Colouring graphs of bounded cliquewidth

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March 31, 2021

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$\omega(G)$: size of a largest clique.

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Not always possible: intersections of segments in the plane.

Good χ-bounding functions

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For every k, every graph of cliquewidth $\leq k$ satisfies $\chi \leq \omega^{2^{2^{O(k \log k)}}}$.

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- $\mathrm{cw} \leq |V(F)| \leq (2^d)^2$ (by considering $R(3, d \cdot 2^d)$)

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Here: Either elementary (single vertex) or there is a cut of "low diversity".

Useful tools

Theorem (Chudnovsky, Penev, Scott, Trotignon '13)

For every hereditary class $\mathcal{G}$, if $\exists c$ such that $\chi \leq \omega^c$, then $\exists d$ such that $\chi \leq \omega^d$ for any graph in the closure of $\mathcal{G}$ under substitutions.

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(Colcombet's version of) Simon's Factorization Forest Theorem.
$\rightsquigarrow$ "decomposition depth" at most $2^{O(k \log k)}$

Conclusion

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Thanks! 😎
(See you at 2pm for Rose's talk)