Dynamic Distance Hereditary Graphs using Split Decomposition

Emeric Gioan

CNRS - LIRMM - Université Montpellier II, France

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Joint work with C. Paul (CNRS - LIRMM)
Dynamic graph representation problem:

Given a representation $R(G)$ of a graph $G$ and a edge or vertex modification of $G$ (insertion or deletion) update the representation $R(G)$. 
Dynamic graph representation problem:

Given a representation \( R(G) \) of a graph \( G \) and a edge or vertex modification of \( G \) (insertion or deletion) update the representation \( R(G) \).

When restricted to a certain graph family \( \mathcal{F} \), the algorithm should:

1. check whether the modified graph still belongs to \( \mathcal{F} \);
2. if so, update the representation;
3. otherwise output a certificate (e.g. a forbidden subgraph).
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Some keys of the problem

Need of a canonical representation (decomposition techniques...) and need of an incremental (dynamic) characterization.
## Some known results

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<td>proper intervals</td>
<td>$O(d + \log n)$ [HSS02]</td>
<td>$O(1)$ [HSS02]</td>
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<td>$O(d)$ [CoPeSt85]</td>
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<td>$O(n)$ [CrPa05]</td>
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<td>distance hereditary</td>
<td>$O(d)$ [GPa07]</td>
<td>$O(1)$ [CoT07]</td>
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</table>

HSS = Hell, Shamir, Sharan  
CoPeSt = Corneil, Perl, Stewart  
SS = Shamir, Sharan  
CrPa = Crespelle, Paul  
GPa = Gioan, Paul  
CoT = Corneil, Tedder  
Cr = Crespelle
1. Revisiting split decomposition

2. Vertex modification of DH graphs

3. Relations with other works
A graph-labelled tree is a pair \((T, \mathcal{F})\) with \(T\) a tree and \(\mathcal{F}\) a set of graphs such that:

- each (internal) node \(v\) of degree \(k\) of \(T\) is labelled by a graph \(G_v \in \mathcal{F}\) on \(k\) vertices
- there is a bijection \(\rho_v\) from the tree-edges incident to \(v\) to the vertices of \(G_v\)
Given a graph labelled tree \((T, \mathcal{F})\), the *accessibility graph* \(G_S(T, \mathcal{F})\) has the leaves of \(T\) as vertices and

- \(xy \in E(G_S(T, \mathcal{F}))\) if and only if \(\rho_v(uv)\rho_v(vw) \in E(G_v)\),
  \(\forall\) tree-edges \(uv, vw\) on the \(x, y\)-path in \(T\)
A split is a bipartition \((A, B)\) of the vertices of a graph \(G = (V, E)\) such that
- \(|A| \geq 2, |B| \geq 2;\)
- for \(x \in A\) and \(y \in B\), \(xy \in E\) iff \(x \in N(B)\) and \(y \in N(A)\).
A **split** is a bipartition \((A, B)\) of the vertices of a graph \(G = (V, E)\) such that

- \(|A| \geq 2, |B| \geq 2;\)
- for \(x \in A\) and \(y \in B\), \(xy \in E\) iff \(x \in N(B)\) and \(y \in N(A)\).
A graph is *prime* if it has no split. The stars and cliques are called *degenerate*.
Split decomposition [Cunningham’82 reformulated]

For any connected graph $G$, there exists a unique graph-labelled tree $(T, \mathcal{F})$ with a minimum number of nodes such that

1. $G = G_S(T, \mathcal{F})$,
2. any graph of $\mathcal{F}$ is prime or degenerate for the split decomposition.

→ We note $(T, \mathcal{F}) = ST(G)$ the split tree of $G$
Distance hereditary graph

A graph is *distance hereditary* if and only if it is totally decomposable for the split decomposition, i.e. its split tree is labelled by cliques and stars.
An intersection model for DH graphs [Gioan and Paul ’07]

The *accessibility set* of a leaf \( a \) in a clique-star labelled tree is the set of paths \((a, b)\) with \( b \) a leaf accessible from \( a \).

A distance hereditary graph is the intersection graph of a family of accessibility sets of leaves in a set of clique-star labelled trees.

*answers a question by Spinrad*
Particular case of cographs

The cographs form the particular case where the centers of all stars are directed towards a root of the split tree.

\[
1 = \text{clique} \\
0 = \text{stable}
\]

\[
1 = \text{clique} \\
(\text{except root}) \\
0 = \text{star} \\
(\text{towards root})
\]

E. Gioan
A subset of vertices $M$ of a graph $G = (V, E)$ is a **module** iff

$$\forall x \in V \setminus M, \text{ either } M \subseteq N(x) \text{ or } M \cap N(x) = \emptyset$$
Revisiting split decomposition
Vertex modification of DH graphs
Relations with other works

**Split decomposition**

**Degenerate graphs**
- cliques and stars

**Totally decomposable graphs**
- Distance hereditary graphs

**Unrooted tree decomposition**
- [Cunningham 82]

**Modular decomposition**

**Degenerate graphs**
- cliques and stables

**Totally decomposable graphs**
- Cographs

**Rooted tree decomposition**
- [Gallai 67]
1. Revisiting split decomposition

2. Vertex modification of DH graphs

3. Relations with other works
Theorem (Gioan and Paul 07)

Let \( G = (V, E) \) be a distance hereditary (DH) graph. It can be tested in

- \( O(|S|) \) whether \( G + (x, S) \), with \( x \notin E \) and \( N(x) = S \), is a DH graph;
- \( O(|S|) \) whether \( G - x \), with \( S = N(x) \), is a DH graph;
Let \((T, \mathcal{F})\) be a graph-labelled tree, and \(S\) be a subset of leaves of \(T\). A node \(u\) of \(T(S)\) is:

- **fully-accessible** by \(S\) if any subtree of \(T - u\) contains a leaf of \(S\);
Let \((T, \mathcal{F})\) be a graph-labelled tree, and \(S\) be a subset of leaves of \(T\). A node \(u\) of \(T(S)\) is:

- **fully-accessible** by \(S\) if any subtree of \(T - u\) contains a leaf of \(S\);
- **singly-accessible** by \(S\) if it is a star-node and exactly two subtrees of \(T - u\) contain a leaf \(l \in S\) among which the subtree containing the neighbor \(v\) of \(u\) such that \(\rho_u(uv)\) is the centre of \(G_u\);
Let \((T, \mathcal{F})\) be a graph-labelled tree, and \(S\) be a subset of leaves of \(T\). A node \(u\) of \(T(S)\) is:

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- **partially-accessible** otherwise.
Theorem (DH incremental characterization [Gioan, Paul ’07] )

Let $G$ be a connected DH graph and $ST(G) = (T, \mathcal{F})$ be its split tree. Then $G + (x, S)$ is a DH graph if and only if:

1. At most one node of $T(S)$ is partially-accessible.
Theorem (DH incremental characterization [Gioan, Paul ’07])

Let $G$ be a connected DH graph and $ST(G) = (T, F)$ be its split tree. Then $G + (x, S)$ is a DH graph if and only if:

1. At most one node of $T(S)$ is partially-accessible.
2. Any clique node of $T(S)$ is either fully or partially-accessible.
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1. At most one node of $T(S)$ is partially-accessible.
2. Any clique node of $T(S)$ is either fully or partially-accessible.
3. If there exists a partially-accessible node $u$, then any star node $v \neq u$ of $T(S)$ is oriented towards $u$ if and only if it is fully-accessible.
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2. Any clique node of $T(S)$ is either fully or partially-accessible.
3. If there exists a partially-accessible node $u$, then any star node $v \neq u$ of $T(S)$ is oriented towards $u$ if and only if it is fully-accessible.
4. Otherwise, there exists a tree-edge $e$ of $T(S)$ towards which any star node of $T(S)$ is oriented if and only if it is fully-accessible.
The insertion fails: the two singly-accessible nodes are oriented towards the partially-accessible node!
The insertion succeeds: in $G_S(T, \mathcal{F})$, we have $N(x) = S$
Insertion algorithm

1. Extract $T(S)$ (require an arbitrary orientation of $ST(G)$);
Insertion algorithm

1. Extract $T(S)$ (require an arbitrary orientation of $ST(G)$);
2. Check the accessibility-type of the nodes and look for an insertion node or edge;
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1. Extract $T(S)$ (require an arbitrary orientation of $ST(G)$);
2. Check the accessibility-type of the nodes and look for an insertion node or edge;
3. Insert the node by either subdividing the insertion edge, or splitting the insertion node, or attaching $\times$ to the insertion node.
Insertion algorithm

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Complexity

1. $O(|N(x)|)$ dynamic recognition
**Insertion algorithm**

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**Complexity**

1. $O(|N(x)|)$ dynamic recognition
2. linear time static recognition
1. Revisiting split decomposition
2. Vertex modification of DH graphs
3. Relations with other works
Edge modification of DH graphs

Theorem (Corneil and Tedder 06)

Let $G = (V, E)$ be a distance hereditary (DH) graph. It can be tested in $O(1)$
whether $G + e$, with $e \notin E$, is a DH graph;
$O(1)$ whether $G - e$, with $e \in E$, is a DH graph.
Another approach for this result [GP 07]

A simple algorithm for this result is given by graph-labelled trees: consider the word between the two leaves $x$ and $y$ where $e = xy$ with $K$ a clique, $L$ resp. $R$ a star with center towards $x$ resp. $y$, and $S$ otherwise.

<table>
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<th>edge insertion</th>
<th>edge deletion</th>
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<tr>
<td>$(R)SS(L)$</td>
<td>$(R)LK(L)$</td>
</tr>
<tr>
<td>$(R)SK(L)$</td>
<td>$(R)LR(L)$</td>
</tr>
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<tr>
<td></td>
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Vertex modification of cographs

Theorem (Corneil, Pearl and Stewart ’85)

Let $G = (V, E)$ be a cograph. It can be tested in

- $O(|S|)$ whether $G + (x, S)$, with $x \notin E$ and $N(x) = S$, is a cograph
- $O(|S|)$ whether $G - x$, with $S = N(x)$, is a cograph
Vertex modification of cographs

**Theorem (Cograph incremental characterization [CPS’85])**

Let $G$ be a cograph and $MD(G) = (T, \mathcal{F})$ be its modular decomposition tree. Then $G + (x, S)$ is a cograph if and only if:

1. At most one node of $T(S)$ is partially-accessible.
2. Any series node of $T(S)$ is either fully or partially-accessible.
3. If a partially-accessible node $u$ exists, then a parallel node $v \neq u$ of $T(S)$ is a descendant of $u$ if and only if it is fully-accessible.
4. Otherwise, a tree-edge $e = uw$ of $T(S)$ exists such that a parallel node $v \neq u$ of $T(S)$ is a descendant of $u$ if and only if it is fully-accessible.

**Another approach for this result [GP 07]**

This result is equivalent to test the insertion/deletion in DH graphs, with the supplementary condition that the split tree is rooted.
Edge modification of cographs

**Theorem (Sharan and Shamir '04)**

Let $G = (V, E)$ be a cograph. It can be tested in

- $O(1)$ whether $G + e$, with $e \notin E$, is a cograph
- $O(1)$ whether $G - e$, with $e \in E$, is a cograph

**Another approach for this result [GP 07]**

This result is equivalent to test the insertion/deletion in DH graphs, with the supplementary condition that the split tree is rooted.
THANKS!