Solving Sparse Rational Linear Systems

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joint work with

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Problem

Let A a non-singular matrix and b a vector defined over $\mathbb{Z}$.

<u>Problem</u> : Compute $x = A^{-1}b$ over the rational numbers

$$A = \begin{pmatrix} 289 & 237 & 79 & -268 \\ 108 & -33 & -211 & 309 \\ -489 & 104 & -24 & -25 \\ 308 & 99 & -108 & 66 \end{pmatrix},\ b = \begin{pmatrix} -131 \\ 321 \\ 147 \\ 43 \end{pmatrix}.$$

$$x = A^{-1}b = \begin{pmatrix} \frac{-5795449}{32845073} \\ \frac{152262251}{98535219} \\ \frac{428820914}{229915511} \\ \frac{1523701534}{689746533} \end{pmatrix}$$

Main difficulty : expression swell

Problem

Let A a non-singular matrix and b a vector defined over $\mathbb{Z}$.

Problem : Compute $x = A^{-1}b$ over the rational numbers

$$A = \begin{pmatrix} -289 & 0 & 0 & -268 \\ 0 & -33 & 0 & 0 \\ -489 & 0 & -24 & -25 \\ 0 & 0 & -108 & 66 \end{pmatrix}, \; b = \begin{pmatrix} -131 \\ 321 \\ 147 \\ 43 \end{pmatrix}.$$

$$x = A^{-1}b = \begin{pmatrix} \frac{-378283}{1282641} \\ \frac{-107}{11} \\ \frac{-4521895}{15391692} \\ \frac{219038}{1282641} \end{pmatrix}$$

Main difficulty : expression swell and take advantage of sparsity

Motivations

Large linear systems are involved in many mathematical applications

Over a finite field : integers factorization [Odlyzko 1999],
discrete logarithm [Odlyzko 1999 ; Thomé 2003].

Over the integers : number theory [Cohen 1993], group theory [Newman 1972],
integer programming [Aardal, Hurkens, Lenstra 1999].

Rational linear systems are central in recent linear algebra algorithms

- Determinant [Abbott, Bronstein, Mulders 1999 ; Storjohann 2005]
- Smith form [Giesbrecht 1995 ; Eberly, Giesbrecht, Villard 2000]
- Nullspace, Kernel [Chen, Storjohann 2005]

Algorithms for non-singular system solving

Dense matrices :

- Gaussian elimination and CRA
  $\hookrightarrow$ $\tilde{O}(n^{\omega+1} \log ||A||)$ bit operations
- P-adic lifting [Monck, Carter 1979 ; Dixon 1982]
  $\hookrightarrow$ $\tilde{O}(n^3 \log ||A||)$ bit operations
- High order lifting [Storjohann 2005]
  $\hookrightarrow$ $\tilde{O}(n^\omega \log ||A||)$ bit operations

Sparse matrices :

- P-adic lifting or CRA [Kaltofen, Saunders 1991]
  $\hookrightarrow$ $\tilde{O}(\gamma n^2 \log ||A||)$ bit operations with γ non-zero elts.

P-adic algorithm with matrix inversion

Scheme to compute $A^{-1}b$ [Dixon 1982] :

$$
\begin{array}{ll}
\text{(1-1)} & B := A^{-1} \bmod p \\
\text{(1-2)} & r := b \\
 & \text{for } i := 0 \text{ to } k \\
\text{(2-1)} & \quad x_i := B.r \bmod p \\
\text{(2-2)} & \quad r := (1/p)(r - A.x_i) \\
\text{(3-1)} & x := \sum_{i=0}^{k} x_i.p^i \\
\text{(3-2)} & \textit{rational reconstruction on } x
\end{array}
$$

P-adic algorithm with matrix inversion

Scheme to compute $A^{-1}b$ [Dixon 1982] :

(1-1)	$B := A^{-1} \bmod p$	$O\tilde{}(n^3 \log \|\|A\|\|)$
(1-2)	$r := b$	
	for $i := 0$ to k	$k = O\tilde{}(n)$
(2-1)	$\quad x_i := B.r \bmod p$	$O\tilde{}(n^2 \log \|\|A\|\|)$
(2-2)	$\quad r := (1/p)(r - A.x_i)$	$O\tilde{}(n^2 \log \|\|A\|\|)$
(3-1)	$x := \sum_{i=0}^{k} x_i.p^i$	
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(3-1)	$x := \sum_{i=0}^{k} x_i.p^i$	
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Main operations : matrix inversion and matrix-vector products

Dense linear system solving in practice

Efficient implementations are available : LinBox 1.0 [www.linalg.org]

- Use tuned BLAS floating-point library for exact arithmetic
  - matrix inversion over prime field [Dumas, Giorgi, Pernet 2004]
  - BLAS matrix-vector product with CRT over the integers
- Rational number reconstruction
  - half GCD [Schönage 1971]
  - heuristic using integer multiplication [NTL library]

random dense linear system with 3 bits entries (P4 - 3.4Ghz)

n	500	1000	2000	3000	4000	5000
Time	0.6s	4.3s	31.1s	99.6s	236.8s	449.2s

performances improvement of a factor 10
over NTL's tuned implementation

What does happen when matrices are sparse ?

We consider sparse matrices with $O(n)$ non zero elements

$\hookrightarrow$ matrix-vector product needs only $O(n)$ operations.

Sparse linear system and P-adic lifting

Computing the modular inverse is proscribed due to fill-in

Solution [Kaltofen, Saunders 1991] :
$\hookrightarrow$ use modular minimal polynomial instead of inverse

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Wiedemann approach (1986)

Let $A \in \mathbb{F}^{n \times n}$ of full rank and $b \in \mathbb{F}^{n}$. Then $x = A^{-1}b$ can be expressed as a linear combination of the Krylov subspace $\{b, Ab, \ldots, A^n b\}$

Let $f^A(\lambda) = f_0 + f_1\lambda + \ldots + f_d\lambda^d \in \mathbb{F}[\lambda]$ be the minimal polynomial of A

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$$A^{-1}b \quad = \quad \frac{-1}{f_0}(f_1 b + f_2 Ab + ... + f_d A^{d-1} b)$$

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$$A^{-1}b \quad = \quad \underbrace{\frac{-1}{f_0}(f_1 b + f_2 Ab + \ldots + f_d A^{d-1} b)}_{x}$$

Applying minpoly in each lifting steps cost $O\tilde{}(nd)$ field operations, then giving a worst case complexity of $O\tilde{}(n^3 \log ||A||)$ bit operations.

Sparse linear system solving in practice

use of LinBox library on Itanium II - 1.3Ghz, 128Gb RAM

- random systems with 3 bits entries and 10 elts/row (plus identity)

	system order				
	400	*900*	*1600*	*2500*	*3600*
Maple	64.7s	849s	11098s	–	–
CRA-Wied	14.8s	168s	1017s	3857s	11452s
P-adic-Wied	10.2s	113s	693s	2629s	8034s
Dixon	**0.9s**	**10s**	**42s**	**178s**	**429s**

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main difference :

$$x_i = B.r \bmod p \tag{2-1} \quad (\textit{Dixon})$$

$$x_i := \frac{-1}{f_0} \sum_{i=1}^{\deg f^A} f_i A^{i-1} r \bmod p \tag{2-1} \quad (\textit{P-adic-Wied})$$

Remark :

n sparse matrix applications is far from level 2 BLAS in practice.

Our objectives

In practice :

Integrate level 2 and 3 BLAS in integer sparse solver

In theory :

Improve bit complexity of sparse linear system solving

$\Longrightarrow$ $O\tilde{}(n^{\delta})$ bits operations with $\delta < 3$?

Our alternative to Block Wiedemann

Express the inverse of the sparse matrix through a structured form
$\hookrightarrow$ block Hankel/Toeplitz structures

Let $u \in \mathbb{F}^{s \times n}$ and $v \in \mathbb{F}^{n \times s}$ s.t. following matrices are non-singular

$$U = \begin{pmatrix} u \\ uA \\ \vdots \\ uA^{m-1} \end{pmatrix}, \quad V = \left(\begin{array}{c|c|c|c} v & Av & \dots & A^{m-1}v \end{array} \right) \in \mathbb{F}^{n \times n}$$

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then we can define the block Hankel matrix

$$H = UAV = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_m \\ \alpha_2 & \alpha_3 & \cdots & \alpha_{m+1} \\ \vdots & & & \\ \alpha_m & \alpha_m & \cdots & \alpha_{2m-1} \end{pmatrix}, \quad \alpha_i = uA^i v \in \mathbb{F}^{s \times s}$$

and thus we have $A^{-1} = VH^{-1}U$

Alternative to Block Wiedemann

- Nice property on block Hankel matrix inverse [Gohberg, Krupnik 1972, Labahn, Choi, Cabay 1990]

$$H^{-1} = \underbrace{\begin{pmatrix} * & \cdots & * \\ \vdots & \iddots & \\ * & & \end{pmatrix}}_{H_1} \underbrace{\begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix}}_{T_1} - \underbrace{\begin{pmatrix} * & \cdots & * \\ \vdots & \iddots & \\ * & & \end{pmatrix}}_{H_2} \underbrace{\begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix}}_{T_2}$$

where H_1, H_2 are block Hankel matrices and T_1, T_2 are block Toeplitz matrices

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where H_1, H_2 are block Hankel matrices and T_1, T_2 are block Toeplitz matrices

- Block coefficients in H_1, H_2, T_1 , T_2 come from Hermite Pade approximants of $H(z) = \alpha_1 + \alpha_2 z + \ldots + \alpha_{2m-1} z^{2m-2}$ [Labahn, Choi, Cabay 1990].

- Complexity of H^{-1} reduces to polynomial matrix multiplication [Giorgi, Jeannerod, Villard 2003].

Alternative to Block Wiedemann

Scheme to compute $A^{-1}b$:

(1-1) $H(z) := \sum_{i=1}^{2m-1} uA^i v.z^{i-1} \bmod p$

(1-2) compute $H^{-1} \bmod p$ from $H(z)$

(1-3) $r := b$

for $i := 0$ to k

(2-1) $x_i := VH^{-1}U.r \bmod p$

(2-2) $r := (1/p)(r - A.x_i)$

(3-1) $x := \sum_{i=0}^{k} x_i.p^i$

(3-2) *rational reconstruction on x*

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Scheme to compute $A^{-1}b$:

(1-1)	$H(z) := \sum_{i=1}^{2m-1} uA^i v.z^{i-1} \bmod p$	$O\tilde{}(sn^2 \log \|\|A\|\|)$
(1-2)	compute $H^{-1} \bmod p$ from $H(z)$	$O\tilde{}(s^2 n \log \|\|A\|\|)$
(1-3)	$r := b$	
	for $i := 0$ to k	$k = O\tilde{}(n)$
(2-1)	$x_i := VH^{-1}U.r \bmod p$	$O\tilde{}((n^2 + sn) \log \|\|A\|\|)$
(2-2)	$r := (1/p)(r - A.x_i)$	$O\tilde{}(n \log \|\|A\|\|)$
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(1-3)	$r := b$	
	for $i := 0$ to k	$k = O\tilde{}(n)$
(2-1)	$x_i := V H^{-1} U.r \bmod p$	$O\tilde{}((n^2 + sn) \log \|\|A\|\|)$
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Not yet satisfying : applying matrices U and V is too costly

Applying block Krylov subspaces

$$V = \left(v \middle| Av \middle| \dots \middle| A^{m-1}v\right) \in \mathrm{I\!F}^{\mathrm{n}\times\mathrm{n}} \text{ and } \mathrm{v} \in \mathrm{I\!F}^{\mathrm{n}\times\mathrm{s}}$$

can be rewrite as

$$V = \left(v \middle| \quad\quad \right) + A\left(\quad \middle| v \middle| \quad\quad \right) + \dots + A^{m-1}\left(\quad\quad \middle| v \right)$$

Therefore, applying V to a vector corresponds to :

- $m-1$ linear combinations of columns of v
- $m-1$ applications of A

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Therefore, applying V to a vector corresponds to :

- $m-1$ linear combinations of columns of v $\quad O(m \times sn \log ||A||)$
- $m-1$ applications of A $\quad O(mn \log ||A||)$

Applying block Krylov subspaces

$$V = \left(v \middle| Av \middle| \ldots \middle| A^{m-1}v \right) \in \mathbb{F}^{n \times n} \text{ and } v \in \mathbb{F}^{n \times s}$$

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How to improve the complexity ?

Applying block Krylov subspaces

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Therefore, applying V to a vector corresponds to :

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- $m-1$ applications of A

How to improve the complexity ?

$\Rightarrow$ using special block projections u and v

Candidates as suitable block projections

Considering $A \in \mathbb{F}^{n \times n}$ non-singular and $n = m \times s$.

Let us denote $\mathcal{K}(A, v) := \left[\ v \mid Av \mid \cdots \mid A^{m-1}v \ \right] \in \mathbb{F}^{n \times n}$

Conjecture :

for any non-singular $A \in \mathbb{F}^{n \times n}$ and $s|n$ there exists a suitable block projection $(R, u, v) \in \mathbb{F}^{n \times n} \times \mathbb{F}^{s \times n} \times \mathbb{F}^{n \times s}$

such that :

1. $\mathcal{K}(RA, v)$ and $\mathcal{K}((RA)^T, u^T)$ are non-singular,
2. R can be applied to a vector with $\tilde{O}(n)$ operations,
3. u, u^T, v and v^T can be applied to a vector with $\tilde{O}(n)$ operations.

A structured block projection

Let v be defined as follow

$$v^T = \begin{pmatrix} v_1 \; \dots \; v_m & & & \\ & v_{m+1} \; \dots \; v_{2m} & & \\ & & \ddots & \\ & & & v_{n-m+1} \; \dots \; v_n \end{pmatrix} \in \mathbb{F}^{s \times n}$$

where v_i's are chosen randomly from a sufficient large set.

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where v_i's are chosen randomly from a sufficient large set.

<u>open question</u> : Let R diagonal and v as defined above,

is $\mathcal{K}(RA, v)$ necessarily non-singular ?

We prooved it for case $s = 2$ but answer is still unknown for $s > 2$

Our new algorithm

Scheme to compute $A^{-1}b$:

(1-1) choose structured blocks u and v

(1-2) choose R and $A := R.A$, $b := R.b$

(1-3) $H(z) := \sum_{i=1}^{2m-1} uA^i v.z^{i-1} \bmod p$

(1-4) compute $H^{-1} \bmod p$ from $H(z)$

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(1-5)	$r := b$	
	for $i := 0$ to k	$k = \tilde{O}(n)$
(2-1)	$x_i := VH^{-1}U.r \bmod p$	$\tilde{O}((mn + sn) \log \|\|A\|\|)$
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taking the optimal $m = s = \sqrt{n}$ gives a complexity of $\tilde{O}(n^{2.5} \log ||A||)$

High level implementation

LinBox project (Canada-France-USA) : `www.linalg.org`

Our tools :

- BLAS-based matrix multiplication and matrix-vector product
- polynomial matrix arithmetic (block Hankel inversion)
  $\hookrightarrow$ *FFT, Karatsuba, middle product*
- fast application of H^{-1} is needed to get $O\tilde{}(n^{2.5} \log ||A||)$

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- BLAS-based matrix multiplication and matrix-vector product
- polynomial matrix arithmetic (block Hankel inversion)
  $\hookrightarrow$ *FFT, Karatsuba, middle product*
- fast application of H^{-1} is needed to get $O\tilde{~}(n^{2.5}\log\|A\|)$
  - Lagrange's representation of H^{-1} at the beginning (*Horner's scheme*)
  - use evaluation/interpolation on polynomial vectors
    $\hookrightarrow$ *use Vandermonde matrix to have dense matrix operations*

Is our new algorithm efficient in practice ?

Comparing performances

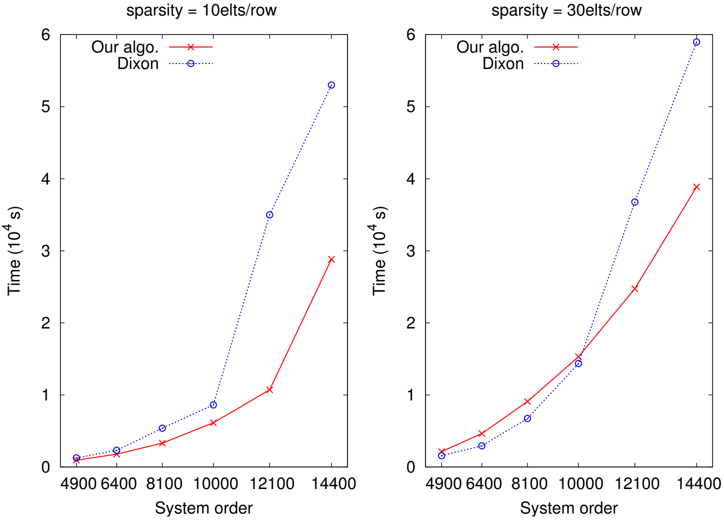
use of LinBox library on Itanium II - 1.3Ghz, 128Gb RAM

- random systems with 3 bits entries and 10 elts/row (plus identity)

	system order					extra memory
	900	*1600*	*2500*	*3600*	*4900*	
Maple 10	849s	11 098s	–	–	–	$O(1)$
CRA-Wied	168s	1 017s	3 857s	11 452s	≈ 28 000s	$O(n)$
P-adic-Wied	113s	693s	2 629s	8 034s	≈ 20 000s	$O(n)$
Dixon	**10s**	**42s**	178s	429s	1 257s	$O(n^2)$
Our algo.	15s	61s	**175s**	**426s**	**937s**	$O(n^{1.5})$

The expected $\sqrt{n}$ improvement is unfortunately amortized by a high constant in the complexity.

Sparse solver vs Dixon's algorithm



Our algorithm performances are depending on matrix sparsity

Practical effect of blocking factors

$\sqrt{n}$ blocking factor value is theoretically optimal

Is this still true in practice ?

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system order = 10 000, optimal block = 100

block size	*80*	*125*	*200*	400	*500*
timing	7213s	5264s	4059s	**3833s**	4332s

system order = 20 000, optimal block ≈ 140

block size	*125*	*160*	*200*	500	*800*
timing	44720s	35967s	30854s	**28502s**	37318s

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best practical blocking factor is dependent upon the ratio of sparse matrix/dense matrix operations efficiency

Conclusions

We provide a new approach for solving sparse integer linear systems :

- improve the complexty by a factor $\sqrt{n}$ (heuristic).
- improve efficiency by minimizing sparse matrix operations and maximizing BLAS use.

<u>drawback</u> : not taking advantage of low degree minimal polynomial

We propose special block projections for sparse linear algebra
$\hookrightarrow$ inverse of sparse matrix in $O(n^{2.5})$ field op.

Conclusions

We provide a new approach for solving sparse integer linear systems :

- improve the complexty by a factor $\sqrt{n}$ (heuristic).
- improve efficiency by minimizing sparse matrix operations and maximizing BLAS use.

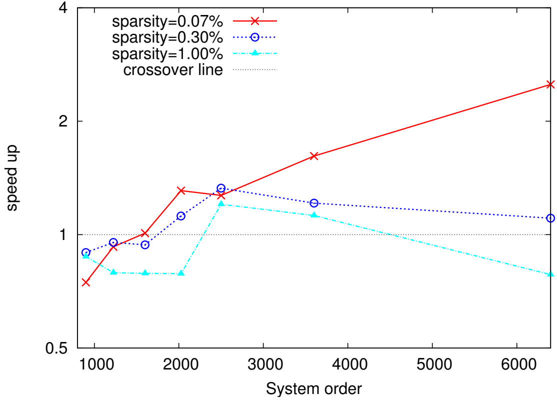
drawback : not taking advantage of low degree minimal polynomial

We propose special block projections for sparse linear algebra
$\hookrightarrow$ inverse of sparse matrix in $O(n^{2.5})$ field op.

Ongoing work :

- provide an automatic choice of block dimension (non square ?)
- prove conjecture for our structured block projections
- handle the case of singular matrix
- introduce fast matrix multiplication in the complexity

Sparse solver vs Dixon's algorithm



The sparser the matrices are, the earlier the crossover appears