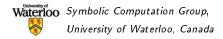
Integer Linear System Solving

Pascal Giorgi



in collaboration with Arne Storjohann

Challenges in Linear and Polynomial Algebra in Symbolic Computation Software October 1-6, 2005.

Motivations

Large linear systems are involved in many mathematical applications

over a field:

- ▶ integers factorization [Odlyzko 1999],
- discrete logarithm [Odlyzko 1999; Thomé 2003],

over the integers:

- number theory [Cohen 1993],
- group theory [Newman 1972],
- ▶ integer programming [Aardal, Hurkens, Lenstra 1999]

Problem

Let A a non-singular matrix and b a vector defined over \mathbb{Z} . Problem: Compute $x = A^{-1}b$ over the rational numbers.

$$A = \begin{bmatrix} -289 & 236 & 79 & -268 \\ 108 & -33 & -211 & 309 \\ -489 & 104 & -24 & -25 \\ 308 & 99 & -108 & 66 \end{bmatrix}, b = \begin{bmatrix} -131 \\ 321 \\ 147 \\ 43 \end{bmatrix}.$$

$$x = A^{-1}b = \begin{bmatrix} \frac{-9591197817}{95078} \\ \frac{131244}{47539} \\ \frac{2909895}{665546} \\ \frac{2909895}{665546} \end{bmatrix}$$

Main difficulty: expression swell

Interest in linear algebra

Integer linear systems are central in recent linear algebra algorithms

Determinant :

[Abbott, Bronstein, Mulders 1999; Storjohann 2005]

▶ Smith Form:

[Eberly, Giesbrecht, Villard 2000]

▶ Nullspace, Kernel:

[Chen, Storjohann 2005]

▶ Diophantine solutions :

[Giesbrecht 1997;Giesbrecht, Lobo, Saunders 1998; Mulders, Storjohann 2003; Mulders 2004]

Algorithms for non-singular system solving

► Gaussian elimination and CRA $O^{\sim}(n^{\omega+1}\log||A||)$ bit operations

► Linear P-adic lifting [Monck, Carter 1979, Dixon 1982] $O^{\sim}(n^3 \log ||A||)$ bit operations

► High order lifting [Storjohann 2005] $O^{\sim}(n^{\omega} \log ||A||)$ bit operations

P-adic algorithm for dense systems

```
Scheme to compute A^{-1}b:
1)
     1.1) B := A^{-1} \mod p
     1.2) r := b
2) for i := 0 to k
     2-1) x_i := Br \mod p
     2-2) r := (1/p)(r - A.x_i)
3)
     3-1) x := \sum_{i=0}^{k} x_i . p^i
     3-2) rational reconstruction on x
```

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                                                                  k = O^{\sim}(n)
                                                                   O^{\sim}(n^2 \log ||A||)
      2-1) x_i := Br \mod p
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                                                                   O^{\sim}(n^2 \log ||A||)
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Dense linear system in practice

Efficient implementations are available :

```
LinBox 1.0 [www.linalg.org]
IML library [www.uwaterloo.ca/~z4chen/iml]
```

Details:

- level 3 BLAS-based matrix inversion over prime field
 - with LQUP factorization [Dumas, Giorgi, Pernet 2004]
 - with Echelon form [Chen, Storjohann 2005]
- ▶ level 2 BLAS-based matrix-vector product
 - use of CRT over the integers
- rational number reconstruction
 - half GCD [Schönage 1971]
 - heuristic using integer multiplication [NTL library]

Timing of Dense linear system solving

use of LinBox library on Pentium 4 - 3.4Ghz, 2Go RAM.

Random dense linear system with coefficients over 3 bits :

n	500	1000	2000	3000	4000	5000
time	0.6s	4.3s	31.1s	99.6s	236.8s	449.2s

Random dense linear system with coefficients over 20 bits :

n	500	1000	2000	3000	4000	5000
time	1.8s	12.9s	91.5s	299.7s	706.4s	MT

performances improvement by a factor 10 compare to NTL's tuned implementation

what does happen when matrices are sparse?

We consider sparse matrices with O(n) non zero elements \hookrightarrow matrix-vector product needs only O(n) operations.

Scheme to compute
$$A^{-1}b$$
:

1)
$$1.1) B := A^{-1} \mod p$$
$$1.2) r := b$$

$$\alpha_i := Br \mod p$$

2-1)
$$x_i := Br \mod p$$

2-2) $r := (1/p)(r - A.x_i)$

3-2) rational reconstruction on x

2) for i :=0 to k
2-1)
$$x_i := Br \mod p$$

2-2) $r := (1/p)(r - A)$

3-1) $x := \sum_{i=0}^{k} x_i . p^i$

3)

P-adic lifting doesn't improve complexity as in dense case.

→ computing the modular inverse is proscribed due to fill-in

```
Solution [Wiedemann 1986; Kaltofen, Saunders 1991] : modular inverse is replaced by modular minimal polynomial
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Let $A \in \mathbb{Z}_p^{n \times n}$ of full rank and $b \in \mathbb{Z}_p^n$. Then $x = A^{-1}b$ can be expressed as a linear combination of the Krylov subspace $\{b, Ab, ..., A^nb\}$

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Let $\Pi^A(\lambda) = c_0 + c_1\lambda + ... + \lambda^d \in \mathbb{Z}_p[\lambda]$ be the minimal polynomial of A

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$$A^{-1}b = \frac{-1}{c_0}(c_1b + c_2Ab + ... + A^{d-1}b)$$

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$$A^{-1}b = \underbrace{\frac{-1}{c_0}(c_1b + c_2Ab + \dots + A^{d-1}b)}_{X}$$

P-adic algorithm for sparse systems

```
Scheme to compute A^{-1}b:
1)
     1.1) \Pi := minpoly(A) \mod p
     1.2) r := b
2) for i := 0 to k
     (2-1) x_i := (-1/\Pi[0]) \sum_{i=1}^{\deg \Pi} \Pi[i] A^{i-1} r \mod p
     2-2) r := (1/p)(r - A.x_i)
3)
     3-1) x := \sum_{i=0}^{k} x_i . p^i
     3-2) rational reconstruction on x
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P-adic algorithm for sparse systems

Scheme to compute $A^{-1}b$:

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1)
      1.1) \Pi := minpoly(A) \mod p
                                                                        O^{\sim}(n^2 \log ||A||)
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2) for i := 0 to k
                                                                        k = O^{\sim}(n)
      (2-1) x_i := (-1/\Pi[0]) \sum_{i=1}^{\deg \Pi} \Pi[i] A^{i-1} r \mod p \quad O^{\sim}(n^2 \log ||A||)
      2-2) r := (1/p)(r - A.x_i)
                                                                         O^{\sim}(n \log ||A||)
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<u>Issue</u>: computation of Krylov space $\{r, Ar, ..., A^{\deg \Pi}r\} \mod p$

Integer sparse linear system in practice

use of LinBox library on Pentium 4 - 3.4Ghz, 2Go RAM.

non-singular sparse linear system with coefficients over 3 bits and 10 non zero elements per row.

n	400	900	1600	2500
CRT + Wiedemann	7.3s	79.2s	464s	1769s
P-adic + Wiedemann	3.1s	32.7s	185s	709s
dense solver				

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2-1)
$$x_i = Br \mod p$$
 (dense case)

2-1)
$$x_i = (-1/\Pi[0]) \sum_{i=1}^{\deg \Pi} \Pi[i] A^{i-1} r \mod p$$
 (sparse case)

Remark:

n sparse matrix applications is far from level 2 BLAS in practice.

Our objectives

In pratice:

Integrate level 2 and 3 BLAS in integer sparse solver

In theory:

Improve bit complexity of sparse integer linear system solving $\implies O^{\sim}(n^{\delta})$ bits operations with $\delta < 3$?

Integration of BLAS in sparse solver

Goal:

- Minimize the number of sparse matrix-vector products.
- Maximize the calls of level 2 and 3 BLAS.

Block Wiedemann algorithm is well designed to incorporate BLAS.

Let k be the blocking factor of Block Wiedemann algorithm. then

- \triangleright the number of sparse matrix-vector product is divided by roughly k.
- order k level 3 BLAS are integrated.

Block Wiedemann and P-adic

Replace vector projections by block of vectors projections

Let N = n/kneed right minimal block generator $P \in \mathbb{Z}_p[X]$ of $\{UV, UAV, UA^2V, ..., UA^{2N}V\}$

the cost to compute P is :

- ▶ $O(k^3 N^2)$ field operations [Coppersmith 1994],
- ► O^{*}(k³ N log N) field operations [Beckermann, Labahn 1994; Kaltofen 1995; Thomé 2002],
- $ightharpoonup O^{\sim}(k^{\omega} \, N \log \, N)$ field operations [Giorgi, Jeannerod, Villard 2003].

Block Wiedemann and P-adic

```
Scheme to compute A^{-1}b:

1) for i := 0 to k

1-1) \Pi := block \ minpoly \ \{UA^iV_i\}_{1..2N} \ mod \ p

1-2) x_i := LC(A^i.V_i, \Pi[i]_{*,1}) \ mod \ p

1-3) r := (1/p)(r - A.x_i)

2)

2-1) x := \sum_{i=0}^k x_i \cdot p^i

2-2) rational \ reconstruction \ on \ x
```

Block Wiedemann and P-adic

Scheme to compute $A^{-1}b$:

1) for i :=0 to k
$$k = O^{\sim}(n)$$

1-1) $\Pi := block \ minpoly \ \{UA^{i}V_{i}\}_{1..2N} \ mod \ p$ $O^{\sim}(k^{2}n \log ||A||)$
1-2) $x_{i} := LC(A^{i}.V_{i}, \Pi[i]_{*,1}) \ mod \ p$ $O^{\sim}(n^{2} \log ||A||)$
1-3) $r := (1/p)(r - A.x_{i})$ $O^{\sim}(n \log ||A||)$
2)
2-1) $x := \sum_{i=0}^{k} x_{i}.p^{i}$
2-2) $rational \ reconstruction \ on \ x$

Not satisfying : computation of block minpoly. at each steps

How to avoid the computation of the block minimal polynomial?

Express the inverse of the sparse matrix throught structured forms. → block Hankel structure

We consider

$$K_{U} = \begin{bmatrix} U \\ UA \\ ... \\ UA^{N-1} \end{bmatrix}, K_{V} = [V|AV|...|A^{N-1}V]$$

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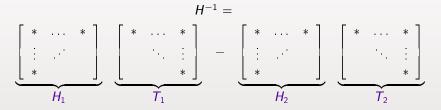
than we have

$$K_UAK_V = H$$
 with H a block Hankel matrix.

Now, we can express A^{-1} with

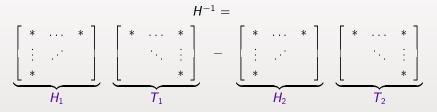
$$A^{-1} = K_V H^{-1} K_U$$

Nice property on block Hankel matrix inverse [Gohberg, Krupnik 1972, Labahn, Koo Choi, Cabay 1990]



where H_1, H_2 are block Hankel matrices and T_1 , T_2 are block Toeplitz matrices

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Block coefficients in H_1 , H_2 , T_1 , T_2 come from Hermite Pade approximant on Block coefficients of H [Labahn, Koo Choi, Cabay 1990].

Complexity of H^{-1} reduces to polynomial matrix multiplication [Giorgi, Jeannerod, Villard 2003].

```
Scheme to compute A^{-1}b:
1)
     1-1) compute S := \{ UA^{i+1}V \}_{i=0..N-1} \mod p
     1-2) compute H^{-1} \mod p from S
     1-3) r = b
2) for i := 0 to k
     2-1) x_i := K_V \cdot H^{-1} \cdot K_U \cdot r \mod p
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3)
     3-1) x := \sum_{i=0}^{k} x_i . p^i
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Scheme to compute A^{-1}b:
1)
      1-1) compute S := \{ UA^{i+1}V \}_{i=0..N-1} \mod p \quad O((n^2k \log ||A||))
      1-2) compute H^{-1} \mod p from S
                                                                O^{\sim}(nk^2 \log ||A||)
      1-3) r = b
2) for i := 0 to k
                                                                    k = O^{\sim}(n)
      (2-1) x_i := K_V \cdot H^{-1} \cdot K_U \cdot r \mod p
                                                           O^{\sim}((n^2 + nk) \log ||A||)
      2-2) r := (1/p)(r - A.x_i)
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```
K_{V} = [V|AV|...|A^{N-1}V]
\downarrow
K_{V} = [V| ] + A[ |V| ] + ... + A^{N-1}[ |V]
```

Applying K_V to a vector corresponds to :

- ullet N-1 linear combinations of columns of V
- ullet N-1 applications of A

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 $O(Nkn\log||A||)$ $O(nN\log||A||)$

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How to improve the complexity?

$$K_{V} = [V|AV|...|A^{N-1}V]$$

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Applying K_V to a vector corresponds to :

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• N-1 applications of A

How to improve the complexity?

 \Rightarrow using special block projections \emph{U} and \emph{V}

Conjecture

Let U and V such that

$$U = \begin{bmatrix} u_1 & \dots & u_k \\ & \ddots & & & \\ & & u_{n-k+1} & \dots & u_n \end{bmatrix}$$

$$V^T = \left[\begin{array}{cccc} v_1 & \dots & v_k \\ & & \ddots & \\ & & v_{n-k+1} & \dots & v_n \end{array} \right]$$

where u_i and v_i are chosen uniformly and randomly from a sufficient large set.

Let D a random diagonal matrix.

Then

the block hankel matrix $H = K_U.A.D.K_V$ is full rank.

Experimental algorithm

```
Scheme to compute A^{-1}b:
1)
      1-1) choose at random special U and V
     1-1) compute S := \{ UA^{i+1}V \}_{i=0..N-1} \mod p
      1-2) compute H^{-1} \mod p from S
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2) for i := 0 to k
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taking $N = k = \sqrt{n}$ gives a complexity of $O(n^{2.5} \log ||A||)$

Prototype implementation

LinBox project (Canada-France-USA) : www.linalg.org

Our tools:

- BLAS-based matrix multiplication and matrix-vector product
- polynomial matrix arithmetic
 Karatsuba algorithm, middle product
- vector polynomial Evaluation/Interpolation using matrix multiplication

Performances

use of LinBox library on Pentium 4 - 3.4Ghz, 2Go RAM.

non-singular sparse linear system with coefficients over 3 bits and $10\ \text{non}$ zero elements per row.

400	900	1600	2500
7.3s	79.2s	464s	1769s
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Performances under progress

use of LinBox library on SMP machine 64 Itanium 2 - 1.6Ghz, 128Go RAM.

non-singular sparse linear system with coefficients over 3 bits and 30 non zero elements per row.

matrix dimension 10.000.

- ▶ $block = 500 \hookrightarrow 10.323s \approx 2h50mn$
- ▶ $block = 400 \hookrightarrow 9.655s \approx 2h40mn$
- ▶ $block = 200 \hookrightarrow 22.966s \approx 6h20mn$
- \Rightarrow using dense solving $4.347s \approx 1h12mn$

matrix dimension 20.000.

▶ $block = 400 \hookrightarrow 47.545s \approx 13h15mn$

Conclusions

We provide an efficient algorithm for solving sparse integer linear system :

- improve the complexty by a factor \sqrt{n} (heuristic).
- allow efficiency by minimizing sparse matrix operation and maximizing BLAS use.

On going improvement:

- optimize the code (use of FFT, minimize the constant)
- provide an automatic choice of block dimension
- proove conjecture on special block projection