Fast interpolation and multiplication of unbalanced polynomials

Pascal Giorgi, Bruno Grenet, Armelle Perret du Cray, Daniel S. Roche

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Motivations

Polynomial multiplication complexity over $\mathcal{R}[x]$ is well understood

Algebraic complexity

- $O(d^{\log_2 3}), \ldots, O(d^{1+o(1)})$ [Karatsuba 1962, Toom 1963, Cook 1966]
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$\Rightarrow$ only few works, e.g. Toom-Cook with structured unbalancedness [Bodrato-Zanoni 2020]

Unbalanced polynomials

Example

$$\begin{aligned} & x^7 + 3x^6 - 1000000000x^5 - 3x^4 - 4x^3 - x^2 + x - 3 \\ \times \quad & x^7 + 3x^6 + 1000000006x^5 - 3x^4 - 4x^3 - 7x^2 + x - 3 \end{aligned}$$

$$= x^{14} + 6x^{13} + 15x^{12} + 12x^{11} - 10000000060000000026x^{10} - 50x^9 - 37x^8 + 6000000018x^7 + 28x^6 + 8x^5 + 17x^4 + 16x^3 + 25x^2 - 6x + 9$$

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Let $f = \sum_{i=0}^{d} f_i x^i \Longrightarrow \begin{cases} s = \text{bitlen}(f) \approx \sum \log |f_i| > d \\ H = \max |f_i| \leq 2^s \end{cases}$

- Balanced case: $\log H = \Theta(s/d) \hookrightarrow$ bit complexity $O^{\sim}(s)$ ☺ quasi-optimal
- Unbalanced case: $\log H = \Theta(s) \hookrightarrow$ bit complexity $O^{\sim}(sd)$ ☹ possibly quadratic

❓ Can we always compute the polynomial product $f \times g$ in time $O^{\sim}(\text{bitlen}(f) + \text{bitlen}(g) + \text{bitlen}(f \times g))$?

Reinterpretation as a sparse interpolation problem

Imagine that you are given most of the coefficients of $f \times g$

$\Delta = x^{14} + 6x^{13} + 15x^{12} + 12x^{11} - 26x^{10} - 50x^9 - 37x^8 + 18x^7 + 28x^6 + 17x^4 + 16x^3 + 25x^2 - 6x + 9$

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Intermediate Problem

Given $f, g \in \mathbb{Z}[x]$

Compute $\Delta \approx f \times g$

Original Problem

Given $f, g \in \mathbb{Z}[x]$

Compute $h = f \times g$

New Problem

Given a way to evaluate $h - \Delta \in \mathbb{Z}[x]$

Interpolate $h - \Delta$ in dense or sparse rep.

Difficulties

- h and Δ may be unbalanced (and sparse)
- Evaluating f, g and Δ might be costly

Sparse interpolation: in a nutshell

Problem definition

Inputs: a way to evaluate $f \in \mathbb{Z}[x]$

bounds (D, H, T) s.t. $D \geq \deg f$; $H \geq \|f\|_\infty$; $T \geq \|f\|_0 = t$

Output: the sparse representation of $f = \sum_{i=1}^{t} f_i x^{e_i}$, with $f_i \neq 0$

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Two main approaches with bit complexity $poly(T, \log D, \log H)$

- (Modular) Blackbox [Prony 1795; Ben-Or-Tiwary 1988, Kaltofen 2010]
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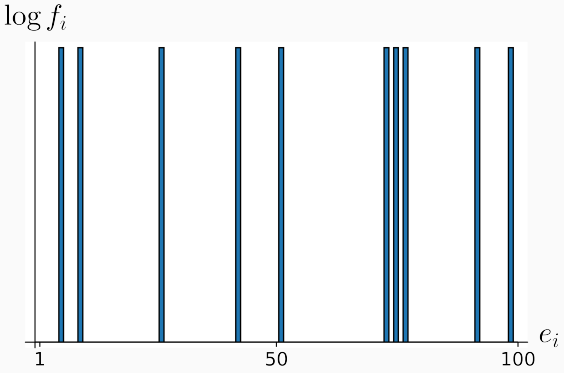
Many works to make more efficient algorithms

⇒ avoiding some bounds [Kaltofen-Lee 2003]
⇒ quasi-linear in one, few, or all parameters [Arnold-Giesbrecht-Roche 2013–2016; Huang 2019]
⇒ first fully quasi-linear [G.-Grenet-Perret du Cray-Roche 2022]

☹ all results assume f is balanced: $\text{bitlen}(f) = O(T \log D + T \log H)$

Fast balanced sparse interpolation [G.-Grenet-Perret du Cray-Roche (ISSAC 2022)]



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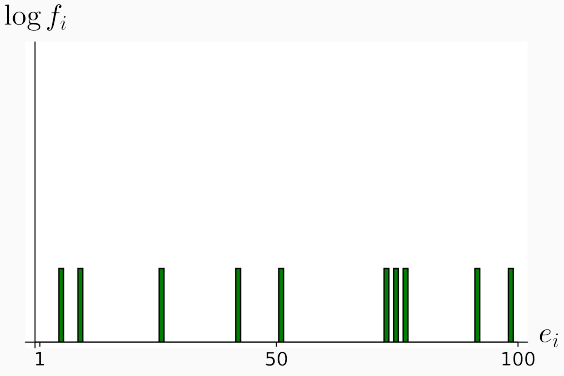

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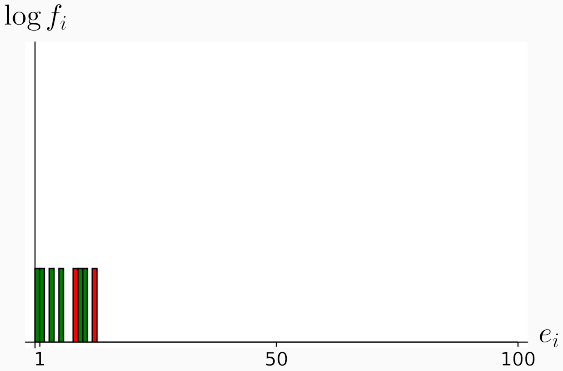

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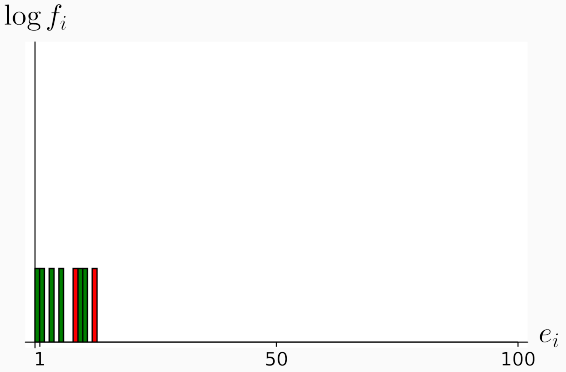

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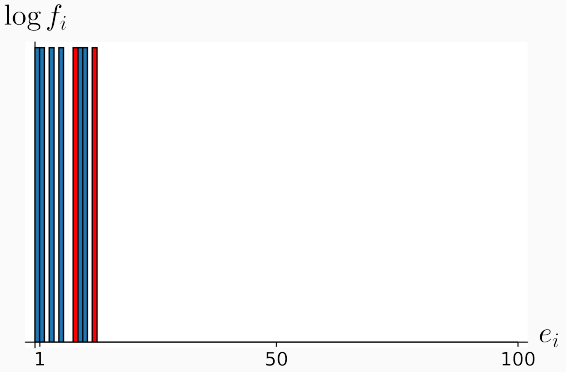

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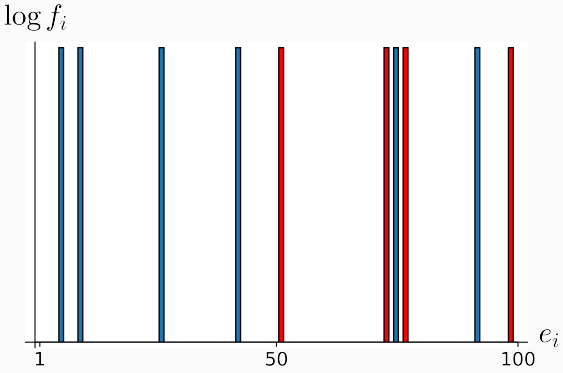

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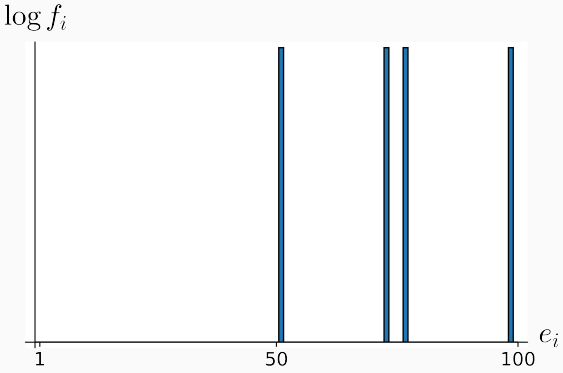

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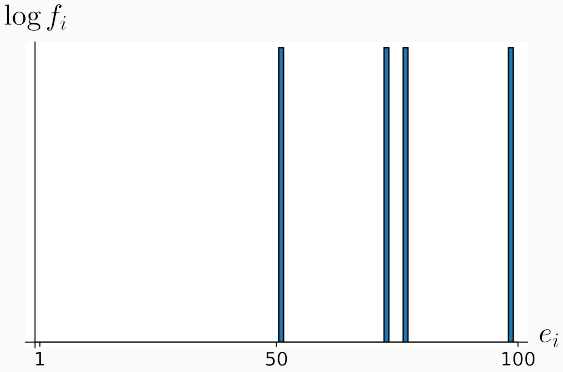

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$\Rightarrow$ Monte-Carlo algorithm (proba $\geq 2/3$), $O(T)$ probes to MBB and $\tilde{O}(T \log(DH))$ bit op.

Our contributions

Interpolation of sparse polynomials with unbalanced coefficients

Given a modular black box for $f \in \mathbb{Z}[x]$ with bounds $s \geq \text{bitlen}(f)$ and $D \geq \deg f$

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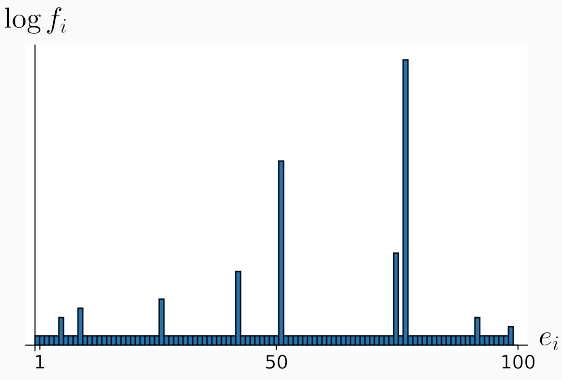
Polynomial multiplication of unbalanced polynomials

Given explicit (dense or sparse) polynomials $f, g \in \mathbb{Z}[x]$

- Probably correct and probably fast algorithm with proba $\geq 1 - \frac{1}{s}$
- Expected bit complexity $\tilde{O}(s \log D)$

where $s = \text{bitlen}(f) + \text{bitlen}(g) + \text{bitlen}(fg)$ and $D = \deg fg$

Unbalanced (sparse) interpolation: the wrong easy solution



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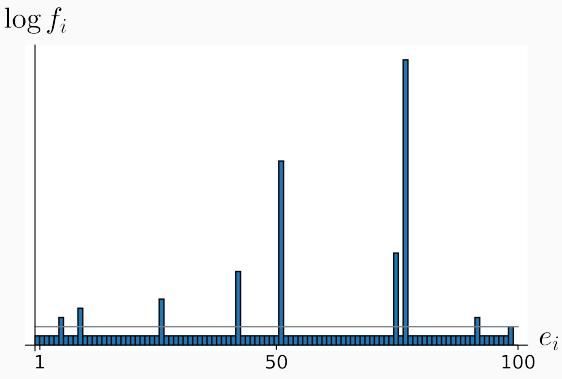

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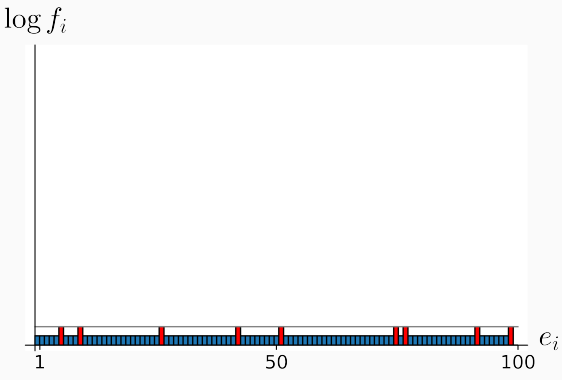

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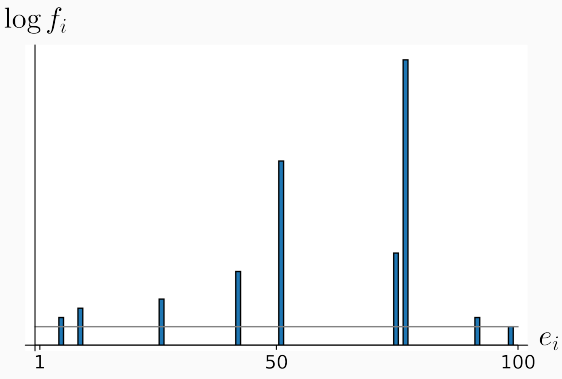

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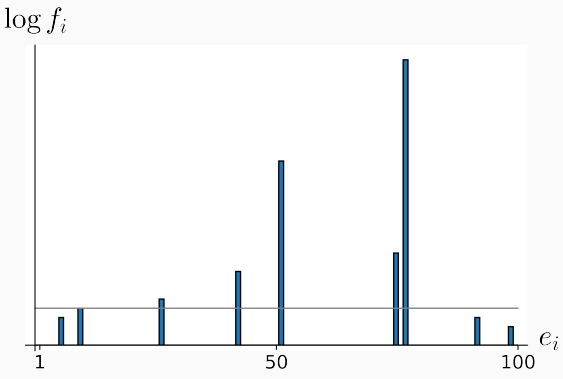

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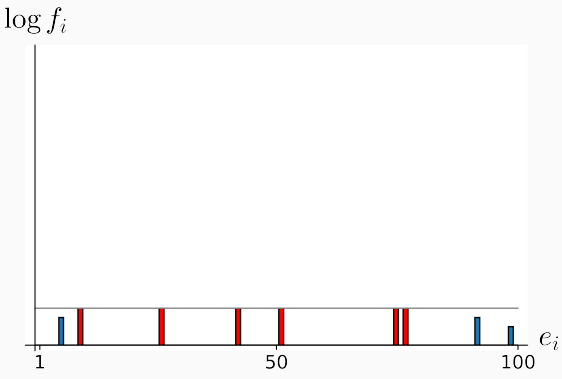

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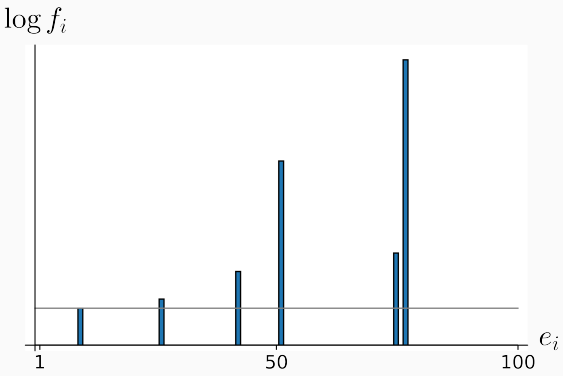

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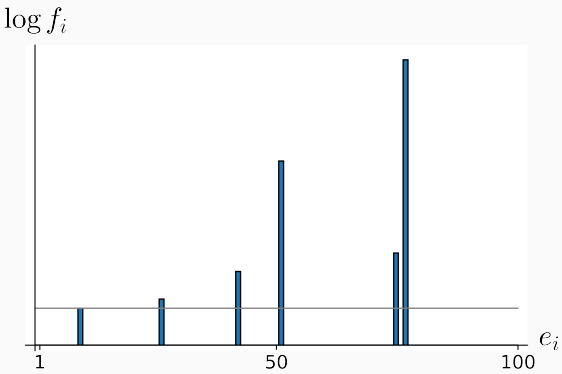

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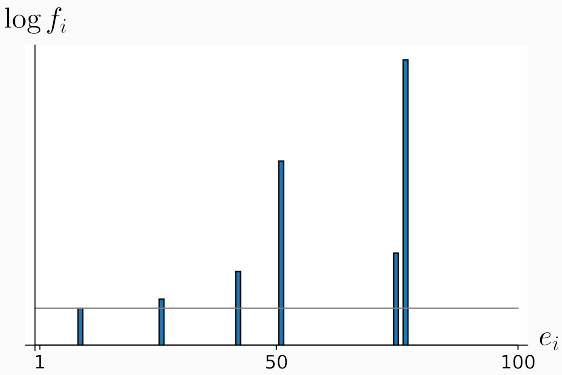

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☹ at some point $m = O(2^s)$, and sparsity of f^* might be $O(s)$
↪ one evaluation of π costs $\tilde{O}(s^2)$ due to explicit representation of f^*

!!! leads to a quadratic time complexity !!!

Unbalanced (sparse) interpolation: our efficient solution

Using a top-down approach

- First recover some *huge terms* f^* of f
- Recursively interpolate $f - f^* \Rightarrow$ *more balanced, smaller coeff, but same sparsity*

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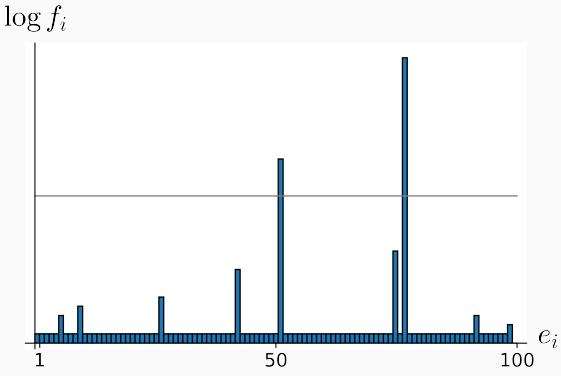

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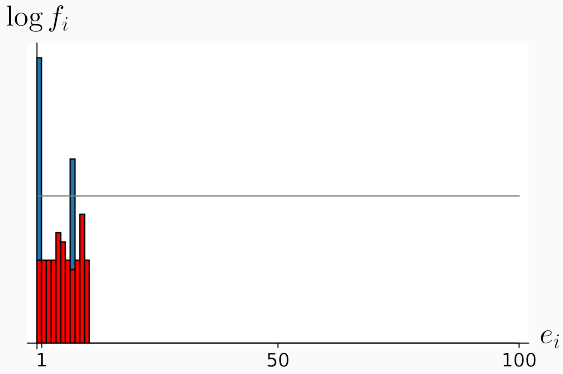

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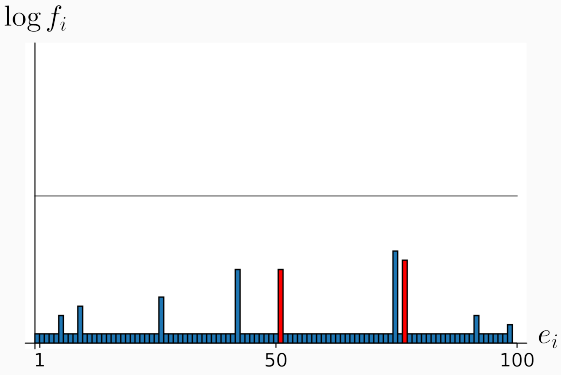

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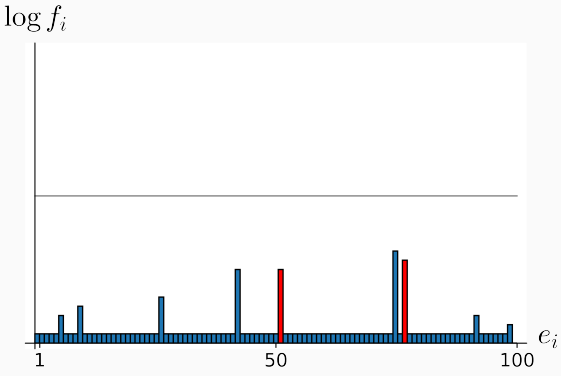
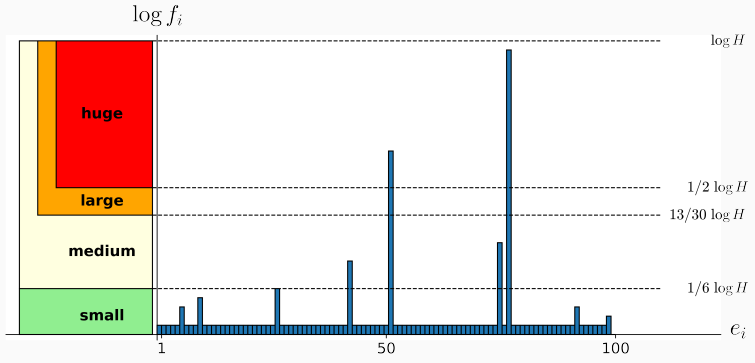


Difficulties

- Expression swell with collisions
- Embedding exponents into coeffs

Recovering the huge terms

huge terms are those beyond half the maximal bit-length (e.g. $\geq 2^{s/2}$)



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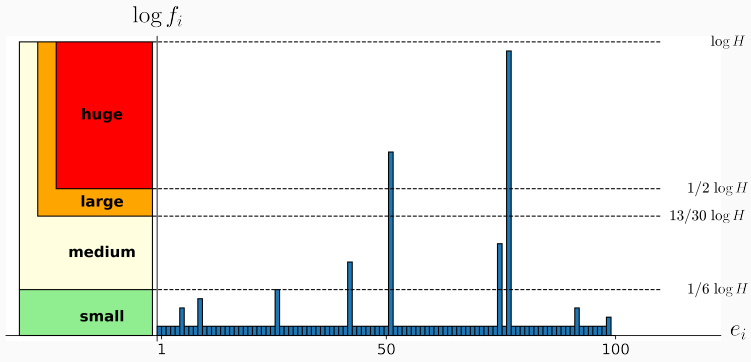

On terms' collision:

- **only** one large/many small still allows exponent decoding using $xf'(x)$
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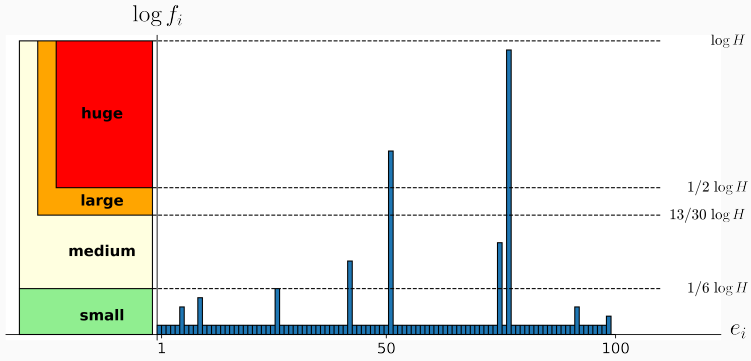

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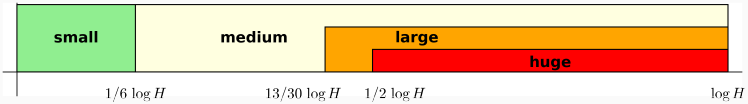
↪ reconstructed huge terms **must** avoid large/medium collisions

Recovering the huge terms: controlling the collisions



Remarks: it is hard to avoid all large / medium collision in $f \bmod x^p - 1$ with $p = O(s)$

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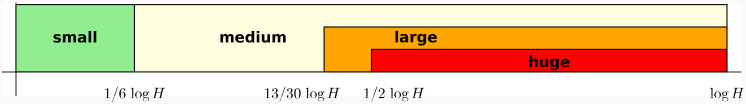

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Our solution

1. Compute a superset $\mathcal{T}$ of the large terms exponents (erroneous= large / medium)
2. Use $\mathcal{T}$ to filter out large / medium collisions in the reconstruction of huge terms
   ↪ (somehow) interpolation with overestimated support

Recovering the huge terms: superset of large terms

Lemma [G.-Grenet-Perret du Cray-Roche 2024]

Let $c_0 x^{e_0}$ be any large terms of f. If $c_0 x^{e_0}$ only collides with many small terms $c_i x^{e_i}$:

$$e_0 = \left\lfloor \frac{c_0 e_0 + \sum_{i \in \mathcal{S}} c_i e_i}{c_0 + \sum_{i \in \mathcal{S}} c_i} \right\rceil \quad \text{assuming } m \geq 4H^{7/6}, \quad \log H = \Omega(\log s + \log D)$$

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Algorithm sketch for computing $\mathcal{T}$:

- Take p so that most collisions are large/small — $p = O(s \log D / \log H)$
- Dense interpolation of $(f, xf') \bmod \langle x^p - 1, m \rangle$ — $\log m = O(\log H)$
  ↪ add to $\mathcal{T}$ the coefficient-wise divisions
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Remark

- The superset is not too large: $\#\mathcal{T} = O(s \log s / \log H)$, overestimated by $O(\log s)$
- Bit complexity: $O^{\sim}(s \log D)$

Algorithm for computing all huge terms

Idea: use $\mathcal{T}$ to detect erroneous huge terms in $f \bmod \langle x^p - 1, m \rangle$

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- Take p so that most terms in $\mathcal{T}$ do not collide — $p = O(\#\mathcal{T} \log D)$
- Let $cx^{e \bmod p}$ be any huge term of $f \bmod \langle x^p - 1, m \rangle$ — $\log m = O(\log H)$
  ↪ add cx^e to f^* **only if** $e \bmod p$ is collision-free in $\mathcal{T} \bmod p$
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Remark

- All huge terms of f are added to f^*, plus some non-large collisions, still $\|f - f^*\|_\infty < \sqrt{H}$
- Bit complexity: $O(s \log D \log^3 s \log\log H) = O^{\sim}(s \log D)$

Our unbalanced sparse interpolation algorithm

Algorithm

Input: Modular black box for $f \in \mathbb{Z}[x]$ and bounds $s \geq \text{bitlen}(f)$, $D \geq \deg(f)$
Output: an explicit $f^* \in \mathbb{Z}[x]$ or error if $\text{bitlen}(f^*) > s$

1. $H \leftarrow 2^s, f^* \leftarrow 0$
2. while $H \geq \max(61, 15 \log s, 6 \log D)$
3. compute the huge terms of $(f - f^*)$ and update f^*
4. $H \leftarrow \sqrt{H}$
5. compute remaining terms of $(f - f^*)$ *via* (balanced) sparse interpolation

Theorem [G.-Grenet-Perret du Cray-Roche 2024]

The algorithm returns an explicit representation of f with probability $\geq 1 - 1/s$, using $\tilde{O}(s \log D)$ bit operations and $O(s \log D \log s)$ MBB evaluations.

Back to our application

Unbalanced polynomial multiplication

Inputs: unbalanced polynomials $f, g \in \mathbb{Z}[x]$ with $\text{bitlen}(f), \text{bitlen}(g) \leq \ell$

Output: $h = f \times g \in \mathbb{Z}[x]$ with $s = \text{bitlen}(h) + 2\ell$

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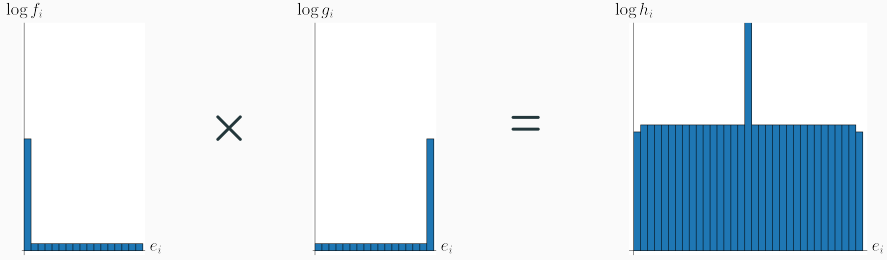
- Otherwise, need a bound estimation: might be quadratic

$$\text{bitlen}(h) \leq 4\ell^2 / \log \ell \quad \text{and} \quad \|h\|_\infty \leq 4^\ell \ell$$



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- Unbalanced sparse interpolation of MBB $f \times g$ with tentative bound $s \overset{?}{\geq} \text{bitlen}(fg)$
- *Probabilistically verify* whether $h = f \times g$ in $\tilde{O}(s \log D)$ [G.-Grenet-Perret du Cray 2023]
- Start with $s = \ell$ and *double* its value until verification succeeds or $4^\ell / \log \ell$ is reached

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Probably correct and probably fast

- Both with probability $\geq 1 - 1/s$
- Expected bit complexity $O\tilde{}(s \log D \log^5 s (\log\log s)^2) = O\tilde{}(s \log D)$

Summary of results

Unbalanced (sparse or dense) polynomials

New complexity of $\tilde{O}(s \log D)$ bit operations

- **Interpolation**: Monte-Carlo algorithm with proba $\geq 1 - 1/s$
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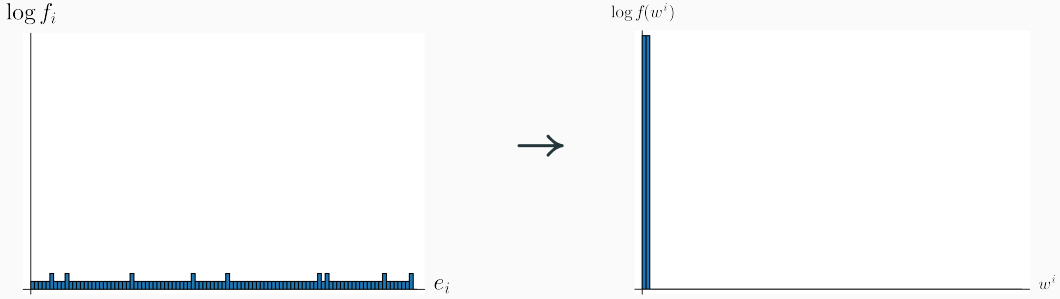
☺ quasi linear for dense and moderately sparse polynomials, i.e. $\log D = poly(\log s)$

☹ not for supersparse polynomials, i.e. $\log D = poly(s)$

Open problems

- Algorithms of complexity $\tilde{O}(s)$ for unbalanced polynomials
  ↪ difficulty $p \geq \log D$ to avoid collisions
- How to remove the *a priori* bounds s and $\log D$ in the interpolation ?
- Fast evaluation of large low-height polynomial at few points to very high precision



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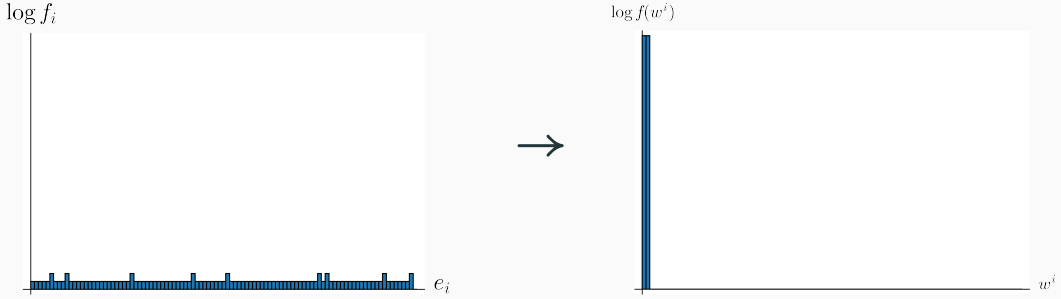

Thank you !

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Under the carpet: MBB/polynomial evaluations

- The input is **always** $\pi = f - f^*$ $\quad \mathrm{bitlen}(f), \mathrm{bitlen}(f^*) \leq s$
- Need to evaluate π on $(1, \omega, \omega^2, \ldots, \omega^{p-1})$ for interpolating $\pi \bmod \langle x^p - 1, m \rangle$
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Evaluation cost for explicit polynomial f^*

1. First, reduce $f^* \bmod \langle x^p - 1, m \rangle \rightarrow O(s \log\log m + s \log\log p)$
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Evaluation cost of the MBB for f

- Let $f(\alpha) \bmod m$ costs $O(B + L \log m \log\log m)$ bit op.
- Our algo need $O^{\sim}((B + L)s \log D)$ bit op.