Grid minor theorem and applications

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Minors and Grids

Large tree-width

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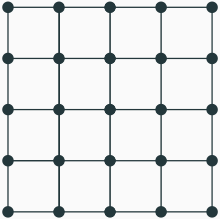
If G has a clique K, then there exists a bag containing K

- If G contains a clique of size k, then $tw(G) \geq k - 1$.
- Is the opposite true?

Question
Does every graph with large treewidth contains a large clique?

Grid



Lemma
The $k \times k$ grid has tree-width exactly k.

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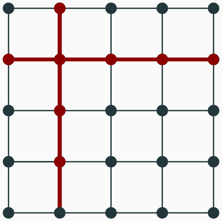

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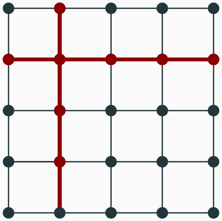

Lemma
The $k \times k$ grid has tree-width exactly k.

- Consider the bramble of all the crosses.
- Hitting all the crosses requires k elements.

Theorem (Robertson and Seymour 1993)
The treewidth of G is at least k if and only if G contains a bramble of order at least $k+1$.

Large grid

Question
Does every graph with large treewidth ***contains*** *a large grid?*

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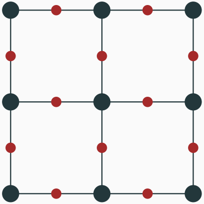
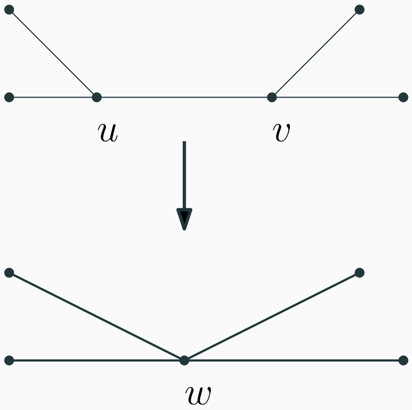
As a **subgraph**, no!



Minor

Definition
A graph H is a **minor** of G ($H \leq_m G$) if it can be obtained from G by deleting vertices, edges and **contracting** edges.



Another definition

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H is a **minor** of G if there exists an function ϕ mapping vertices of H to **connected subgraph** of G s.t:

- $\phi(u) \cap \phi(v) = \emptyset$ if $u \neq v$.
- If $uv \in E(H)$, then $\phi(u)$ $\phi(v)$ are **adjacent**.

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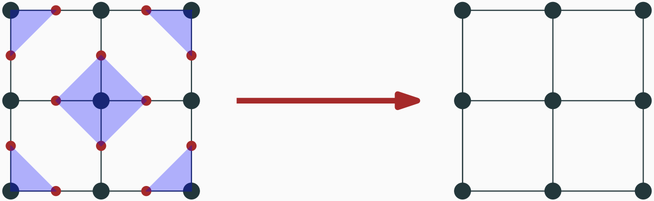

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- When contracting the edge uv into w: replace u and v by w in the bags.

Tree-width and minors

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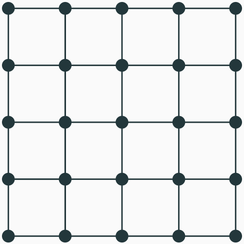
- When removing a vertex/edge: same decomposition
- When contracting the edge uv into w: replace u and v by w in the bags.
- If $H \leq_m G$, then $tw(H) \leq tw(G)$

Lemma
If G contains a clique of size k as a minor, then $tw(G) \geq k$.

Kuratowski

Theorem (Kuratowski)
A graph is planar if and only if it doesn't contain K_5 or $K_{3,3}$ as a minor.

- The grid is planar, so we have large treewidth without K_5 minors.



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- Also show a **polynomial time algorithm** to find this grid.

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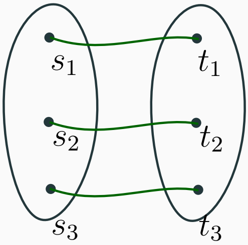
- Also show a **polynomial time algorithm** to find this grid.

Very important result with a lot of algorithmic applications!

Disjoint paths problem

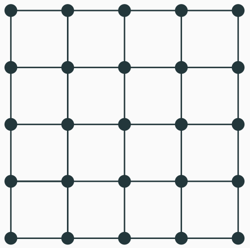
Problem (Disjoint paths problem)
Given a graph G and k pairs of vertices $(s_1, t_1), \dots, (s_k, t_k)$, does there exists k disjoint paths $P_1, \dots, P_k$ such that every P_i is a path between s_i and t_i



Irrelevant vertex

Theorem (Robertson and Seymour 1995)
The k disjoint paths problem has an algorithm in $f(k)n^3$.

- Small treewidth: DP; or
- Large grid: **irrelevant vertex**



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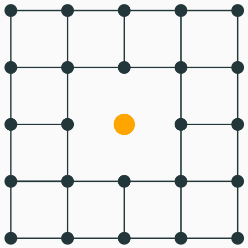

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There exists an algorithm deciding if G ***contains a minor*** *of H and running in time $f(|H|) \cdot |G|^3$*

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- Planar: K_5 and $K_{3,3}$ are forbidden.
- Every minor-closed graph family $\mathcal{F}$, deciding if a graph G belongs to $\mathcal{F}$ is **polynomial**.

Planar vertex deletion

Problem
*Let G be a graph and k be an integer. Does there exists a set of k vertices X such that $G - X$ is **planar**?*

- The class of graph for which the answer is **yes** is minor-closed. The problem is thus FPT.

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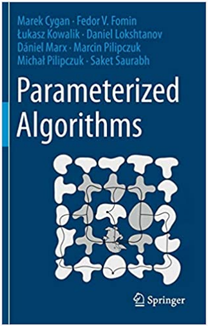
Let G be a graph and k be an integer. Does there exists a set of k vertices X such that $G - X$ is ***planar****?*

- The class of graph for which the answer is **yes** is minor-closed. The problem is thus FPT.
- Easier proofs exist.



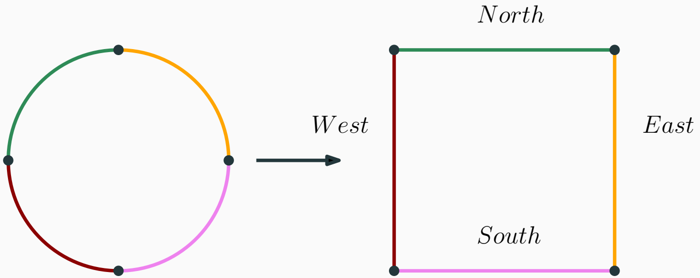
Planar case

Excluded grid: Planar graphs

Theorem (Robertson and Seymour)
For any integer t, every ***planar graph*** *of treewidth at least $\frac{9}{2}t$ contains a $t \times t$ grid as a minor. Moreover, there exists a polynomial time algorithm to find the model.*

Planar embedding

Consider a planar embedding of the graph and partition the outer face into North east south and west.

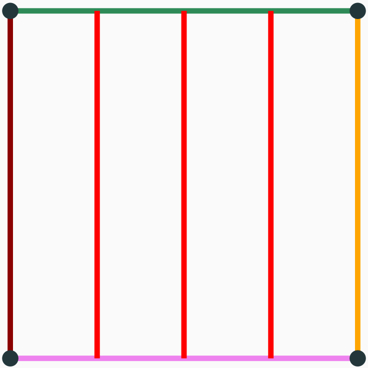


Theorem (Menger 1927)
Let G be a connected graph and x and y two vertices, the size of minimum (x, y)-cut is equal to the maximum number of pairwise vertex-disjoint paths between x and y.

Finding a grid

Lemma

If we can find k vertex disjoint paths from North to South and k vertex disjoint paths from West to East, then there exists a $k \times k$ grid.



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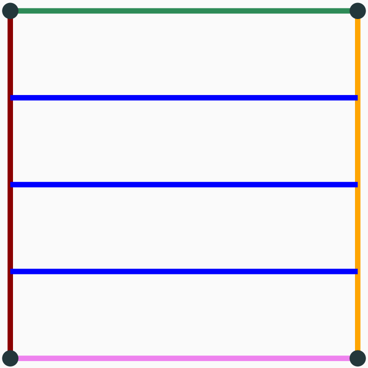

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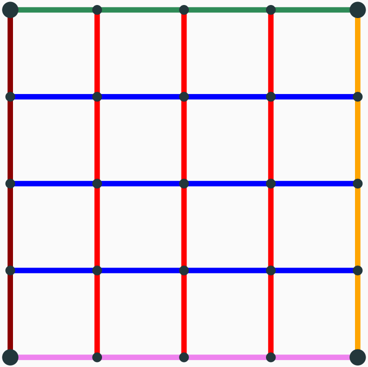
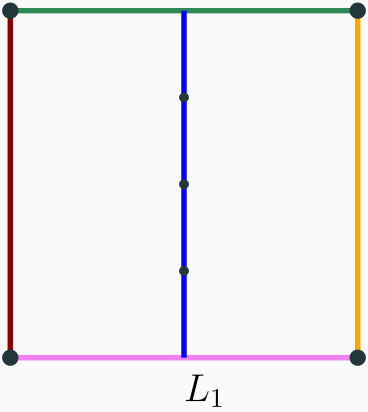


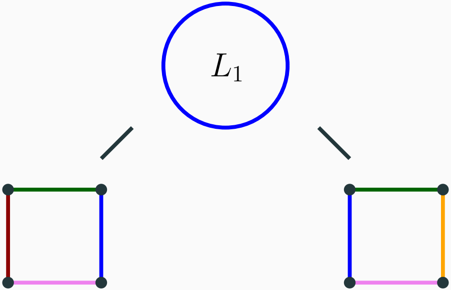
Cut

So either we find a $k \times k$ grid or there exists a k vertex cut L_1 cutting West and East



Recursion

The idea is to set the root of the tree decomposition as L_1, and recurse on both side when you contract L_1 into a single vertex.



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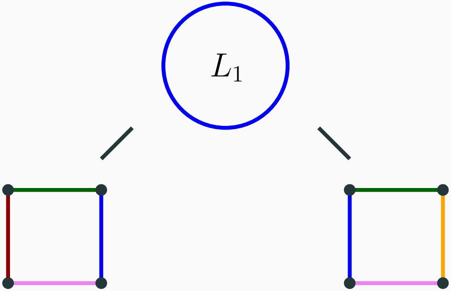
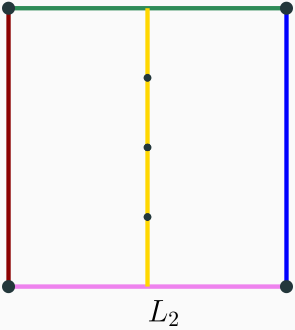


We need to remember on each side, that L_1 is contracted.

West-East cut

Suppose we have another East-West cut L_2



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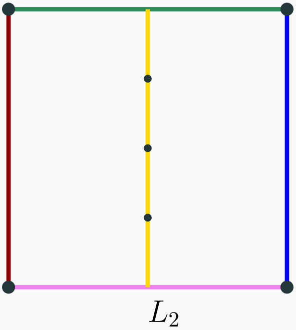

- The root of this subtree will contain $L_1 \cup L_2$
- When recursing on the right, we need to remember L_1 and L_2
- When recursing on the left, we need to remember **only** L_2

Grid minor theorem, end

- Overall we only need to remember one cut per side.
- Since each cut has size at most k, it makes $4k$ vertices.
- So overall all bags have size at most $5k$.

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Theorem
There exists an polynomial time algorithm that, taking a planar graph G and an integer k as input, computes either:

- *A tree decomposition of width $5k$; or*
- *A model of the $k \times k$ grid.*

Vertex cover in planar graphs

Problem (Vertex Cover)
Given a graph G, find the smallest set X of vertices incident to all edges.

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Theorem
There is an algorithm in $2^{tw} \cdot n$ for vertex cover.

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Given a graph G and an integer k, does there exists a vertex cover of size k?

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Theorem
*There is a $2^{\sqrt{k}} \cdot n$ algorithm for k-Vertex Cover on **planar graphs**.*

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Lemma

The vertex cover of a $k \times k$ *grid is at least* $\lfloor \frac{ks^2}{2} \rfloor$

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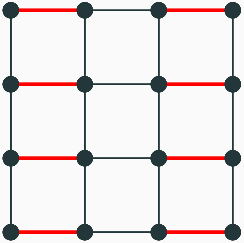
- A matching of size $\lfloor \frac{k^2}{2} \rfloor$
- Needs at least one vertex per edge of the matching



VC and minors

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If $H \leq_m G$*, then* $VC(H) \leq VC(G)$

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Lemma
If G *contains a* $k \times k$ *grid as a minor, then* $\lfloor \frac{k^2}{2} \rfloor \leq VC(G)$.

Algorithm

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Proof.
If $tw(G) \geq 10\sqrt{k}$, then by the grid minor theorem, G contains a grid of size $2\sqrt{k} \times 2\sqrt{k}$ as a minor. Thus $VC(G) \geq 2k$. $\square$

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The algorithm for vertex cover on planar graphs: Find a decomposition of width $O(\sqrt{k})$ (running time: $2^{O(tw)} \cdot n$)

- If it doesn't exists: Answer no!
- If it exists, run the algorithm in time $2^{tw} \cdot n = 2^{O(\sqrt{k})} \cdot n$.

Paths

Problem (k-paths)
Let G be a graph, does there exists a path of length k?

- There is a $2^{O(k)}n^{O(1)}$ algorithm in general graphs
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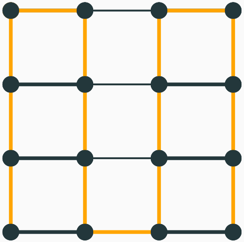

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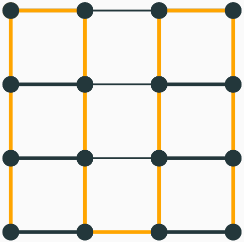

So the algorithm tries to compute a tree decomposition of width $O(\sqrt{k})$. If tw is larger answers **YES**, if not do DP.

The general approach:

- Compute the treewidth (approx.) of the graph.
- If it is at least $c \cdot \sqrt{k}$ answers NO (for minimisation) or YES (maximisation)
- If the treewidth is at most $c \cdot \sqrt{k}$, do DP.

Theorem (Demaine et al. 2005)
There exists a ***subexponential algorithm*** *on planar graphs for: Vertex cover, independent set, dominating set, feedback vertex set, longest path ...*

Bidimensionality

Can also be used for:

- PTASes on planar graphs for vertex cover, feedback vertex set ...
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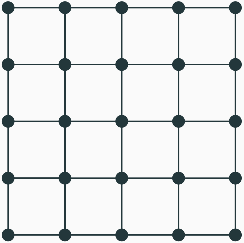
Remark
*Works on any class of graph where the relation between size of the grid minor and treewdith is **linear**.*

So H-free graphs for example.

Conclusion

Theorem (Robertson and Seymour)
Every graph with large treewidth has a large grid as a minor



- Small tree width: DP
- Large treewidth: use the grid