Rank-width, circle graphs, and vertex-minors

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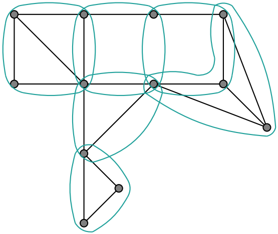
UNIVERSITY OF WATERLOO

Width Parameters
March 2021

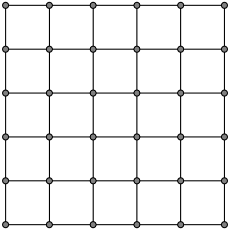
Theorem (Robertson-Seymour-86)

Every graph of tree-width $\geq f(t)$ has a $t \times t$ grid as a minor.

No



$\Longrightarrow$



as a minor

Theorem (Geelen-Kwon-McCarty-Wollan-20)

Every graph of **rank-width** $\geq f(t)$ *has a* $t \times t$ **comparability grid** *as a* **vertex-minor**.

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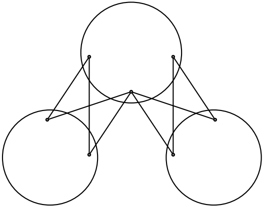

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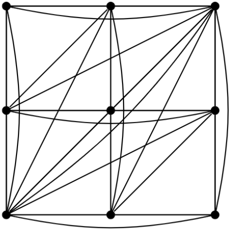

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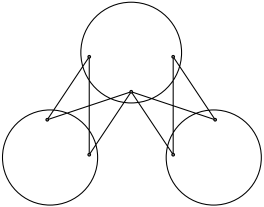

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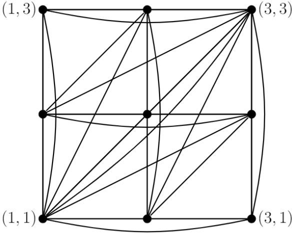

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- $\mathrm{rw}(G) \leq \text{clique-width}(G) \leq 2^{\mathrm{rw}(G)+1}$ (Oum-Seymour-06)
- H a vertex-minor of $G \implies \mathrm{rw}(H) \leq \mathrm{rw}(G)$.
- Comparability grids have $\mathrm{rw} = \Theta(t)$.

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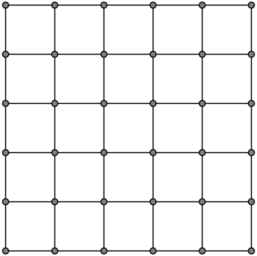
A class of graphs has unbounded

- tree-width iff it has all planar graphs as minors.
- rank-width iff it has all **circle graphs** as vertex-minors.



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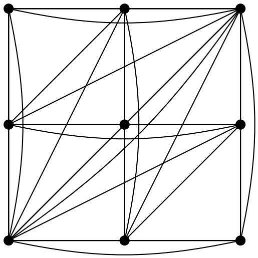

Cut-rank(X) is the rank (over the binary field) of the matrix $\mathrm{adj}[X, V(G) \setminus X]$.

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\begin{array}{cc}
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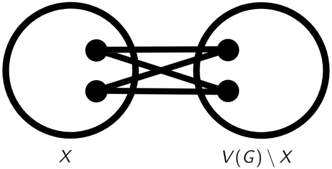

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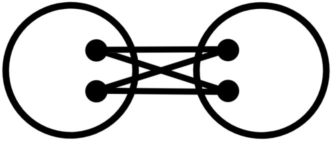
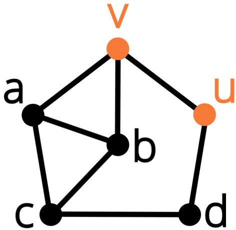
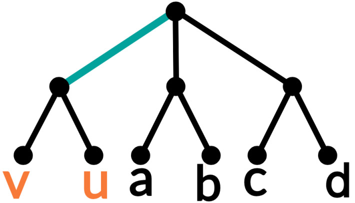


$$\textbf{cut-rank}(X) = \textbf{cut-rank}(V(G) \setminus X)$$

Cut-rank(X) is the rank (over the binary field) of the matrix $\mathrm{adj}[X, V(G) \setminus X]$.

Rank-width(G) is the minimum **width** of a subcubic tree T with leafs $V(G)$.





$$\textbf{width}(T) = \max_{e \in E(T)} \text{cut-rank}(X_e)$$

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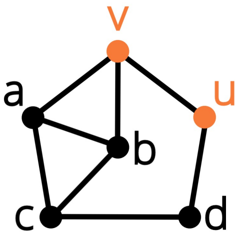

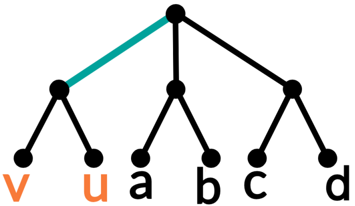

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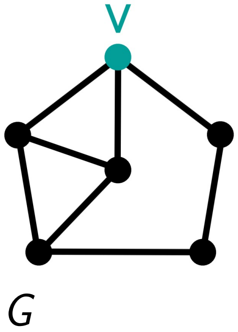
Locally complementing at v replaces the induced subgraph on the neighbourhood of v by its complement. This yields **local equivalence** classes of graphs.

The **vertex-minors** of G are the induced subgraphs of graphs in the local equivalence class of G.



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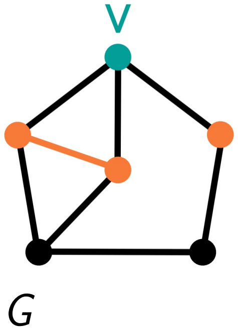

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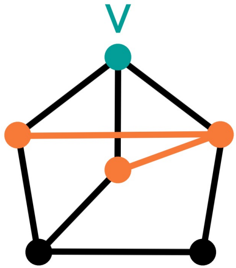

$G * v$

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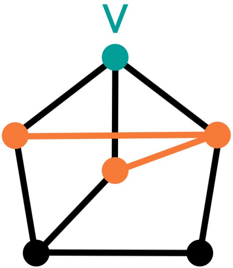

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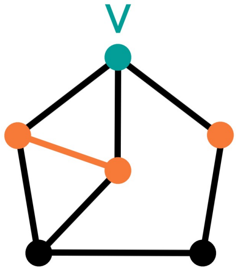

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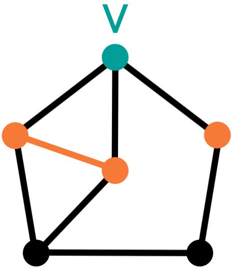

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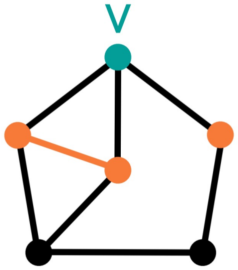

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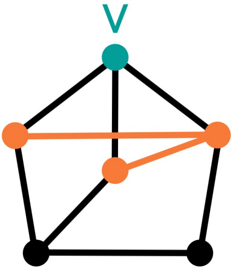

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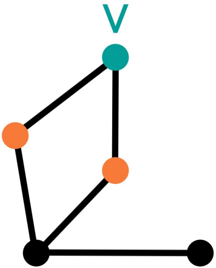

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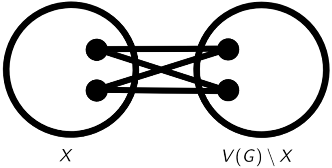

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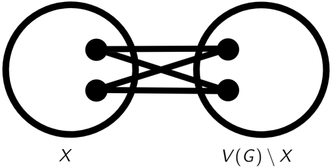

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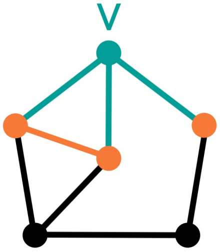

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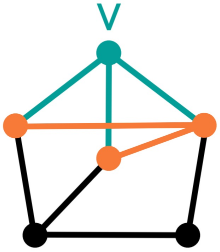

$G * v$

The following are equivalent for any graph class.

- It has unbounded clique-width.
- It has unbounded rank-width.
- It has all **comparability grids** as vertex-minors.
- It has all **circle graphs** as vertex-minors.

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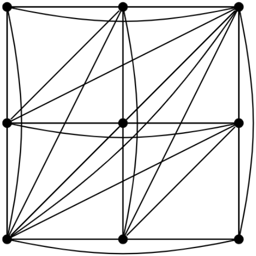

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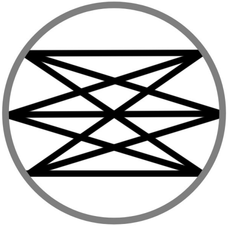
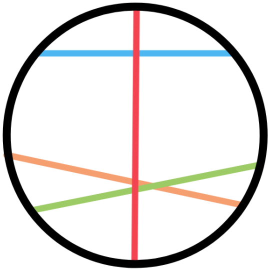
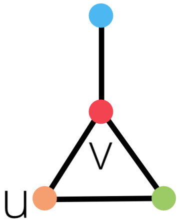


A **circle graph** is the intersection graph of chords on a circle. They are closed under local complementation. Every circle graph is a vertex-minor of a comparability grid.



chord diagram



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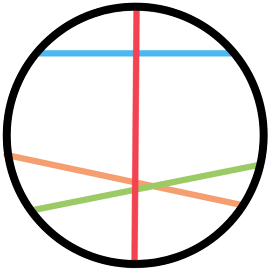

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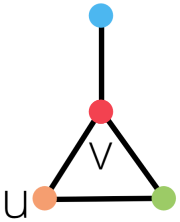

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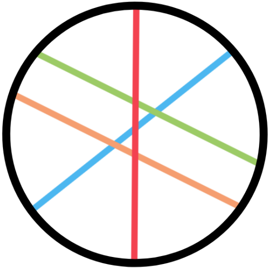

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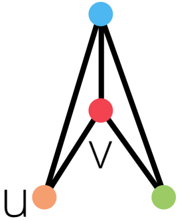

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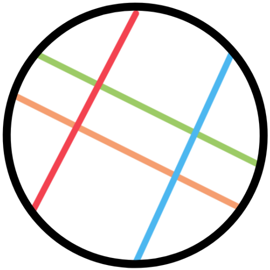

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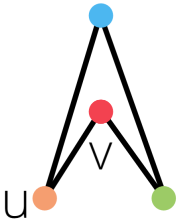

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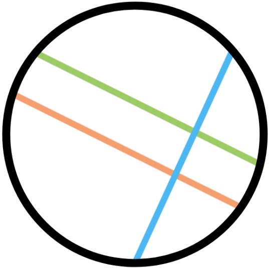

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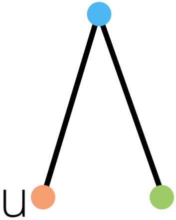

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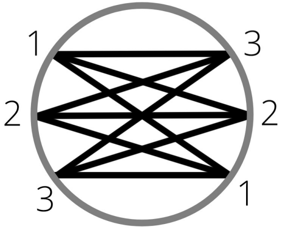

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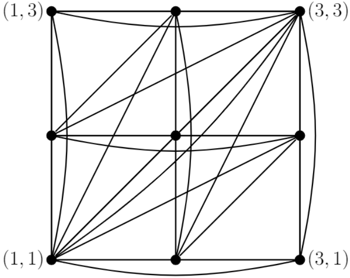
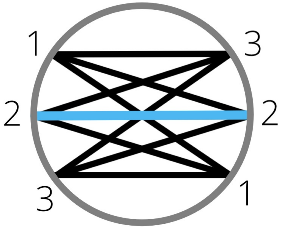
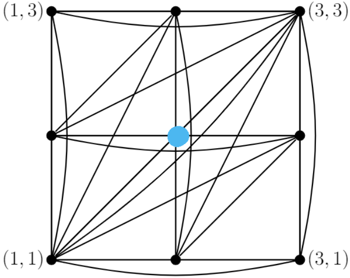


comparability grid

A **circle graph** is the intersection graph of chords on a circle. They are closed under local complementation. Every circle graph is a vertex-minor of a comparability grid.

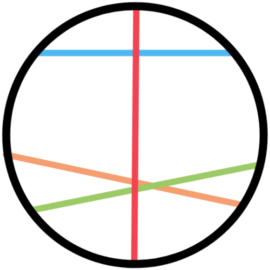


chord diagram



comparability grid

View a **chord diagram** as a 3-regular graph and contract the chords to get the **tour graph**. It is invariant under local complementation, and vertex-deletion works nicely.



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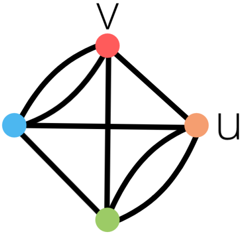

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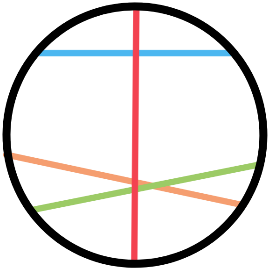

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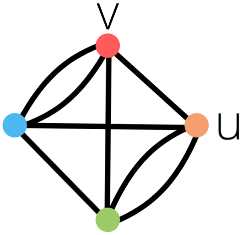

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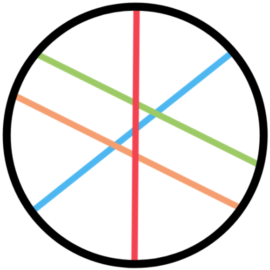

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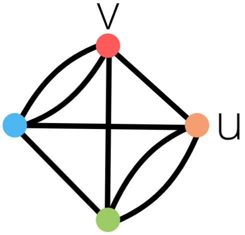

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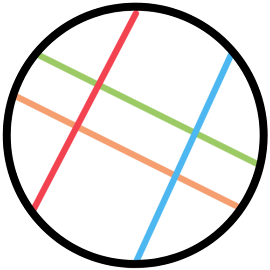

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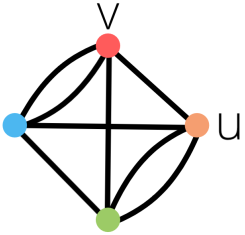

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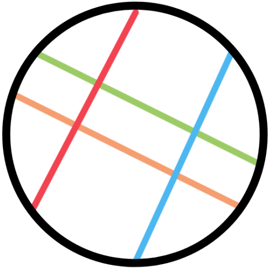

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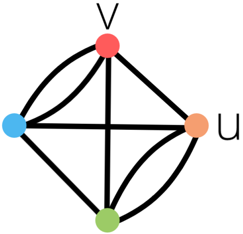

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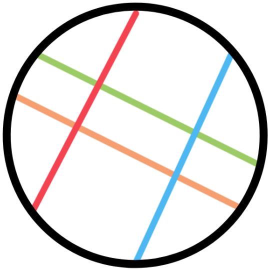

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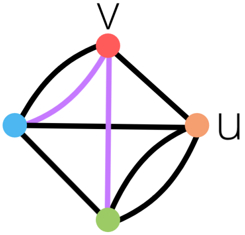

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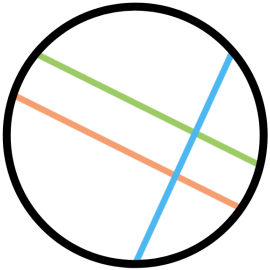

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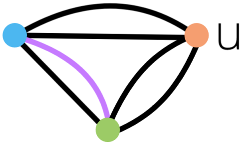

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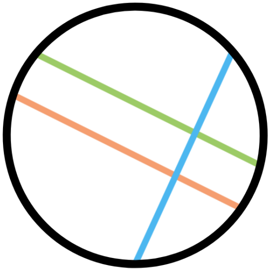

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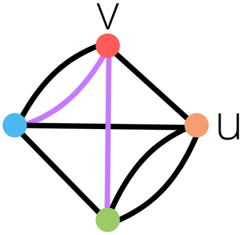

tour graph

Lemma

If H is a minor of G and $e \notin E(H)$, then H is a minor of either $G - e$ or G/e.

Theorem (Bouchet-88)

If H is a vertex-minor of G and $v \in V(G) \setminus V(H)$, then H is a **vertex-minor** *of either*

- $G - v$,
- $G * v - v$, or
- $G * v * u * v - v$ for each neighbour u of v.

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minor	$\sim$	vertex-minor
grid	$\sim$	comparability grid
planar graphs	$\sim$	circle graphs

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Pause :)

Kuratowski's Theorem

A graph is planar iff and only if it has no K_5 or $K_{3,3}$ minor.

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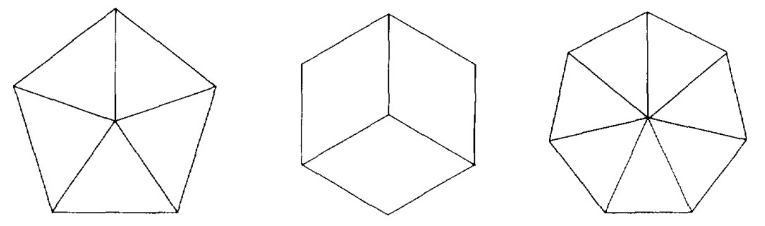

Menger's Theorem

For any $S, T \subseteq V(G)$ and edge e, either $G - e$ or G/e has no smaller (S, T)-separator than G.

Theorem (Oum-05)

For any disjoint $S, T \subseteq V(G)$ and vertex $v \notin S \cup T$, at least two of the three graphs $G - v$, $G * v - v$, $G * v * u * v - v$ have no smaller **cut-rank** *(S, T)-cut than G.*

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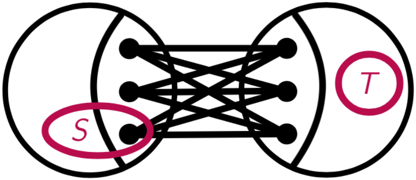

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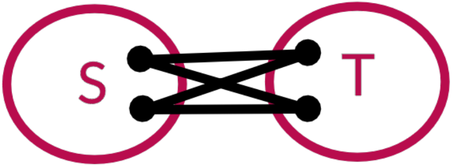

branch-width	$\sim$	rank-width
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$$\leftrightsquigarrow$$

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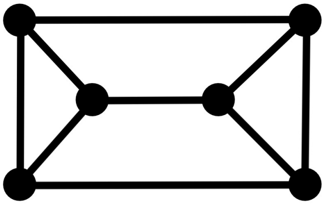
Consider a planar graph with a spanning tree **T**. Draw a curve closely around **T**. So $E(G) \setminus E(\mathbf{T})$ yields one set of non-crossing chords and $E(\mathbf{T})$ yields another. The circle graph is the **fundamental graph** $\mathcal{F}(\mathbf{T})$. What is $\mathcal{F}(\mathbf{T'})$?



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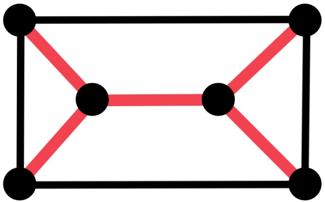

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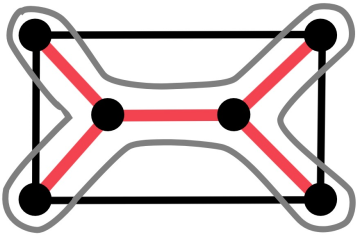

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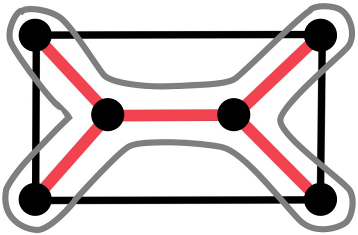

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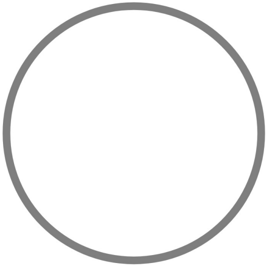

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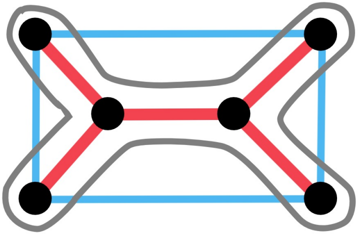

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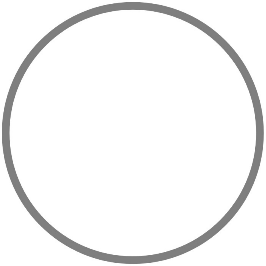

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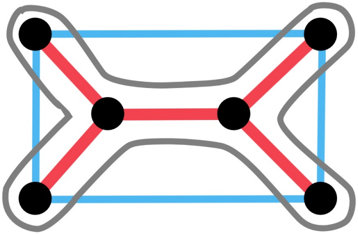

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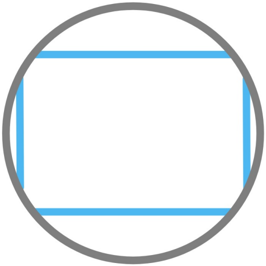

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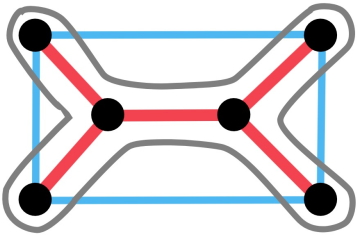

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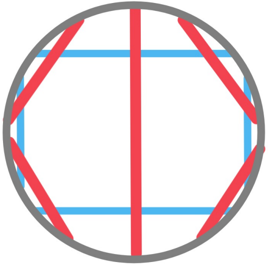

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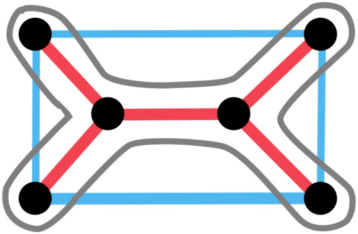

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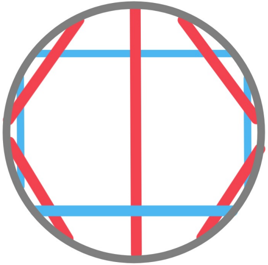

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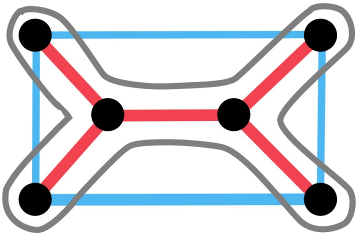

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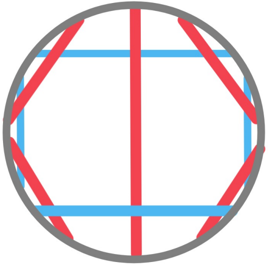

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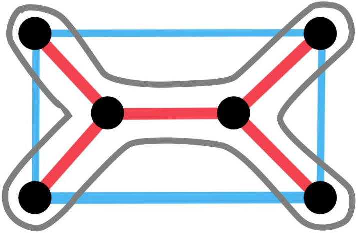

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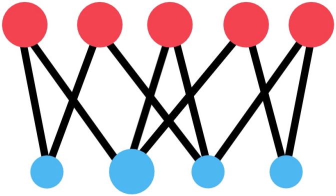

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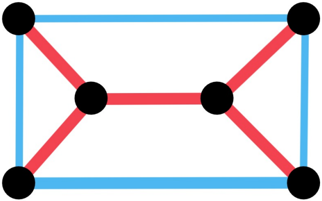

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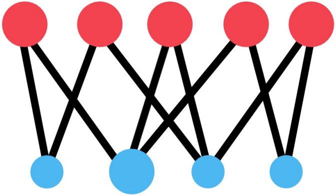

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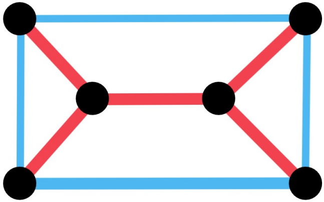

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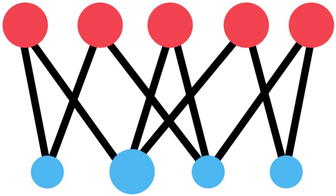

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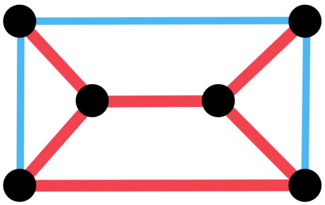
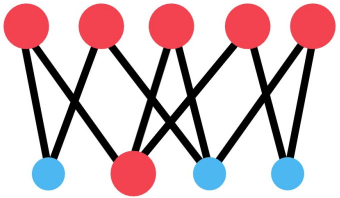


planar graph



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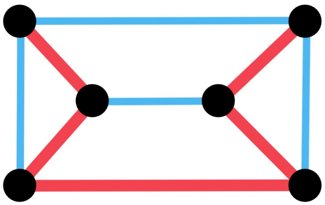
Consider a planar graph with a spanning tree **T**. Draw a curve closely around **T**. So $E(G) \setminus E(\mathbf{T})$ yields one set of non-crossing chords and $E(\mathbf{T})$ yields another. The circle graph is the **fundamental graph** $\mathcal{F}(\mathbf{T})$. What is $\mathcal{F}(\mathbf{T'})$?



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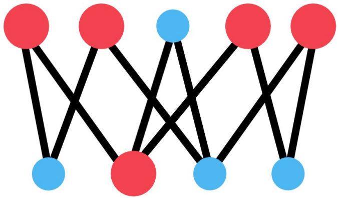

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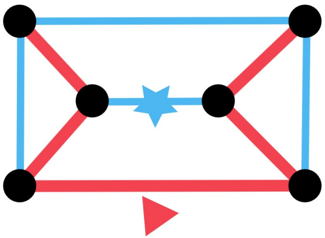

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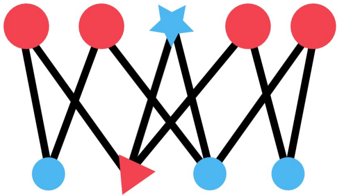

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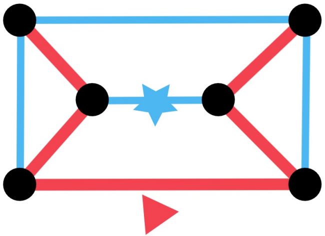

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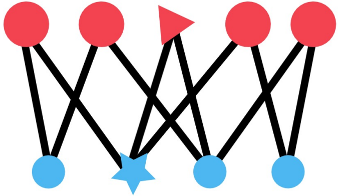

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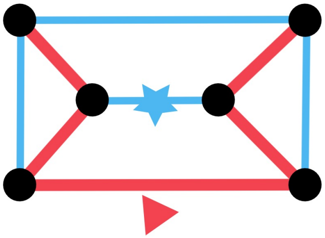

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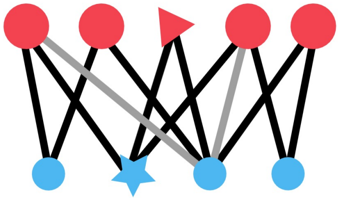

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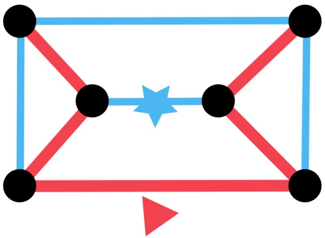

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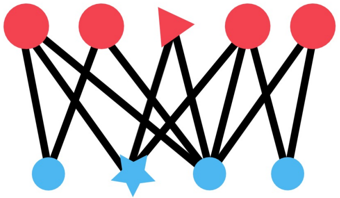

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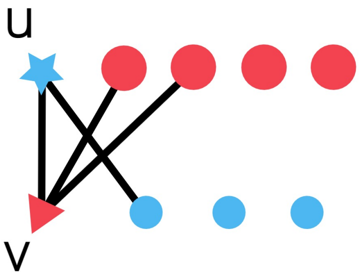
How do we switch out u and v?

1) Exchange their labels.

2) Complement between $N(u) - \{v\}$ and $N(v) - \{u\}$.



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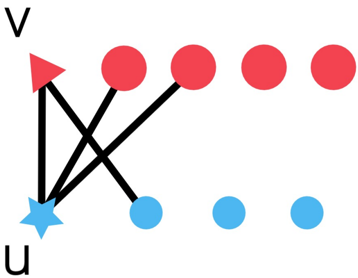

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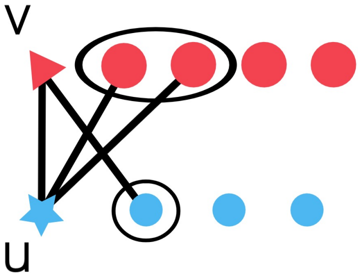

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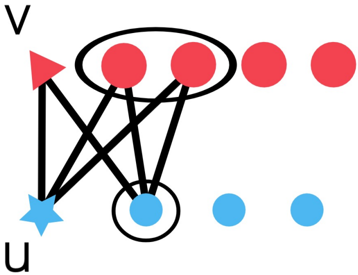

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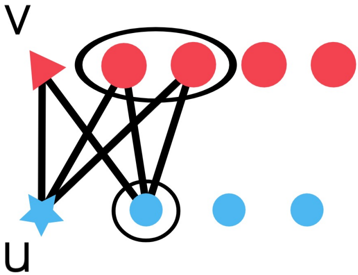

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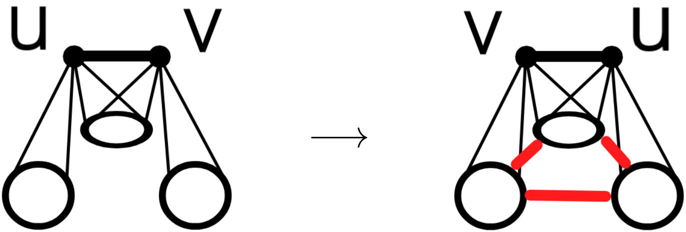

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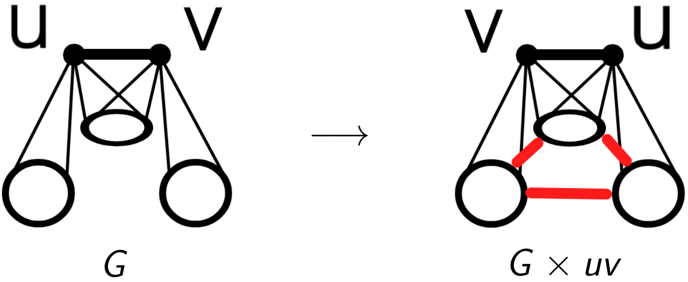

planar graphs $\rightsquigarrow$ pivot-equivalent bipartite circle graphs

via **fundamental graphs**

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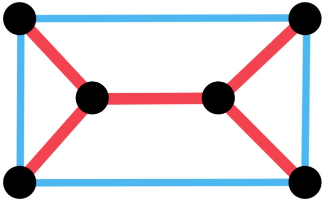
The fundamental graphs of two distinct, 2-connected planar graphs are pivot equivalent iff the planar graphs are **dual**.



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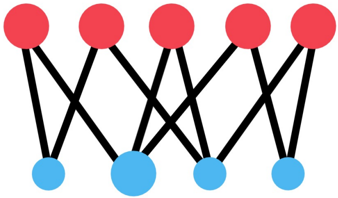

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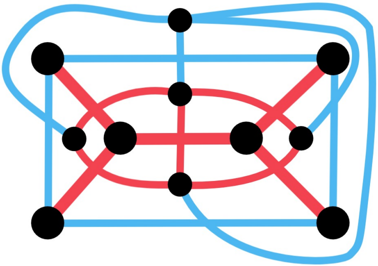

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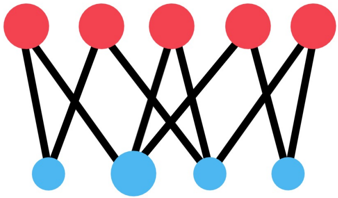

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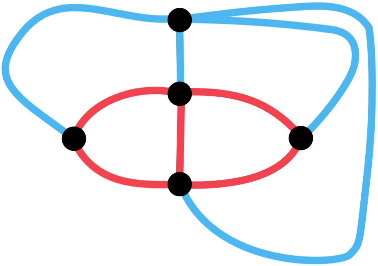

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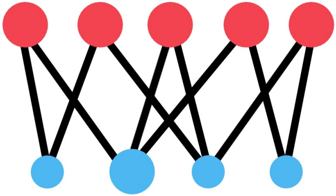

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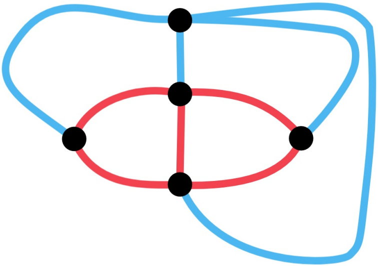

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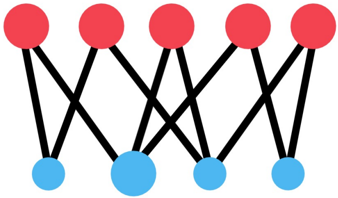

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Every bipartite circle graph is the fundamental graph of a planar graph, and every circle graph is a vertex-minor of one that is bipartite.

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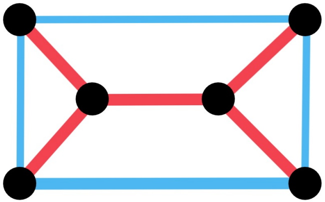

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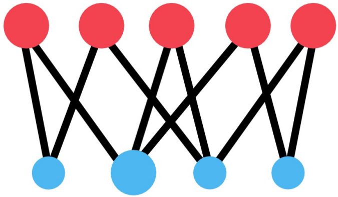

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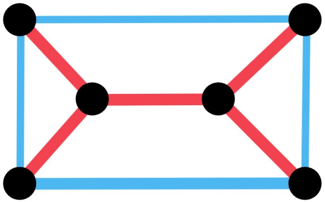

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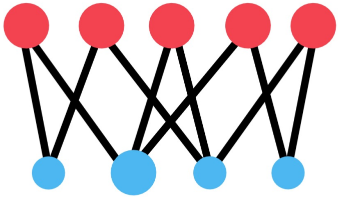

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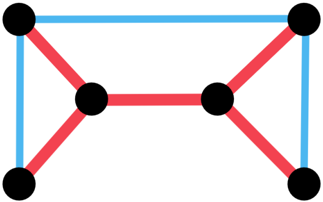

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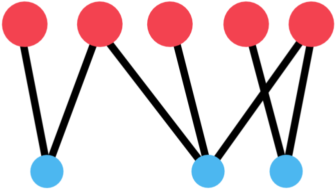

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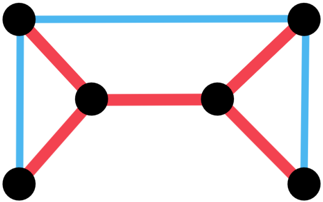

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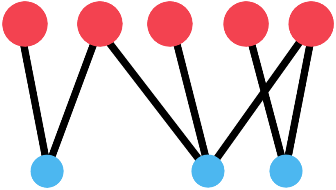

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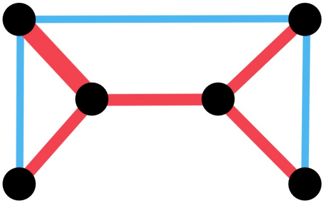

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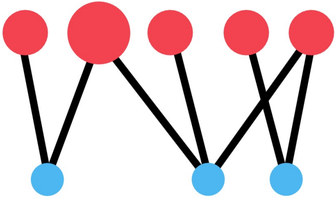

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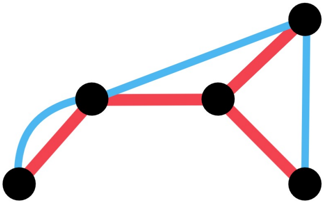

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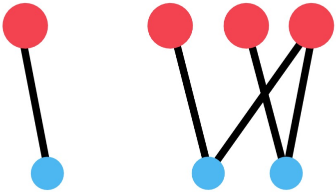

fundamental graph
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branch-width	$\sim$	rank-width
minor	$\sim$	vertex-minor
grid	$\sim$	comparability grid
planar graphs	$\sim$	circle graphs
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Pause

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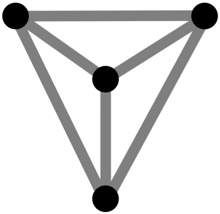
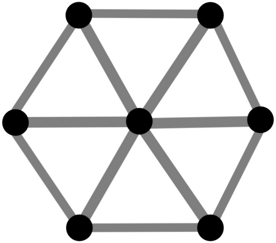


K_4



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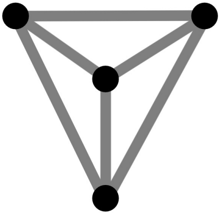

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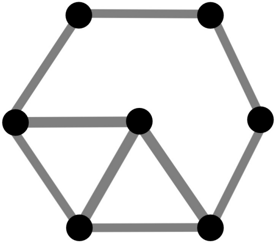

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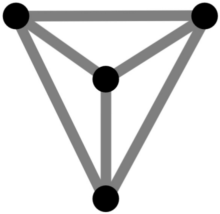

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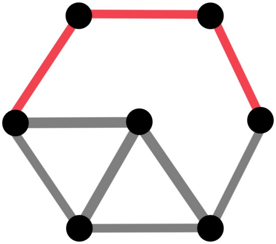

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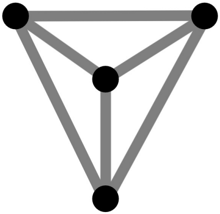

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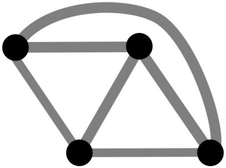

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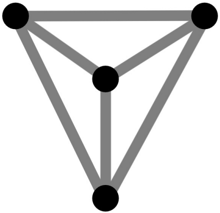

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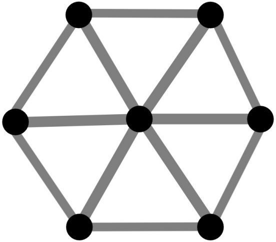

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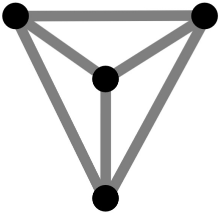

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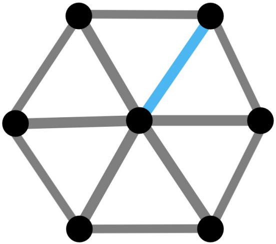

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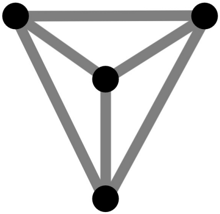
Pivot-minors seem truly harder...

There are large graphs with a "uniquely obtainable" minor.

Each edge can only be deleted in one way maintaining minor.



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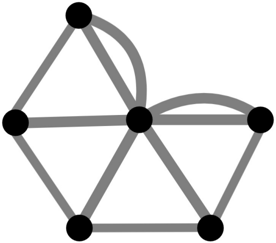

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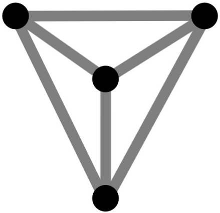

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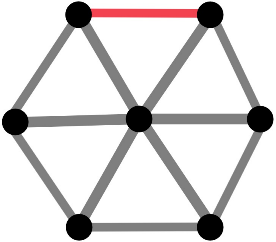

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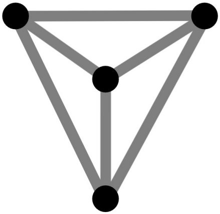

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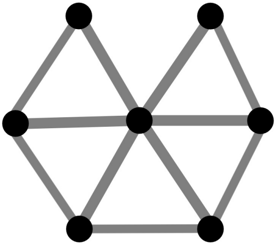

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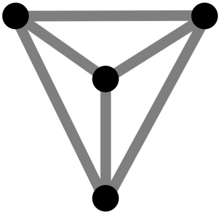

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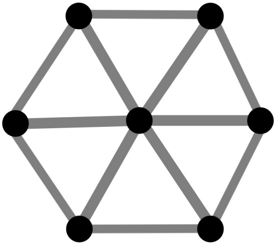

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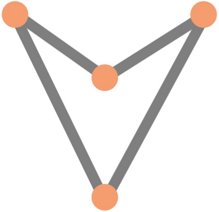

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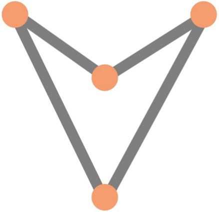

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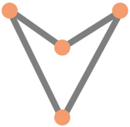

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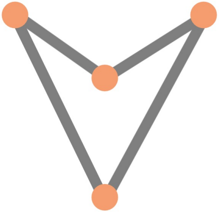

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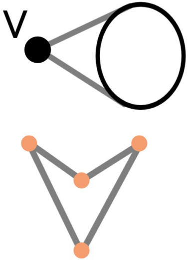

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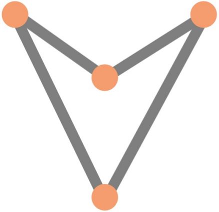

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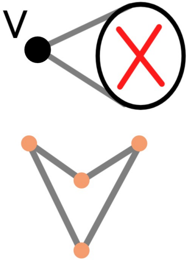

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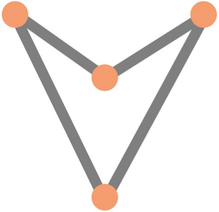

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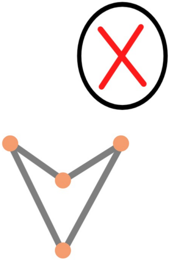

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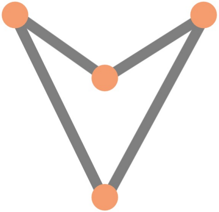

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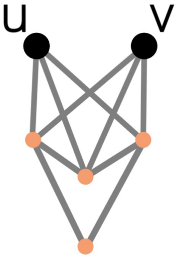

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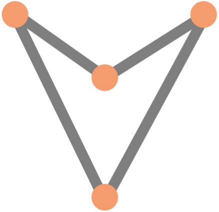

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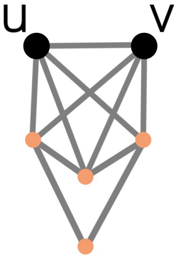

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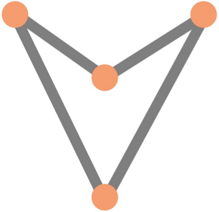

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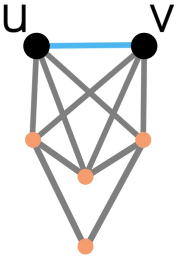

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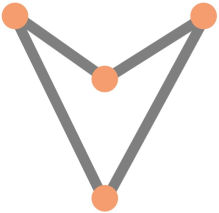

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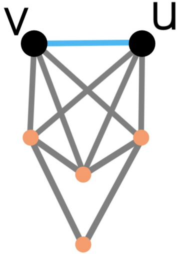

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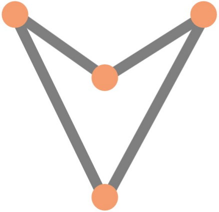

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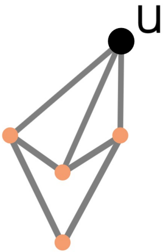

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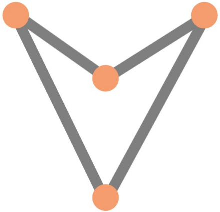

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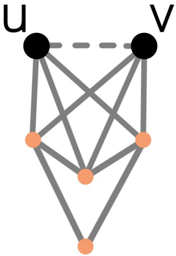

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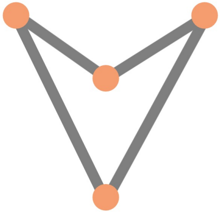

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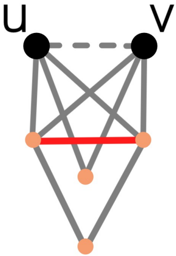

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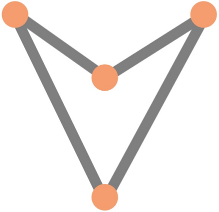

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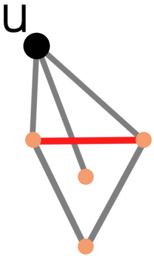

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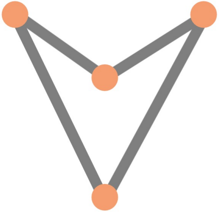

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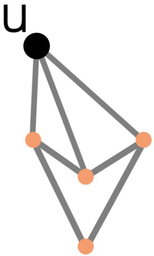

$(G * v - v) * u$

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- $\mathrm{depth}(K_1) = 0$,
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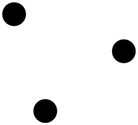

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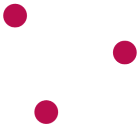

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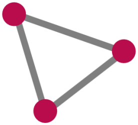

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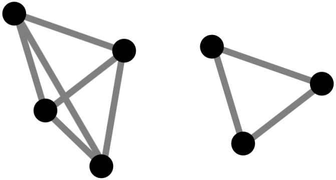

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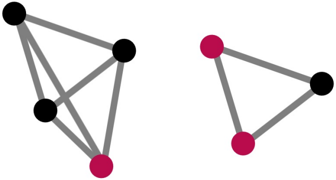

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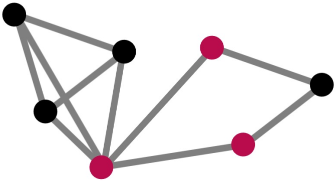

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Theorem (Kwon-McCarty-Oum-Wollan-21)

A class of graphs has **unbounded shrub-depth** *iff it has all* **paths** *as vertex-minors.*

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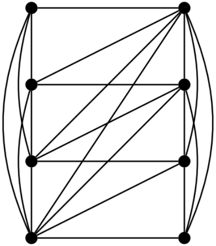
Theorem (Kwon-McCarty-Oum-Wollan-21)

A class of **bipartite** *graphs has unbounded shrub-depth iff it has all paths as* **pivot-minors***.*

Yet there are classes of unbounded shrub-depth **without** all paths as pivot-minors.



H_n

Conjecture

A class of graphs has unbounded shrub-depth iff it has all **paths** *or all* H_n *as pivot-minors.*

Is it true when rank-width is bounded?!?
See Nešetřil-Ossona de Mendez-Pilipczuk-Rabinovich-Siebertz.

Conjecture (Oum-09)

A class of graphs has unbounded rank-width iff it has all **bipartite circle graphs** *as pivot-minors.*

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Every proper **vertex-minor-closed** *class can be characterized by a* **finite** *list of forbidden vertex-minors.*

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Every proper **vertex-minor-closed** *class can be characterized by a* **finite** *list of forbidden vertex-minors.*

Conjecture

A class of graphs has unbounded shrub-depth iff it has all **paths** *or all* H_n *as pivot-minors.*

Is it true when rank-width is bounded?!?
See Nešetřil-Ossona de Mendez-Pilipczuk-Rabinovich-Siebertz.

Conjecture (Oum-09)

A class of graphs has unbounded rank-width iff it has all **bipartite circle graphs** *as pivot-minors.*

Conjecture

Every proper **vertex-minor-closed** *class can be characterized by a* **finite** *list of forbidden vertex-minors.*

Thank you!