Algorithms and efficient implementations in exact linear algebra

Pascal Giorgi

soutenance HDR - 28 oct. 2019

Linear algebra is a widespread tool

- Assessing security in cryptography : NFS algo. [Lenstra(s) 93] 

- Coding theory in practice: list decoding [Guruswami, Sudan 98] 
- Weather forecast simulation 

Numerical linear algebra

⇒ approximated solutions : float

✓ dedicated hardware

✗ pb of stability

✓ mature developments

Exact linear algebra

⇒ exact solutions: $\mathbb{Z}, Z_p, \mathbb{Q}, \mathbb{Z}[X]$

✗ no dedicated hardware

✓ no stability issue

✗ slower development

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✓ improved over past 20 years

at the end of the talk

High performance linear algebra

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  - reductions to core problems
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Matrix multiplication

- $O(n^{2.81})$ [Strassen 69]
- $\vdots$
- $O(n^{2.37})$ [Le Gall 2014]

Polynomial/Integer multiplication

- $O(d^{1.58})$ [Karatsuba 62]
- $\vdots$
- $O(d \log(d))$ [Harvey, Hoeven 2019]

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$\Rightarrow$ finding best algo., reductions and thresholds is rather complicated

Exact linear algebra versatility

$$
\begin{bmatrix} 993 & 512 & 509 \\ 106 & 978 & 690 \\ 946 & 442 & 832 \end{bmatrix}^{-1} = \begin{cases} \begin{bmatrix} 648 & 98 & 16 \\ 648 & 839 & 305 \\ 31 & 193 & 516 \end{bmatrix} \text{ over } \mathbb{Z}_{997} \\ \\ \begin{bmatrix} \frac{14131}{9642515} & -\frac{11167}{19285030} & -\frac{8029}{19285030} \\ \frac{141137}{86782635} & \frac{172331}{173565270} & -\frac{157804}{86782635} \\ -\frac{219584}{86782635} & \frac{22723}{173565270} & \frac{458441}{173565270} \end{bmatrix} \text{ over } \mathbb{Q} \end{cases}
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- algebraic *vs* bit (or word) complexity → different algo. reductions
- sparse *vs* dense approach

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Main goal:

implement fast algorithms

but take care of

constants, memory, parallelism

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Our work in Exact Linear Algebra

- design "faster" algorithms
- provide their efficient implementation into libraries

$\Rightarrow$ 3 major libraries: Givaro [BCD+16]; FFLAS-FFPACK [DGP04; DGP08; DGLS18]; LinBox [DGG+02; BDG10; BDG+14]

Important dates in exact linear algebra

Major algorithmic breaktrough

- Gaussian Elimation in $O(n^\omega)$ with $\omega < 3$ [Strassen 69]
  $\Rightarrow$ influence algebraic complexity: reduction to matrix mult.
- Sparse linear algebra in $O(n^2)$ [Wiedemann 86, Coppersmith 90]
  $\Rightarrow$ provide iterative methods to finite fields
- Polynomial linear algebra in $\tilde{O}(n^\omega d)$ [Storjohann 02]
  $\Rightarrow$ influence bit complexity: reduction to matrix mult.

Outline

Dense linear algebra (post Strassen)

Sparse linear algebra - Polynomial linear algebra (a short round trip)

Beyond time: space and confidence

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Beyond time: space and confidence

Dense linear algebra mod p

Problems reduces to matrix mult. : $O(n^\omega)$ op. in $\mathbb{Z}_p$, $\omega < 2.3728$ [Le Gall14]

$\Rightarrow$ linsys, det, inv [Strassen 69], rank [Ibarra et al. 82], minpoly, charpoly [Pernet et. al 07]

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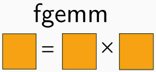
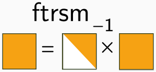
FFLAS-FFPACK $\rightarrow$ reductions effective in practice [DGP08; DGP04; DGLS18]

- which matrix mult./reductions and how ?



$\longrightarrow$ subcubic algo., architecture optim. (BLAS)



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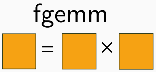

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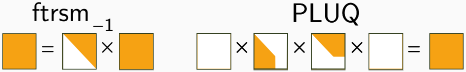


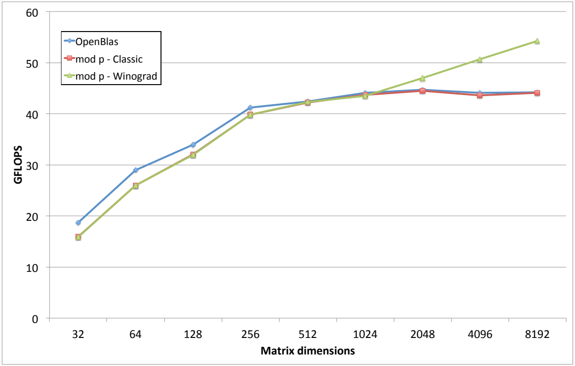
$\longrightarrow$ minimize mod p

- optimized half wordsize prime $p \Rightarrow$ larger p throught CRT

Matrix multiplication $\bmod p$ ($< 26 bits$)

FFLAS `fgemm` $\Rightarrow$ rely on numerical computation [Dumas et. al 02],[DGP08]

- delayed reductions mod p ✓ $O(n^2)$
- adaptative multiplication over $\mathbb{Z}$
  - ↪ t levels of Strassen-Winograd if $9^t \lfloor \frac{n}{2^t} \rfloor (p-1)^2 < 2^{53}$ ✓ $\omega < 3$
  - ↪ use BLAS as base case ✓ cache+simd



benchmark on Intel i7-4960HQ (2014), $p < 20$ bits

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Most efficient solutions $\Rightarrow$ reduction to smaller prime(s) matrix mult.

- convert problem to polynomial matrix mult. $\bmod q$ (Kronecker)
  $\mathbb{Z} \rightarrow \mathbb{Z}_m \rightarrow \mathbb{Z}_q[X]_{<d} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}_p$

- convert problem to many matrix multiplication $\bmod p_i$ (CRT)
  $\mathbb{Z} \rightarrow \mathbb{Z}_m \rightarrow \underbrace{\mathbb{Z}_{p_1 \times \cdots \times p_d} \rightarrow \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_d} \rightarrow \mathbb{Z}}_{\text{RNS conversions}} \rightarrow \mathbb{Z}_p$
  $AB \bmod (p_1 \times \cdots \times p_d) \longleftrightarrow (AB \bmod p_1, \ldots, AB \bmod p_d)$

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How to improve the reduction ? especially RNS

RNS conversions in practice

Fast RNS conversions $O(d \log(d) \log\log(d))$ word op. [Borodin, Moenck 74]

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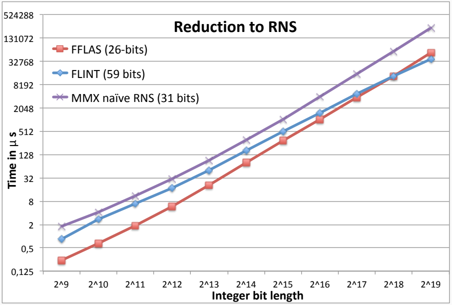
Naive RNS conversions $O(d^2)$ word op.
$\Rightarrow$ can be reduced to matrix mult. for many conversion [DGLS18]

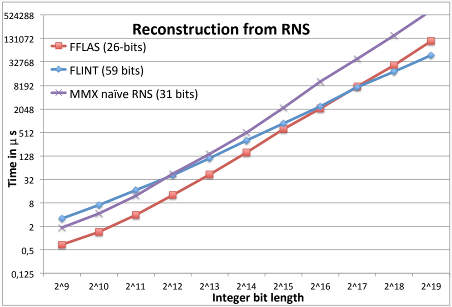
- pseudo-reduction:

$$A_0 + A_1\beta + \cdots A_{d-1}\beta^{d-1} \longrightarrow \begin{bmatrix} A_0 & \dots & A_{d-1} \end{bmatrix} \times (\beta^i \bmod p_j)_{i,j}$$
$$O(d) \longrightarrow O(\log d)$$

- r RNS conversions: $O(\mathbf{rd}^{\omega-\mathbf{1}}) + O(d^2)$ word op. (practical $\omega \simeq 2.81$)

Simultaneous conversions with RNS: in practice

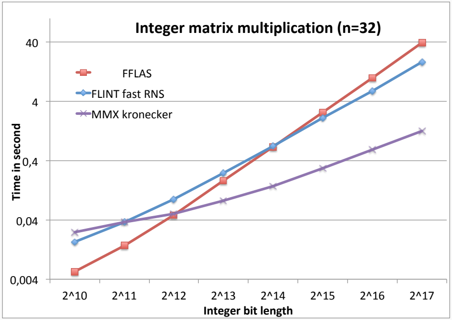


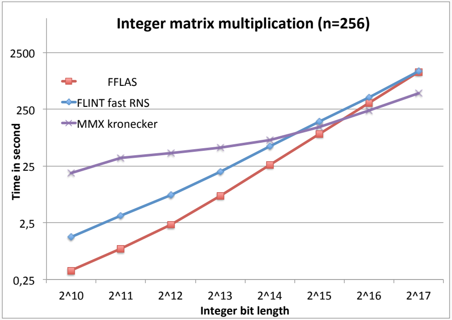


benchmark on Intel Xeon-E52697 (2017) - for matrix multiplication ($n = 128$)

- for good β and $p_j < 2^{26} \Rightarrow$ re-use BLAS (and possibly Strassen)
- can extend the p_j without sacrifying performance [DGLS18]

FFLAS integer matrix multiplication





benchmark on Intel Xeon-E52697 (2017)

- our solution: reduce everything to matrix mult. $O(n^\omega d + n^2 d^{\omega-1})$
- over $\mathbb{Z}_p \Rightarrow$ reduce afterward (slight slowdown)

Dense linear algebra modulo p: in practice

FFLAS approach: algo. reduction to `fgemm` [DGP04; DGP08]
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Example with ftrsm:

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- $A_3 X_2 = B_2 \bmod p$
- $D = B_1 - A_2 X_2$ over $\mathbb{Z}$
- $A_1 X_1 = D \bmod p$

reduce r.h.s $\bmod p$ when $n = 1$

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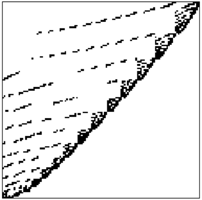
- p multi-precision : only $O(n^2)$ RNS conversions [G. HDR 19]
  $T(n,n) = 2T(n/2,n) + n^\omega d + \cancel{n^2 d^{\omega-1}} \rightarrow$ save a $\log(n)$

✓ practical performance ~ `fgemm` (matching up with constants)

Outline

Exact sparse linear algebra



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- Gauss fill-in : still $O(n^\omega)$ op in $\mathbb{K}$

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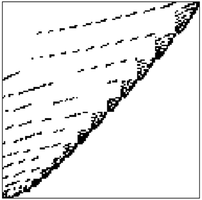

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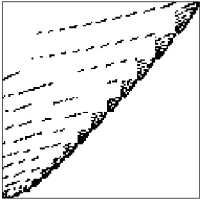

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$\Rightarrow$ trade a sequence of integers to a shorter one of matrices $(UA^i V)_{i\geq 0}$

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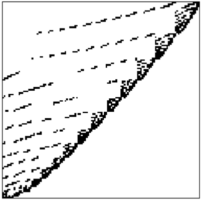

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How to improve in theory/practice:
→ fast linear algebra with polynomials (fast on degree/dim)

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  reduced basis of $\mathbb{K}[X]$-module , e.g. $\{qF = 0 \bmod X^d\}$
  $\Rightarrow$ central step in block Wiedemann method
- many derived reductions:
  - row reduction [GJV03]; Hermite [Gupta, Storjohann 11; Labahn et. al 17]
  - Popov [Beckerman et. al 06; Neiger 16]; inverse [Jeannerod, Villard 05; Zhou et. al 15]
  - rank, nullspace [Storjohann, Villard 05; Zhou et. al 12]

Fast minimal approximant basis in LinBox

state of the art implem. of PM-Basis algo. [GJV03] ($\mathbb{K} = \mathbb{Z}_p$)

- base case : kernel basis over $\mathbb{Z}_p$ → FFLAS-FFPACK
- fast matrix mult. in $\mathbb{Z}_p[X]$ [DEGU07; GL14] (ANR HPAC)

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$\Rightarrow$ some (UA^iV) are not necessary when rank deficient

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Variant of PM-Basis minimizing dependency on F [GL14]

- iterative DAC algo.
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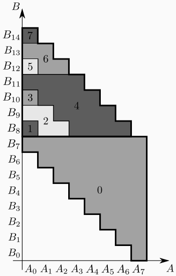

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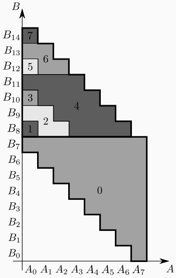
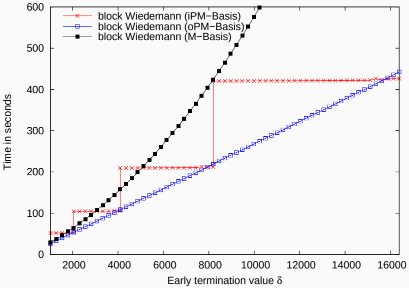


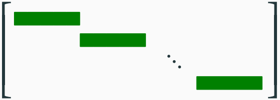


Main application:
rank deficiency with block Wiedemann
E.T. $\Rightarrow$ reduce staircase effect

Improving sparse linear algebra

key point: re-use fast polynomial linear algebra [EGG+06; EGG+07]

- prove existence of sparse projections for block Wiedemann

$U =$  with  a vector of $1's$

- blackbox matrix inverse over $\mathbb{K}$: $\tilde{O}(n^{2.5})$ op in $\mathbb{K}$

  $\Rightarrow$ sparse projection and PM-Basis

- solve $Ax = b$ over $\mathbb{Q}$ in $\tilde{O}(n^{2.5} \log \|A\|)$ word op.
  - improve classical approaches: $\tilde{O}(n^3 \log \|A\|)$ word op.
  - practical algo → thanks to PM-Basis (dim $\simeq$ deg)

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Improving the confidence in computed results

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How to sustain confidence:

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- verify the solution ⇒ e.g. $Ax = b$ or $uC = (uA)B$ [Freivalds 79]

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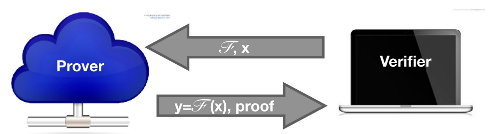
last solution is adapted to linear algebra (no linear complexity)

Verified computation model



- generating proof negligible
- verifying proof easier than $\mathcal{F}(x)$

Generic approaches in linear algebra [Goldwasser et. al 08, Kaltofen et. al 11]

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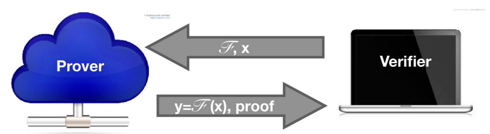

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Optimal verification algorithms exist [Dumas, Kaltofen 14]:

e.g. over $\mathbb{Z}_p$, $\quad \mathrm{rank}(A) = r \iff A = PLUQ$ and $\mathrm{rank}(U) = r$

$\Rightarrow$ optimal prover/verifier time + "independency" from implementation

Optimal certificates in linear algebra

Most classical problems have an "optimal" probabilistic certificate:

- sparse, dense over $\mathbb{K}$, $\mathbb{Z}$ and $\mathbb{Q}$
- very often with interactive protocols

Optimal certificates in linear algebra

Most classical problems have an "optimal" probabilistic certificate:

- sparse, dense over $\mathbb{K}$, $\mathbb{Z}$ and $\mathbb{Q}$
- very often with interactive protocols

Our concerns: computation over $\mathbb{K}[X]$

- minimal approximant basis as a good candidate
- is interactivity really required ?

verifying $A(x)B(x) = 0 \bmod X^k$ not so easy

Our main tool: multiplications as linear maps

Example:

$$f = 3X^2 + 2X + 1, \quad g = X^2 + 2X + 4 \qquad \Rightarrow \qquad \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 3 & 2 & 1 \\ & 3 & 2 \\ & & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 17 \\ 8 \\ 3 \end{bmatrix}$$

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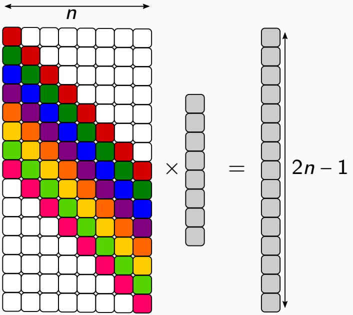
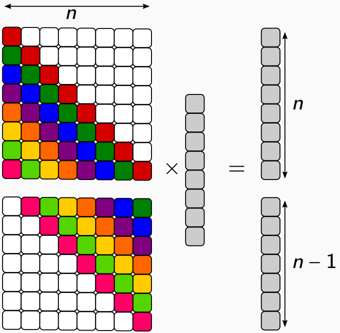


Full product (FP)

$\Longrightarrow$



Short products (SP_{lo}, SP_{hi})

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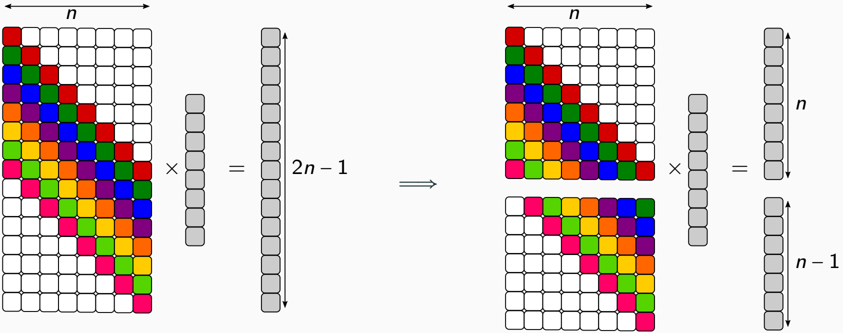

Full product (FP) — Short products (SP_{lo}, SP_{hi})

→ eval. $X = \alpha \quad \simeq \quad$ mult. by $\begin{bmatrix} 1 & \alpha & \dots & \alpha^{n-1} \end{bmatrix}$

Verifying truncated polynomial products

Low short product over $\mathbb{K}[X]$:
verify $C(X) = A(X)B(X) \bmod X^k$ in optimal time [Gio18]

$$\begin{bmatrix}1 & \alpha & \dots & \alpha^{k-1}\end{bmatrix}\begin{bmatrix} a_0 & & & \\ a_1 & \ddots & & \\ \vdots & \ddots & \ddots & \\ a_{k-1} & \cdots & a_1 & a_0 \end{bmatrix}\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{k-1}\end{bmatrix} = \begin{bmatrix}1 & \alpha & \dots & \alpha^{k-1}\end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{k-1}\end{bmatrix}$$

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- $O(k)$ op. in $\mathbb{K}$ → evaluating $\alpha^{k-1}\mathrm{rev}(A)$ at $X = \alpha^{-1}$
- no interactivity

⇒ similar result for Toeplitz matrix: e.g. middle product
⇒ optimal also for polynomial matrix mult. → use Freivald

Verifying minimal approximant basis

Optimal certificate [GN18]

$P \in \mathbb{K}[X]^{n\times n}$ is a minimal approximant basis of $\mathcal{A}_d(F)$ iff

- P is row-reduced (**minimality**)
- $PF = 0 \bmod X^d$ (**approximant**)
- $\det(P) = X^\delta$
- $\begin{bmatrix} P(0) & C(0) \end{bmatrix}$ is full rank with $C = PFX^{-d}$
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- prover: P in $\tilde{O}(n^\omega d)$ and $C(0)$ in $O(n^\omega d)$
- verifier: almost $O(\mathrm{size}(P) + \mathrm{size}(F))$, almost no-interactivity

$\Rightarrow$ work for non-uniform truncation: $\bmod X^{\mathbf{d}}$ with $\mathbf{d} = (d_1, \ldots, d_m)$

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$$\text{polynomial mult.} = \begin{cases} \text{naive} \rightarrow O(n^2) \text{ time and } O(1) \text{ extra space} \\ \text{Karatsuba} \rightarrow O(n^{1.58}) \text{ time and } O(n) \text{extra space} \end{cases}$$

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Our concern: generic approach for in-place multiplication ?

Algorithms	Time complexity	Space complexity
naive	$2n^2 + 2n - 1$	O(1)
Karatsuba ['62]	$< 6.5n^{1.58}$	$\leq 2n + 5\log(n)$
Karatsuba [(Roche'09)]	$< 10n^{1.58}$	$\leq 5\log(n)$
Toom-3 ['63]	$< \frac{73}{4}n^{1.46}$	$\leq 2n + 5\log_3(n)$
FFT [CT'65]	$9n\log(2n)$	$2n$
FFT [Roche'09]	$11n\log(2n)$	$O(1)$

Our technique: again, multiplication as a linear map

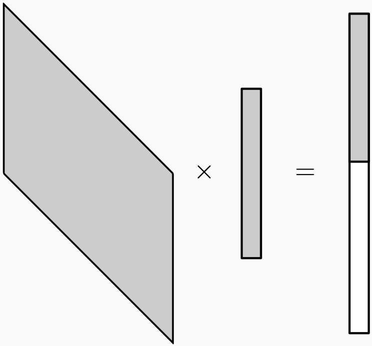
In-place polynomial multiplication

A first positive answer from [GGR19]

Given algo. with $\mathsf{M}(n)$ time complexity and cn space complexity

- in-place full product (half-additive) in time $O((2c+7)\mathsf{M}(n))$

$$(f_0 + X^k \hat{f}) \cdot (g_0 + X^k \hat{g}) = f_0 g_0 + X^k (f_0 \hat{g} + \hat{f} g_0) + X^{2k} \hat{f} \hat{g}$$



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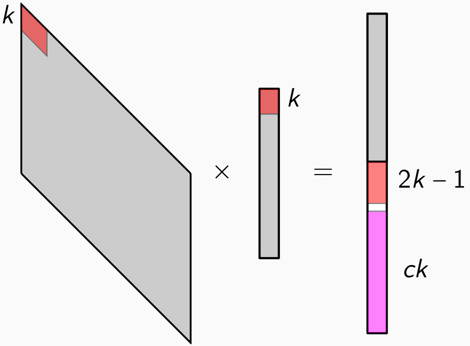

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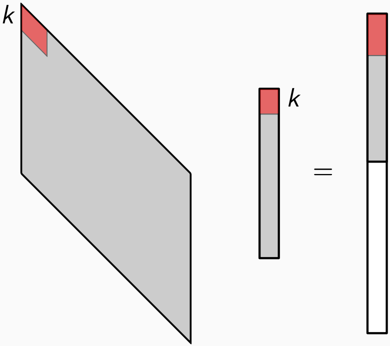

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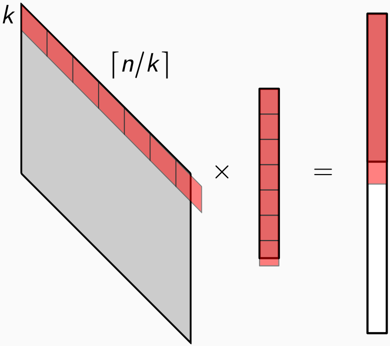

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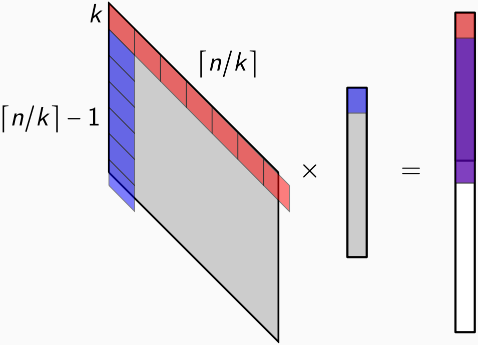

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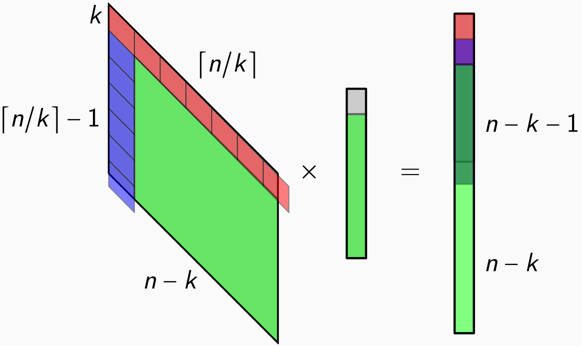

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- in-place full product (half-additive) in time $O((2c+7)\mathsf{M}(n))$
- in-place short product in time $O((2c+5)\mathsf{M}(n))$
- in-place middle product in time $O(\mathsf{M}(n)\log(n))$

Take-home message

Efficient dense linear algebra mod p

fast integer matrix multiplication + delayed mod p

$\Rightarrow$ influence libraries in computer algebra (e.g. Maple, SageMath, ...)

Efficient reductions in theory/practice

Polynomial linear algebra	$\Leftarrow$	High Order lifting
$\Uparrow$		$\Uparrow$
PM-Basis algorithm†	$\Leftarrow$	**polynomial matrix mult.** †‡
$\Downarrow$		$\Uparrow$
Sparse linear algebra (BW)		**FFT +** `fgemm`**+ RNS**

(†) with optimal certificate; (‡) with in-place variant

Perspectives

Toward better space complexity

- in-place reductions to polynomial mult.
  - series inversion, division
  - multipoint evaluation/interpolation
  - any linear maps $\Rightarrow$ find generic approach
- hard problems: composition of linear maps, use input as output

$\Rightarrow$ applicable in practice ? usefullness in BW ?

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Optimal certificates in linear algebra

- linear maps: what transposition tells us about certification ?
- extension to other bases of $\mathbb{K}[X]$-submodule (e.g. interpolant basis)
  ⇒ modular multiplication ?

Perspectives

Further software improvements

- effective reduction to polynomial matrix mult. [Hyun, Neiger, Schost 19]
- linear algebra mod p for "medium range" p

Sparse polynomial arithmetic

starting thesis (10/2019): Armelle Perret Du Cray

$\Rightarrow$ practical quasi-linear time algo. (proba.)

- product certification, multiplication
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Thank you !!!

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