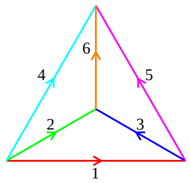
A *bipolar orientation* of a graph w.r.t. two adjacent vertices (s, s') is an acyclic orientation with unique source s and unique sink s'



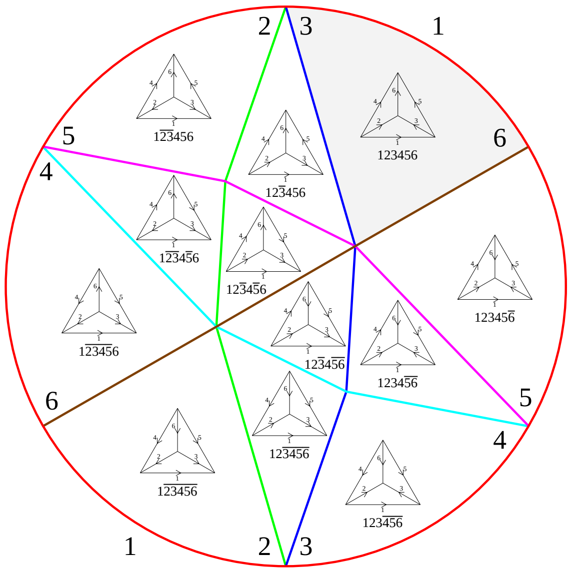
The number of bipolar orientations does not depend on the choice of the edge $e = (s, s')$.

A FAMOUS CURIOUS PROPERTY

The property in graphs is a particular case of the property in hyperplane arrangements



$\longrightarrow$



edge $v_i v_j \longrightarrow$ hyperplane $x_i - x_j = 0$
spanning trees $\longrightarrow$ bases
acyclic orientations $\longrightarrow$ regions
bipolar orientations $\longrightarrow$ bounded regions

A FAMOUS CURIOUS PROPERTY

More generally the property is true in oriented matroids.

Theorem [Zaslavsky 75, Las Vergnas 77]
The number of bounded regions of an oriented matroid (with no loop nor isthmus), w.r.t. to a given element e, on the positive side on e, does not depend on e and equals

$$\beta(M) = t_{1,0}(M)$$

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$$\beta(M) = t_{1,0}(M)$$

This number is the coefficient of x (or y) in the Tutte polynomial of M:

$$t(M; x, y) = \sum_{i,j} t_{i,j} x^i y^j$$

Theorem [Tutte 54]
If the ground set of M is linearly ordered, then the number of (i, j)-active bases w.r.t. the linear ordering is an invariant and equals $t_{i,j}$.

Let M be an ordered oriented matroid on E
(or an ordered hyperplane arrangement E, or a graph with an ordered set of edges E).

The ground set $E = e_1 < ... < e_n$ is linearly ordered,
and a bounded region (or a bipolar orientation) is always thought of w.r.t. e_1.

The *(bounded) active bijection* of M is a bijective mapping

from the set of bounded regions w.r.t. e_1

onto the set of $(1,0)$-active bases w.r.t. $(E,<)$.

This bijection has a very simple definition...

...but, first, what is a $(1,0)$-active basis?

Let B be a basis of M.

C_e = *fundamental circuit* of $e \notin B$ w.r.t. B = unique circuit in $B \cup e$

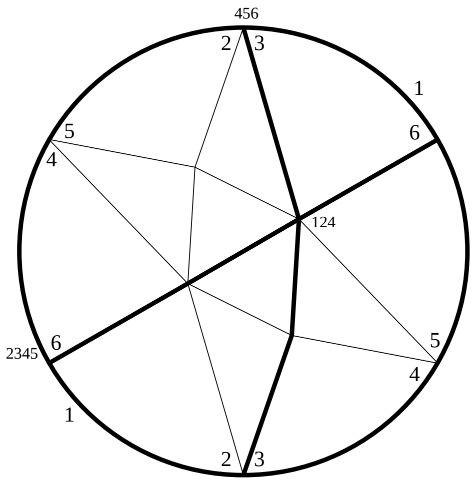
C_b^* = *fundamental cocircuit* of $b \in B$ w.r.t. B = unique cocircuit in $(E \setminus B) \cup b$

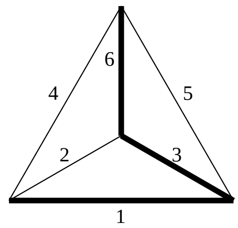
fundamental tableau of B = matrix $n \times n$ on $\{0, \mathrm{x}\}$ with

C_e as non-zero elements of row $e \notin B$,

C_b^* as non-zero elements of column $b \in B$

Ex. Fundamental tableau of basis 136





	C_1^*	2	C_3^*	4	5	C_6^*
1	x					
C_2	x	x	x			
3			x			
C_4	x		x	x		x
C_5			x		x	x
6						x

THE ACTIVE BIJECTION IN THE BOUNDED CASE

Assume the ground set E of M is ordered: $E = e_1 < ... < e_n$

B is $(1,0)$-*active* if

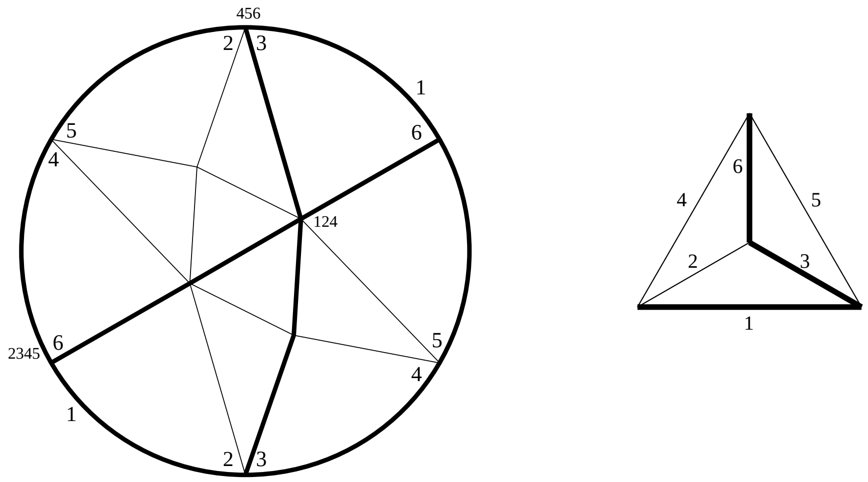
the smallest non-zero element of a row is non-zero in a previous one

i.e. $min\ C_e \in \cup_{\ a<e,\ a\notin B}\ C_a$

the smallest non-zero element of a column is non-zero in a previous one (except first column)

i.e. $min\ C^*_b \in \cup_{\ a<b,\ a\in B}\ C^*_a$ for $b \neq e_1$

Ex. Basis 136 is $(1,0)$-active



	C_1^*	2	C_3^*	4	5	C_6^*
1	x					
C_2	x	x	x			
3			x			
C_4	x		x	x		x
C_5			x		x	x
6						x

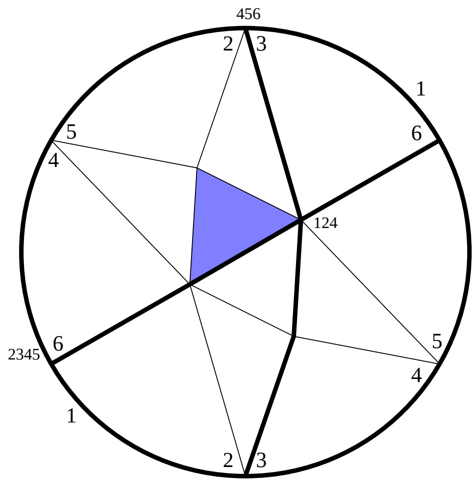
If M is an **oriented** matroid, or a **signed** arrangement, or a **directed** graph,

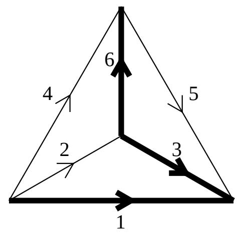
signed *fundamental tableau* of B = matrix $n \times n$ on $\{0, +, -\}$ with

C_e as non-zero elements of row $e \notin B$, with its signs in $\{+, -\}$,
and (by convention) e signed $-$

C^*_b as non-zero elements of column $b \in B$, with its signs in $\{+, -\}$,
and (necessarily) b signed $+$

Ex. Signed fundamental tableau of 136, w.r.t. to the blue region.





	C_1^*	2	C_3^*	4	5	C_6^*
1	$+$					
C_2	$+$	$-$	$-$			
3			$+$			
C_4	$+$		$-$	$-$		$+$
C_5			$+$		$-$	$-$
6						$+$

THE ACTIVE BIJECTION IN THE BOUNDED CASE

Let M be an ordered oriented matroid (or signed arrangement, or directed graph).

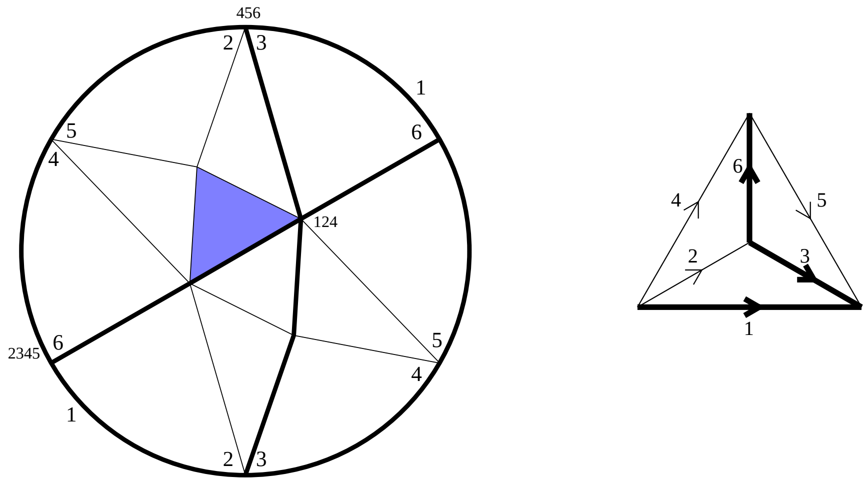
A basis B of M is *fullly optimal* in M if

the smallest element of each row is signed $+$

the smallest element of each column is signed $-$ (except the first).

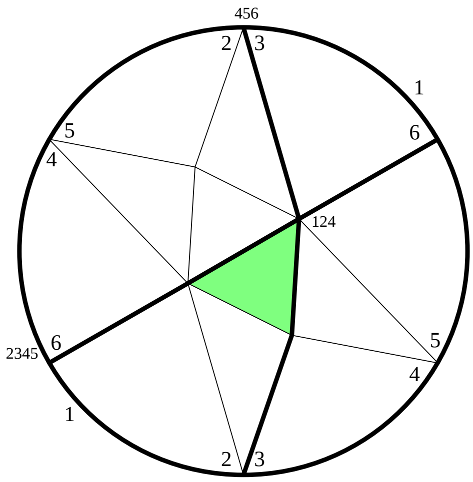
Rk. This implies M is bounded acyclic and B is $(1, 0)$-active

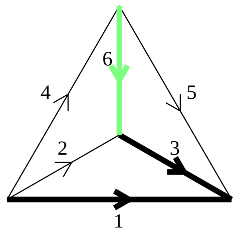
Ex. The basis 136 is not fully optimal w.r.t. to the blue region.



	C_1^*	2	C_3^*	4	5	C_6^*
1	$+$					
C_2	$+$	$-$	$-$			
3			$+$			
C_4	$+$		$-$	$-$		$\oplus$
C_5			$+$		$-$	$-$
6						$+$

Ex. The basis 136 is fully optimal w.r.t. to the green region (obtained by reversing 6).





	C_1^*	2	C_3^*	4	5	C_6^*
1	+					
C_2	+	−	−			
3			+			
C_4	+		−	−		−
C_5			+		−	+
6						+

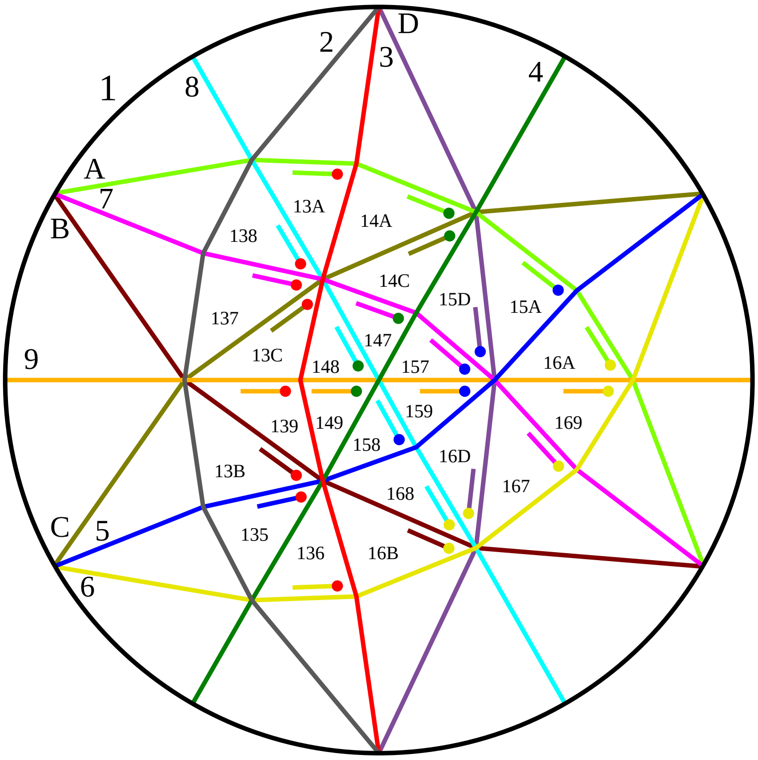
THE ACTIVE BIJECTION IN THE BOUNDED CASE

Main Theorem.

A bounded acyclic ordered oriented matroid M has **one and only one** *fully optimal basis, denoted by $\alpha(M)$.*

The mapping α is a bijection between bounded regions of M and $(1,0)$-active bases of M.

In particular, a bounded region of an ordered hyperplane arrangement, or a bipolar orientation of an ordered graph, has a unique fully optimal basis.


1
2
3
D
4
8
A
7
B
9
C
5
6
13A
14A
138
14C
15D
15A
137
147
13C
148
157
16A
139
149
159
169
158
13B
16D
167
168
135
136
16B

Construction of the bijection

From bases to regions: just sign successively the elements one by one the good way!

From regions to bases: linear programming refinements...

Rk. There exist a deletion/contraction construction, and other characterisitc properties.

Linear programming refinements

In usual linear programming, the vertex intersection of $B \setminus e_1$ is optimal

if and only if

in the signed tableau of B

the first column $C^*_{e_1}$ is positive

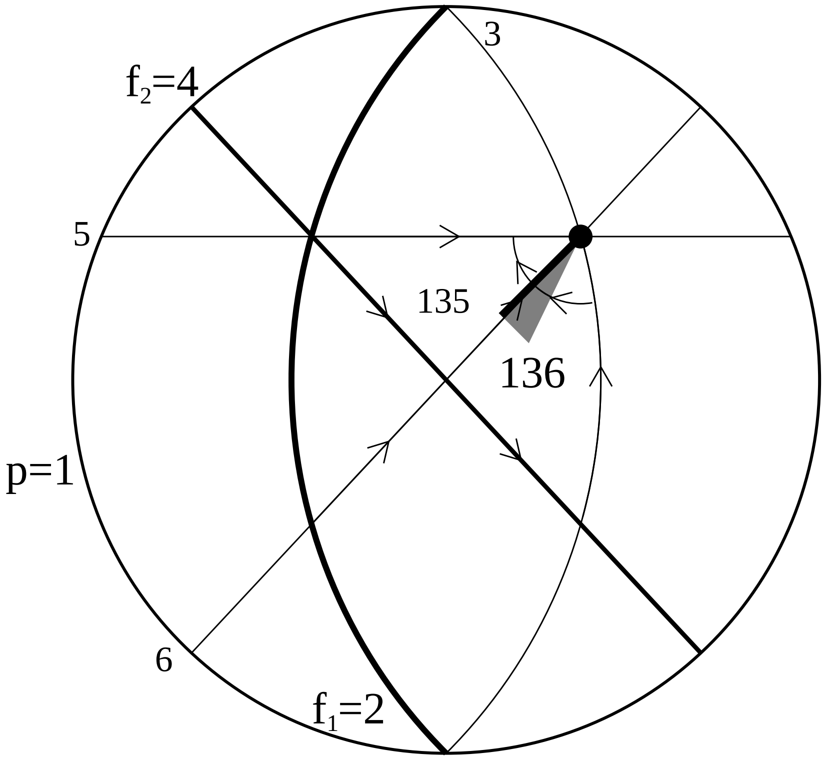
the second line C_{e_2} is negative (except on e_1)

Here we take into account the whole fundamental tableau, i.e.

all lines: multiobjective programming (instead of one objective function)

all columns: flag optimization (instead of one optimal face)

Ex. The region $\alpha^{-1}(136)$ has one fully optimal basis
but it has two optimal vertices in usual LP
and the same optimal vertex as the region $\alpha^{-1}(135)$



basis 136

	1	2	3	4	5	6
1	+					
2	+	-	-			
3			+			
4	+		-	-	-	
5					+	
6			+		-	-

THE ACTIVE MAPPING IN THE GENERAL CASE

Th [Tutte 54]

$$t(M;x,y) = \sum_{i,j} b_{i,j} x^i y^j$$

where $b_{i,j} = \#\ (i,j)$-active bases

Th [Las Vergnas 84]

$$t(M;x,y) = \sum_{i,j} o_{i,j} (\frac{x}{2})^i (\frac{y}{2})^j$$

where $o_{i,j} = \#\ (i,j)$-active reorientations

$$o_{i,j} = 2^{i+j} b_{i,j}$$

THE ACTIVE MAPPING IN THE GENERAL CASE

The *active mapping* α maps an ordered oriented matroid onto one of its bases.

It is defined by

1) $\alpha(M)$ is the fully optimal basis of M if it is bounded acyclic

2) $\alpha(M^*) = E \setminus \alpha(M)$

3) $\alpha(M) = \alpha(M/A) \ \cup \ \alpha(M(A))$
where A is the union of all positive circuits of M whose smallest element is the greatest possible minimal element of a positive cocircuit of M.

For a given oriented matroid,
we get a $2^{i+j} - 1$ activity preserving correspondence
between all orientations and all bases

and, more specifically,

an activity preserving bijection
between all subsets (related to bases) and all orientations

between no-broken-circuit subsets and acyclic orientations

$$\alpha(M) \quad = \quad \uplus_{1\leq k\leq \iota} \;\; \alpha(M(F''_k)/F''_{k-1}) \;\; \biguplus \;\; \uplus_{1\leq k\leq \iota} \;\; \alpha(M(F'_{k-1})/F'_k)$$

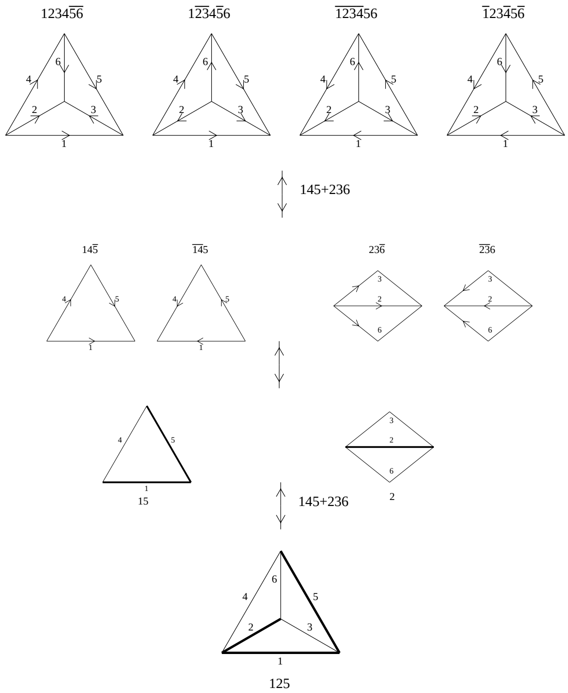
active decomposing sequence of M:

$$\emptyset = F'_\varepsilon \subset ... \subset F'_0 = F_c = F''_0 \subset ... \subset F''_\iota = E$$

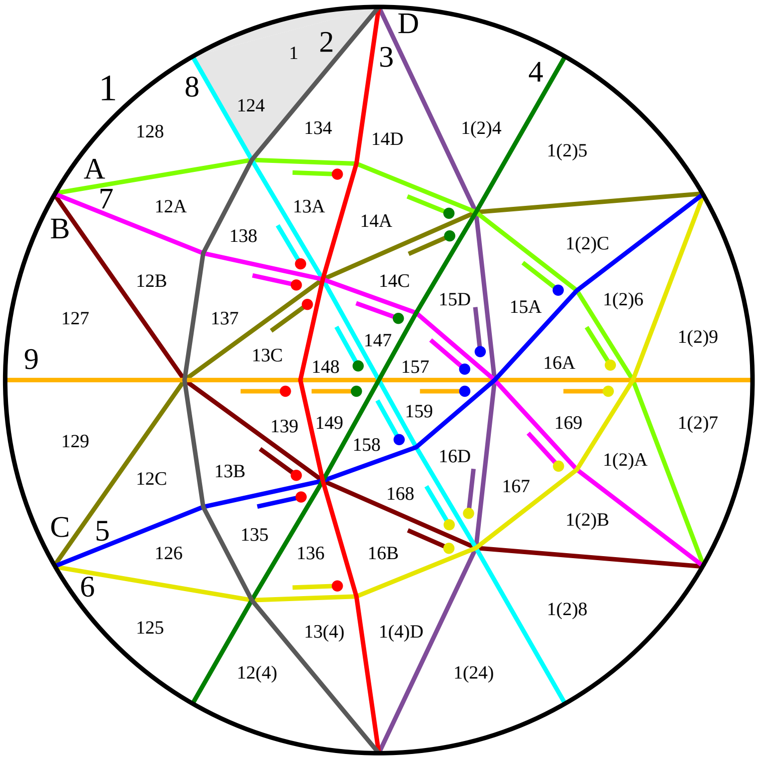
Theorem

$$t(M;x,y) = \sum \left(\prod_{1\leq k\leq \iota} \beta\big(M(F'_k)/F'_{k-1}\big)\right) \left(\prod_{1\leq k\leq \varepsilon} \beta\big(M(F''_{k-1})/F''_k\big)\right) x^\iota \; y^\varepsilon$$

where the sum is over all active decomposing sequences of bases of M



$1234\bar{5}\bar{6}$
$1\bar{2}\bar{3}4\bar{5}6$
$\bar{1}\bar{2}\bar{3}\bar{4}56$
$\bar{1}23\bar{4}5\bar{6}$
145+236
$14\bar{5}$
$\bar{1}\bar{4}5$
$23\bar{6}$
$\bar{2}\bar{3}6$
15
2
145+236
125



1
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D
4
8
1
124
134
14D
1(2)4
1(2)5
128
A
7
B
12A
13A
14A
138
1(2)C
12B
14C
15D
15A
1(2)6
127
137
147
1(2)9
9
13C
148
157
16A
129
139
149
159
169
1(2)7
158
16D
1(2)A
12C
13B
168
167
C
5
1(2)B
126
135
136
16B
6
1(2)8
125
13(4)
1(4)D
12(4)
1(24)

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Sum up

structure	active bijection	
oriented matroids	activity classes of reorientations	bases
	act. cl. of acyclic reorientations	internal bases
	act. cl. of totally cyclic reor.	external bases
	bounded acyclic reorientations	$(1,0)$-active bases
	reorientations	subsets
	acyclic reorientations	no-broken-circuit subsets
hyperplane	reorientations = signatures	bases = simplices
arrangements	acyclic reorientations = regions	
graphs	reorientations = orientations	bases = spanning trees
	unique sink acyclic orientations	internal spanning trees
	bipolar orientations	$(1,0)$-active spanning trees
uniform o.m.	bounded regions	LP optimal vertices
supersolvable A_n	permutations	increasing trees
supersolvable B_n	signed permutations	signed increasing trees