Memory-efficient polynomial arithmetic

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Multiplication of polynomials

- **Input.** $F = \sum_{i=0}^{n-1} F[i]X^i$ and $G = \sum_{j=0}^{n-1} G[j]X^j$
- **Output.** $H = F \times G = \sum_{k=0}^{2n-2} H[k]X^k$

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  $= f_0 g_0 + ((f_0 + f_1)(g_0 + g_1) - f_0 g_0 - f_1 g_1) X^{\frac{n}{2}} + f_1 g_1 X^n$

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- Toom-Cook algorithm: split F and G in three or more parts
- FFT-based algorithms:
  $(F, G) \xrightarrow{\text{eval.}} (F(\omega^i), G(\omega^i))_i \xrightarrow{\text{mult.}} FG(\omega^i)_i \xrightarrow{\text{interp.}} FG$

Time complexity of polynomial arithmetic

- Multiplication: $\mathsf{M}(n)$
  - Naïve: $O(n^2)$
  - Karatsuba: $O(n^{\log_2 3}) = O(n^{1.585})$ Karatsuba (1962)
  - Toom-3: $O(n^{\log_3 5}) = O(n^{1.465})$ Toom (1963), Cook (1966)
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What about space complexity?

Space complexity of polynomial arithmetic

First thought: count extra memory apart from input/output

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However, need to precise the complexity model !!!

Space-complexity models

Algebraic-RAM machine:

- $\rightarrow$ *Standard* registers of size $O(\log n)$
- $\rightarrow$ *Algebraic* registers containing one coefficient

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Previous results

Karatsuba's algorithm:

$$\left(f_0+X^{\frac{n}{2}}f_1\right)\cdot\left(g_0+X^{\frac{n}{2}}g_1\right)=f_0g_0+((f_0+f_1)(g_0+g_1)-f_0g_0-f_1g_1)X^{\frac{n}{2}}+f_1g_1X^n$$

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- Thomé (2002) : space of $n + O(\log n)$
  $\rightarrow$ careful use output + n temp. registers + $O(\log n)$ stack
- Roche (2009): space of only $O(\log n)$
  $\rightarrow$ *half-additive* version ($h \leftarrow h_\ell + fg$ where $\deg(h_\ell) < n$)

Previous results

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space of $2n$: FFT is in-place (overwriting) but # points $\approx 2n$

- Roche (2009): space of $O(1)$ when $n = 2^k$ and $\omega^{2n} = 1$
  → compute half of the result + recurse
- Harvey-Roche (2010): space of $O(1)$ when $\omega^{2n} = 1$
  → same with TFT v.d. Hoeven (2004)

Previous results

Summary of complexities

Algorithms	**Time complexity**	**Space complexity**
naive	$2n^2 + 2n - 1$	$O(1)$
Karatsuba ('62)	$< 6.5n^{\log(3)}$	$\leq 2n + 5\log(n)$
Karatsuba (Thomé'02)	$< 7n^{\log(3)}$	$\leq n + 5\log(n)$
Karatsuba (Roche'09)	$< 10n^{\log(3)}$	$\leq 5\log(n)$
Toom-3 ('63)	$< \frac{73}{4}n^{\log_3(5)}$	$\leq 2n + 5\log_3(n)$
FFT (CT'65)	$9n\log(2n) + O(n)$	$2n$
FFT (Roche'09)	$11n\log(2n) + O(n)$	$O(1)$
TFT (HR'10)	$O(n\log(n))$	$O(1)$

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Can *every* polynomial multiplication algorithm be performed without extra memory?

- $O(1)$-space Karatsuba's algorithm?
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- What about other products (short and middle)?

Results:

- Yes!
- Almost (for other products)

Outline

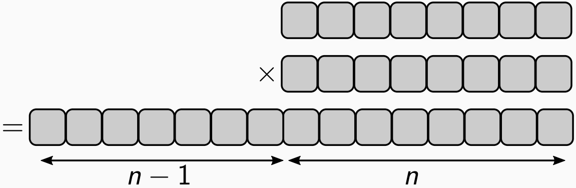
Polynomial products and linear maps

Space-preserving reductions

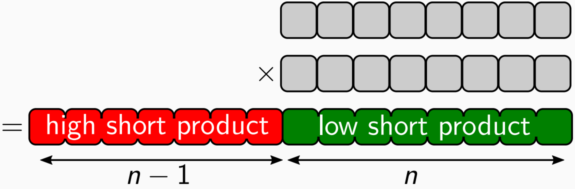
In-place algorithms from out-of-place algorithms

Polynomial products and linear maps

Short product



Short product



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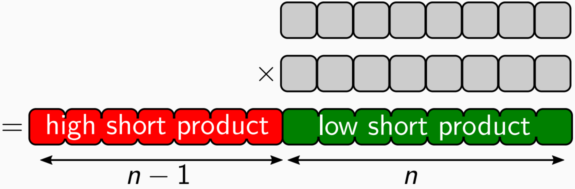
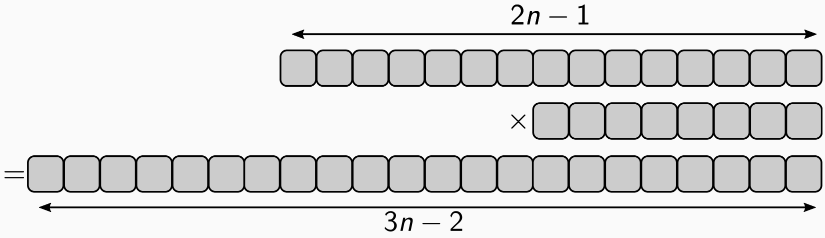


- Low short product: product of truncated power series
- Useful in other algorithms
- Time complexity: $\mathrm{M}(n)$
- Space complexity: $O(n)$

Middle product



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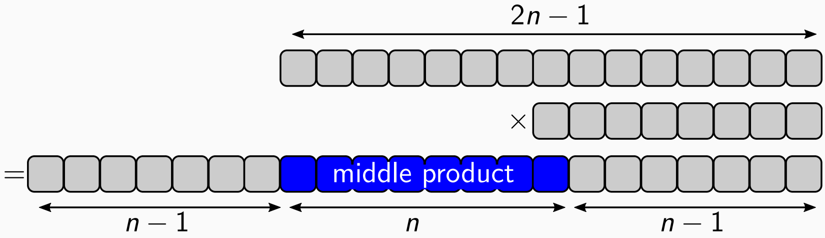

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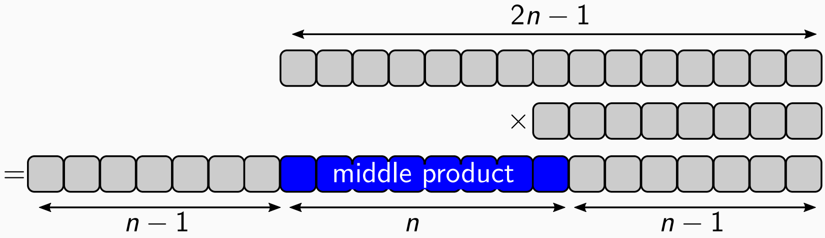


- Useful for Newton iteration
    - $G \leftarrow G(1-GF) \bmod X^{2n}$ with $GF = 1 + X^n H$
    - division, square root, ...
- Time complexity: $\mathsf{M}(n) \rightarrow$ Tellegen's transposition
- Space complexity: $O(n)$

Multiplications as linear maps

Example:

$$
\begin{aligned}
f &= 3X^2 + 2X + 1 \\
g &= X^2 + 2X + 4 \\
fg &= 3X^4 + 8X^3 + 17X^2 + 10X + 4
\end{aligned}
$$

Multiplications as linear maps

Example:

$$f = 3X^2 + 2X + 1$$

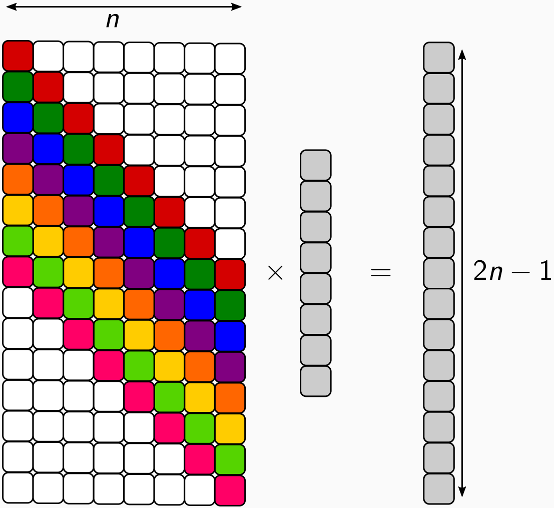
$$g = X^2 + 2X + 4$$

$$fg = 3X^4 + 8X^3 + 17X^2 + 10X + 4$$

$$\begin{bmatrix} 1 & & \\ 2 & 1 & \\ 3 & 2 & 1 \\ & 3 & 2 \\ & & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 17 \\ 8 \\ 3 \end{bmatrix}$$

Multiplications as linear maps

Full product:



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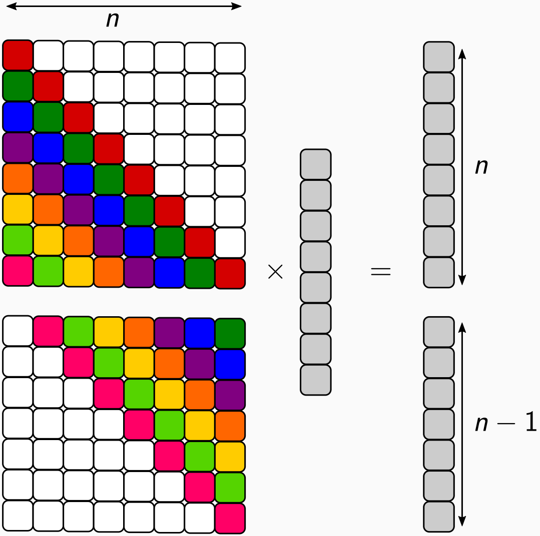

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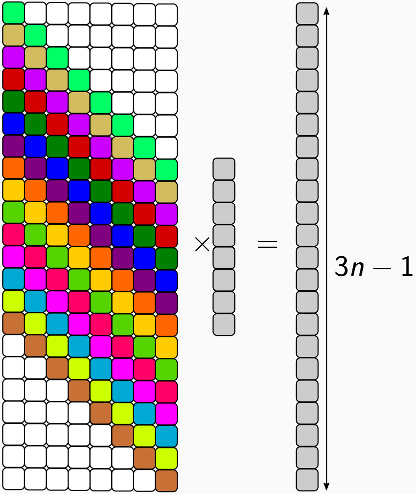

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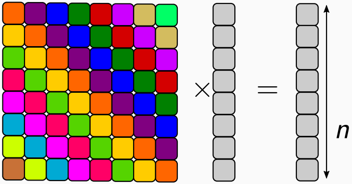
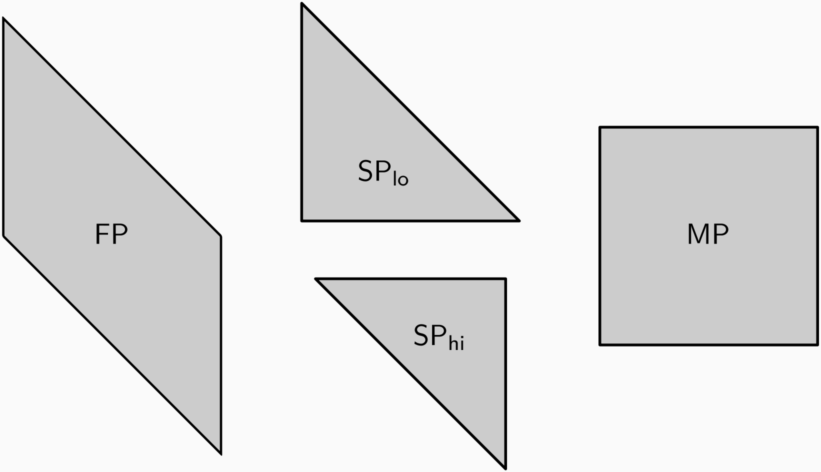


Multiplications as linear maps

For simplicity in the presentation we assume



Full product Short products Middle product

Space-preserving reductions

Relative difficulties of products

- Without space restrictions:
    - $\mathsf{SP} \leq \mathsf{FP}$ and $\mathsf{FP} \leq \mathsf{SP}_{\mathsf{lo}} + \mathsf{SP}_{\mathsf{hi}}$
    - $\mathsf{MP} \equiv \mathsf{FP}$ (transposition)
    - $\mathsf{MP} \leq \mathsf{SP}_{\mathsf{lo}} + \mathsf{SP}_{\mathsf{hi}} + (n-1)$ additions

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    - MP $\equiv$ FP (transposition)
    - MP $\leq \mathrm{SP}_{\mathrm{lo}} + \mathrm{SP}_{\mathrm{hi}} + (n-1)$ additions

- Size of inputs and outputs:
    - FP : $(n, n) \rightarrow 2n-1$
    - $\mathrm{SP}_{\mathrm{lo}}$: $(n, n) \rightarrow n$
    - $\mathrm{SP}_{\mathrm{hi}}$: $(n-1, n-1) \rightarrow n-1$
    - MP : $(2n-1, n) \rightarrow n$

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✓ We provide space/time preserving reductions

A relevant notion of reduction

Definitions

- $\mathrm{TISP}(t(n), s(n))$: computable in time $t(n)$ and space $s(n)$
- $A \leq_c B$: A is computable with oracle B

  if $B \in \mathrm{TISP}(t(n), s(n))$ then

$$A \in \mathrm{TISP}(c\,t(n) + o(t(n)), s(n) + O(1))$$

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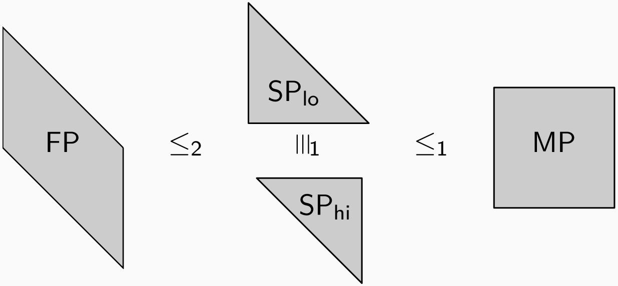
Example

$A \equiv_1 B$ means A and B are equivalent for both time and space

First results in a nutshell

Theorem



$\mathrm{FP} \le_2 \mathrm{SP}_{\mathrm{lo}}$

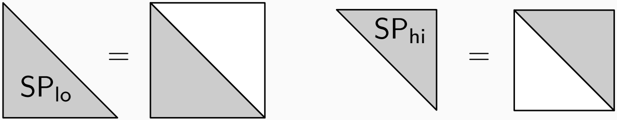
$|||_1$

$\mathrm{SP}_{\mathrm{hi}} \le_1 \mathrm{MP}$

Visual proof

Use of *fake padding* (in input, **not** in output!)

- $\mathrm{SP}_{\mathrm{lo}}(n) \leq \mathrm{MP}(n)$; $\mathrm{SP}_{\mathrm{hi}}(n) \leq \mathrm{MP}(n-1)$



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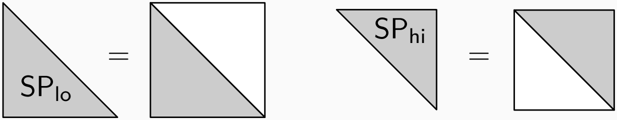
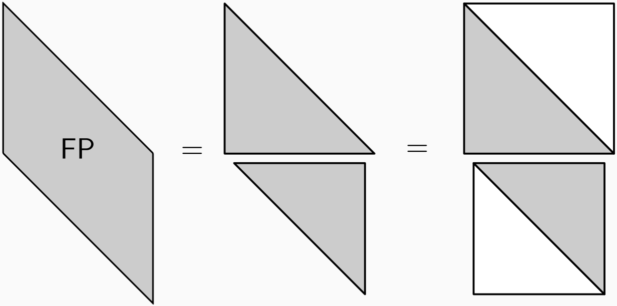
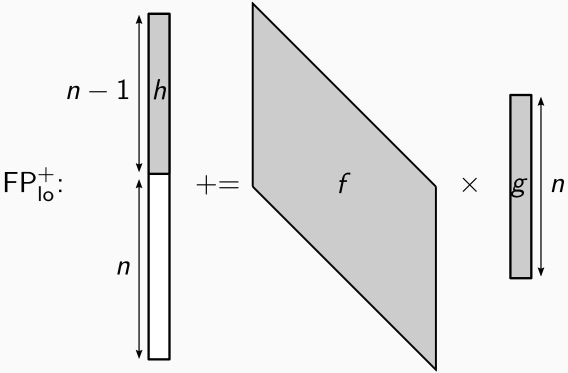


- $\mathrm{FP}(n) \leq \mathrm{SP}_{\mathrm{hi}}(n) + \mathrm{SP}_{\mathrm{lo}}(n) \leq \mathrm{MP}(n) + \mathrm{MP}(n-1)$



Half-additive full product: $h \leftarrow h + f \cdot g$



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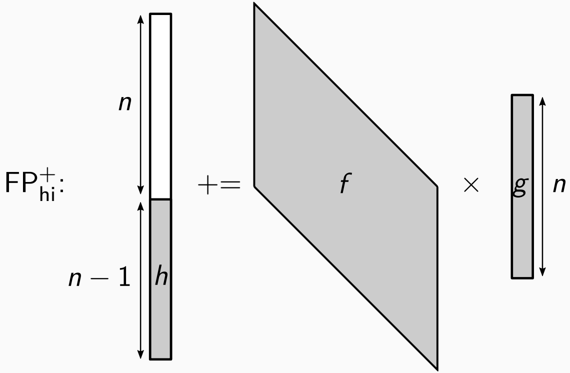

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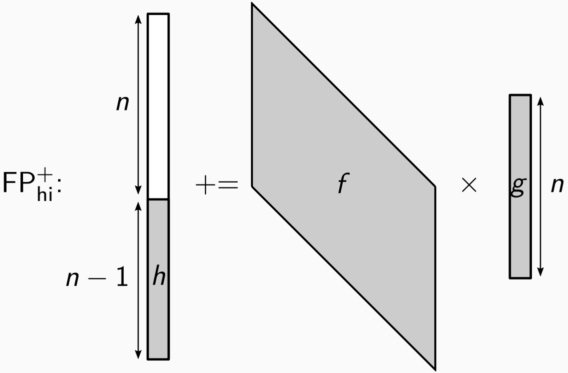

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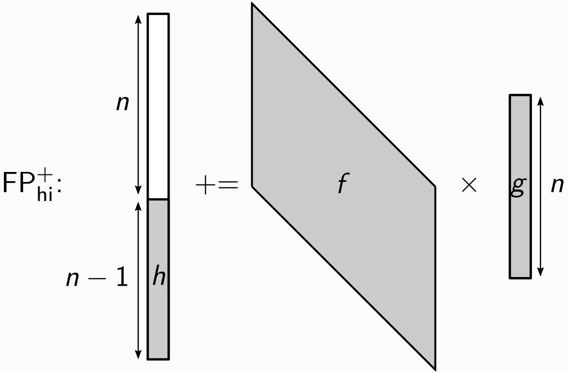

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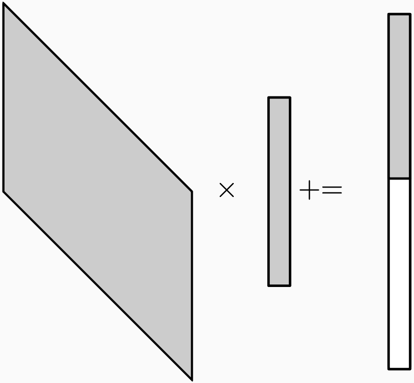
Theorem $\mathsf{FP}^+ \leq_2 \mathsf{SP}$ and $\mathsf{SP} \leq_{3/2} \mathsf{FP}^+$

From SP to FP$^+$



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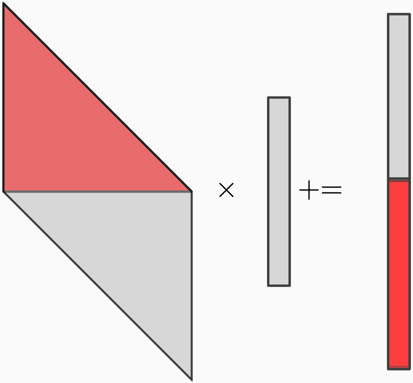

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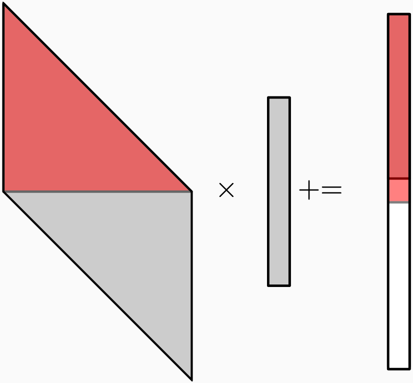

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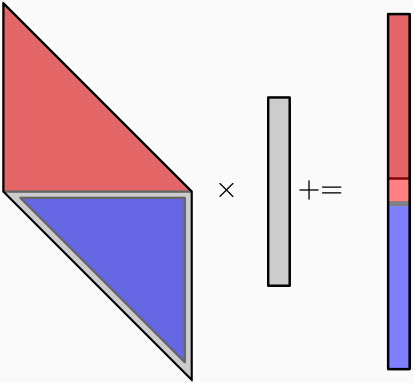

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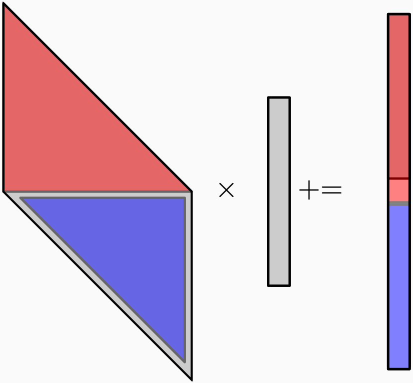

$$\mathsf{FP}^+_{\mathsf{lo}}(n) \leq \mathsf{SP}_{\mathsf{lo}}(n) + \mathsf{SP}_{\mathsf{hi}}(n) + n - 1$$

From FP$^+$ to SP

$$\left(f_0 + X^{\lceil n/2 \rceil} f_1\right) \cdot \left(g_0 + X^{\lceil n/2 \rceil} g_1\right) = f_0 g_0 + X^{\lceil n/2 \rceil}(f_0 g_1 + f_1 g_0) \mod X^n$$

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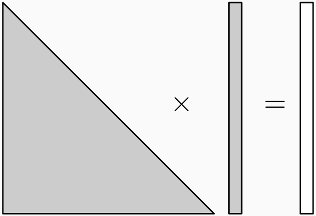

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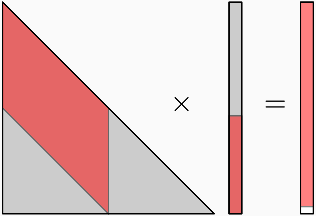

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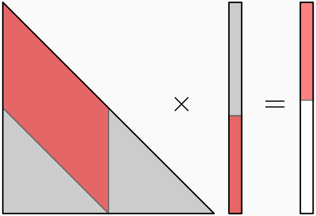

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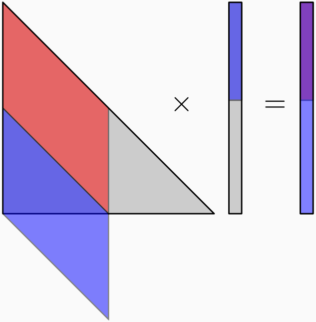

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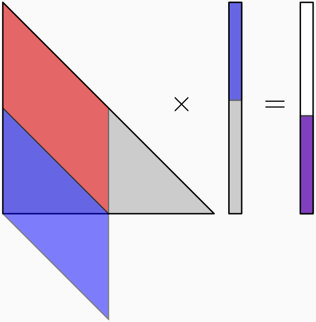

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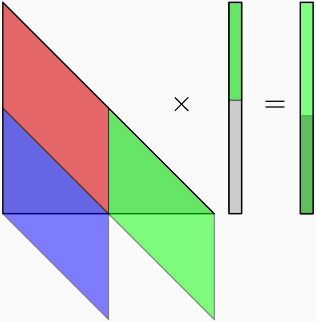

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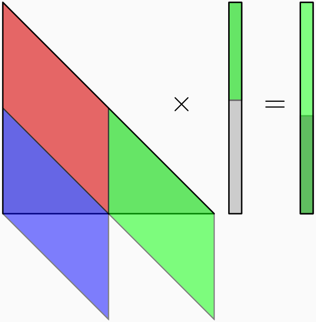

$$\mathrm{SP}_{\mathrm{lo}}(n) \leq \mathrm{FP}(\lfloor n/2 \rfloor) + \mathrm{FP}^+_{\mathrm{lo}}(\lfloor n/2 \rfloor) + \mathrm{FP}^+_{\mathrm{hi}}(\lceil n/2 \rceil)$$

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- From FP to SP:
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- From FP to SP:
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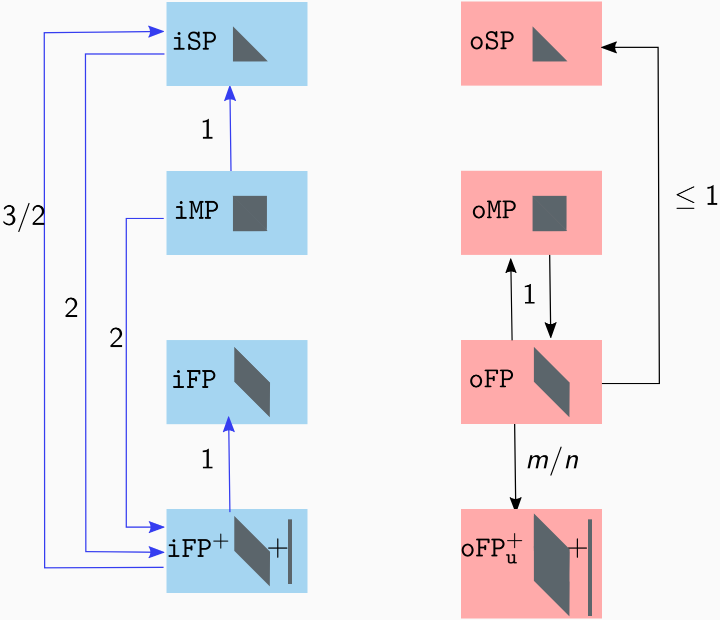
- From SP to MP:
  - partial result:
    - up to $\log(n)$ increase in time complexity
    - techniques from next part
  - without space restriction
    - FP to MP through Tellegen's transposition principle

Summary of results so far



In-place algorithms from out-of-place algorithms

Framework

- In-place algorithms parametrized by out-of-place algorithm
    - Out-of-place: uses cn extra space
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Similar approach for matrix mul. : Boyer, Dumas, Pernet, Zhou (2009)

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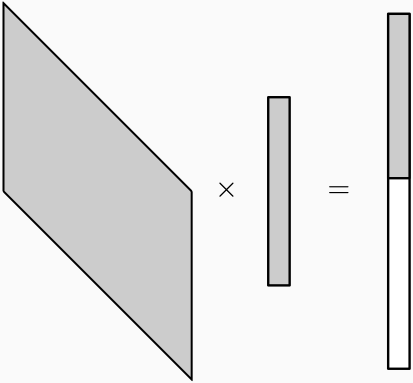
- *Fake* padding:
    - Pretend to pad inputs with zeroes
    - Make the data structure responsible for it
        - $O(1)$ increase in memory
        - *Cf.* strides in dense linear algebra
    - OK in inputs, not in outputs!

Our results

- In-place full product (half additive) in time $(2c+7)\mathsf{M}(n)$
- In-place short product in time $(2c+5)\mathsf{M}(n)$
- In-place middle product in time $O(\mathsf{M}(n)\log n)$

In-place FP^+ from out-of-place FP

$$(f_0 + X^k \hat{f}) \cdot (g_0 + X^k \hat{g}) = f_0 g_0 + X^k (f_0 \hat{g} + \hat{f} g_0) + X^{2k} \hat{f} \hat{g}$$



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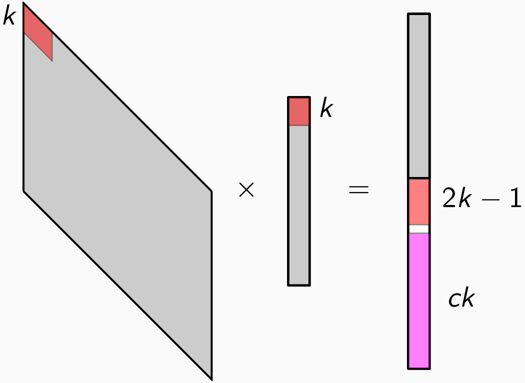
$$(f_0 + X^k \hat{f}) \cdot (g_0 + X^k \hat{g}) = \color{red}{f_0 g_0} \color{black} + X^k(f_0\hat{g} + \hat{f} g_0) + X^{2k}\hat{f}\hat{g}$$



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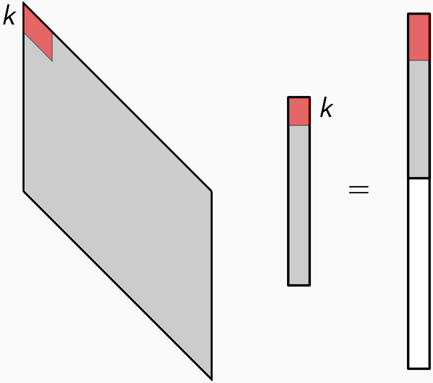

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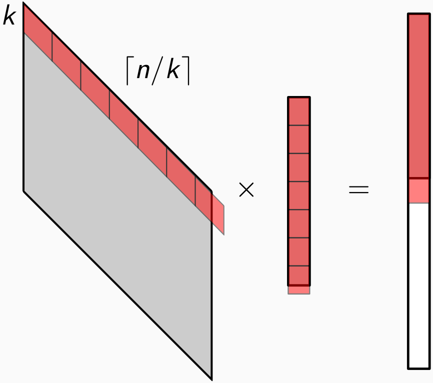

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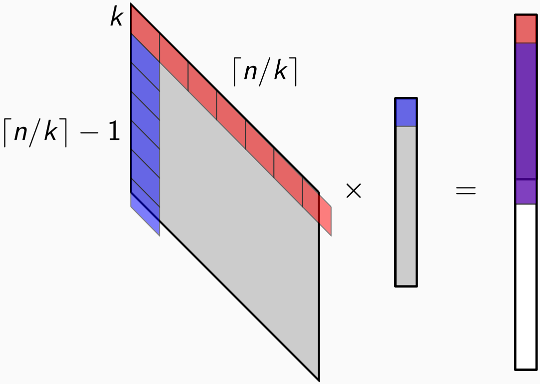

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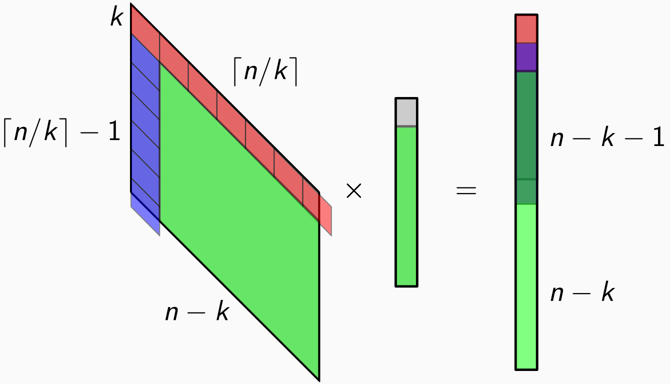
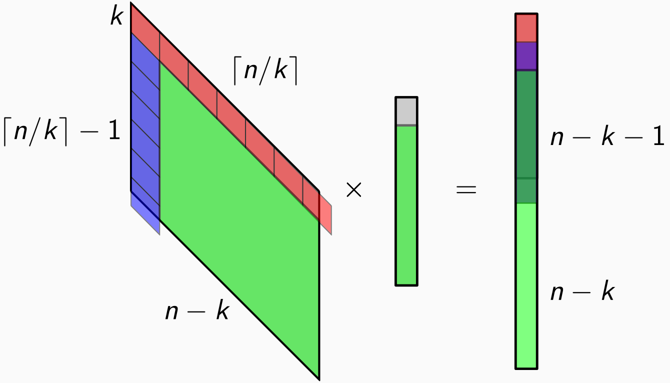


Analysis



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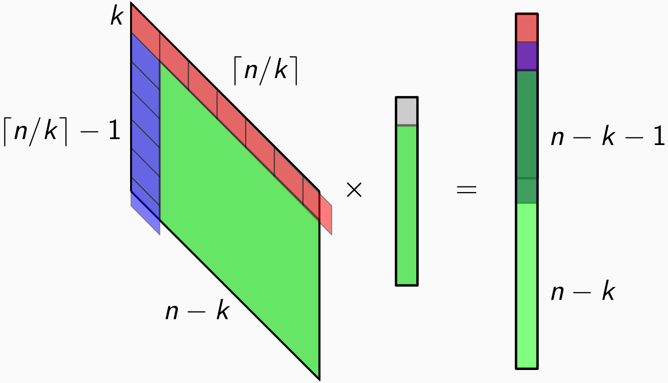

- $ck + 2k - 1 \leq n - k \implies k \leq \frac{n+1}{c+3}$
- $T(n) = (2\lceil n/k \rceil - 1)(\mathrm{M}(k) + 2k - 1) + T(n-k)$

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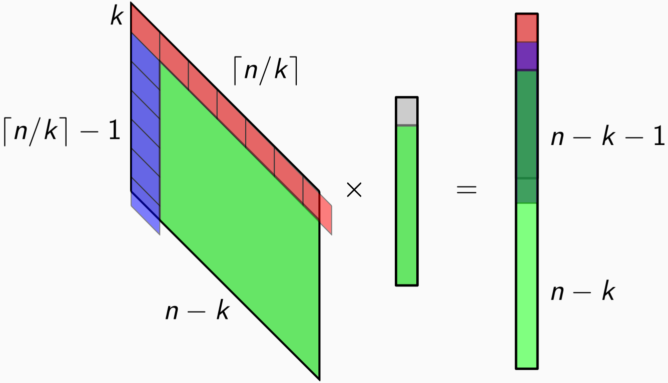

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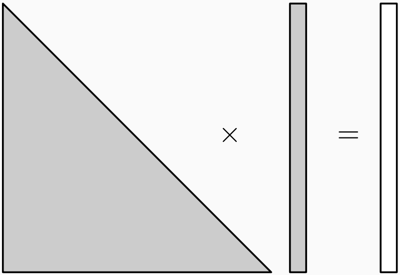
In-place short product



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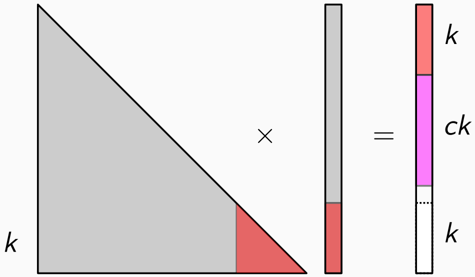

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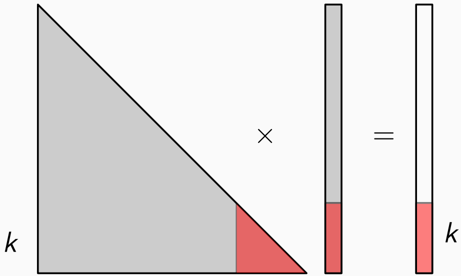

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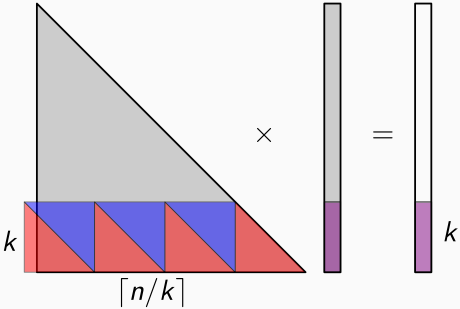

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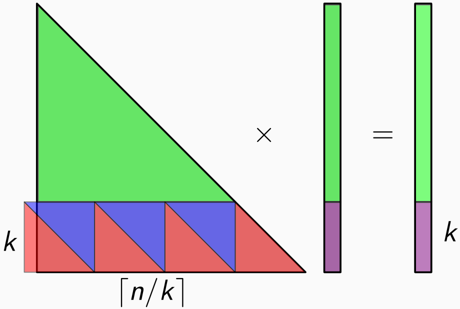

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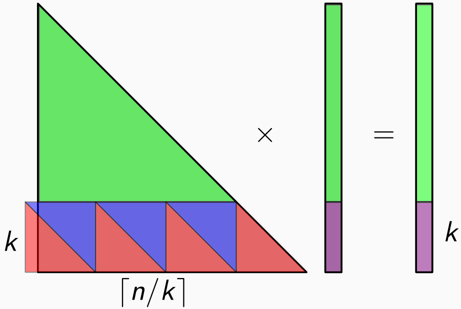

- $k \leq n/(c+2)$
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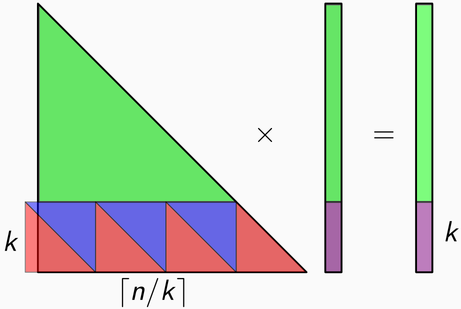

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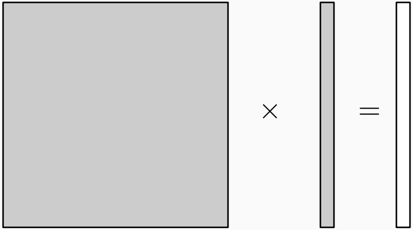

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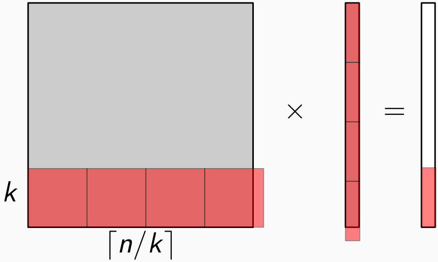

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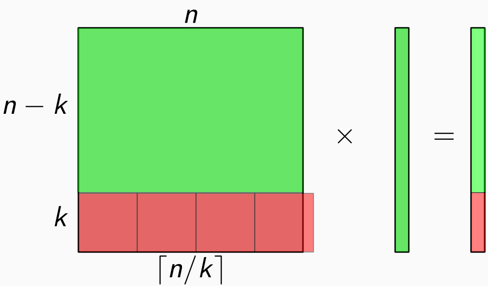

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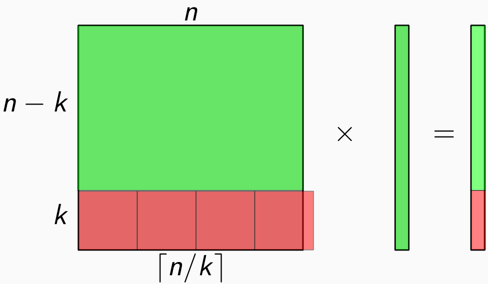

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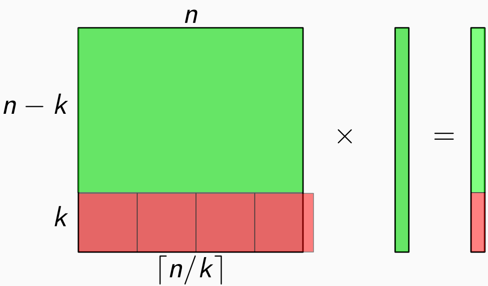

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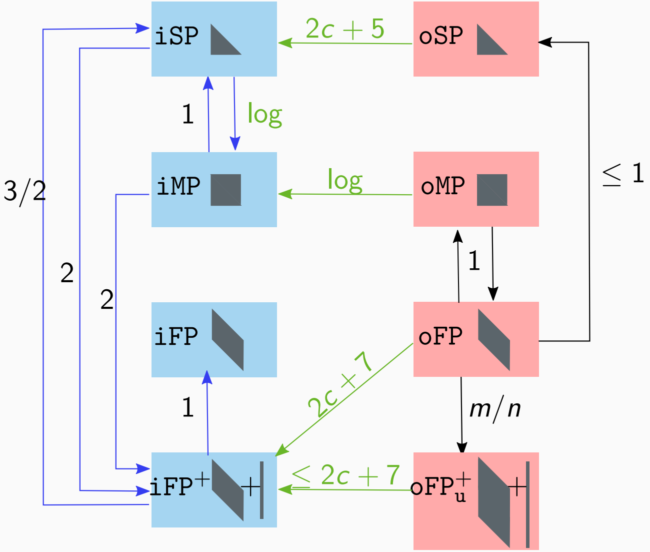
Remark

- In place: division with remainder
- Only quotient or only remainder: not clear
- Main difficulty: size of the output

Summary of the results



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