Some algorithmic applications of twin-width

Rémi Watrigant (LIP, Lyon)

Results mainly from:
Twin-width III, *É. Bonnet, C. Geniet, E.J. Kim, S. Thomassé, R. W.*
arxiv.org/abs/2007.14161

Journées CALAMAR
2 avril 2021

Outline:

- Maximum Independent Set
- Minimum Coloring
- Minimum Dominating Set

Maximum Independent Set (MIS)

Theorem [Tww I]

Given a FO formula φ and a n-vertex graph G with a d-sequence of G, one can decide $G \models \varphi$ in time $f(|\varphi|, d)n$ for some computable function f

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"$\alpha(G) \geqslant k$" is equivalent to:

$$\exists x_1 \exists x_2 \cdots \exists x_k \bigwedge_{1 \leqslant i < j \leqslant k} \neg(x_i = x_j) \wedge \neg E(x_i, x_j) \wedge \neg E(x_j, x_i)$$

$\Rightarrow$ Deciding MIS is FPT in k and $d := tww(G)$

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But the function f is a tower of exponentials ☹

$\rightarrow$ Now: $O(k^2 d^{2k} n)$ for MIS

Maximum Independent Set (MIS)

Before twin-width: cographs: twin-decomposition

$$G_n \to G_{n-1} \to \quad \dots \quad \to G_{i+1} \to G_i \quad \dots \quad \to G_1$$

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Solving MIS:

- for $i = n, \dots, 1$, for each $u \in V(G_i)$, compute

$$OPT(u) := OPT(G[u(G)])$$

$\rightarrow$ initialization ok

$\rightarrow$ in G_1: $OPT(u) = OPT(V(G))$

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- when contracting u, v into z:

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With a d-contraction sequence:

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- each G_i is a trigraph : (V_i, E_i, R_i)

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Solving MIS:

- for $i = n, \dots, 1$
  for each $T \subseteq V(G_i)$ connected red induced subgraph of size $\leqslant k$

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$OPT(T) :=$ OPT of $G[\bigcup_{u \in T} u(G)]$
intersecting each $u(G)$, for all $u \in T$

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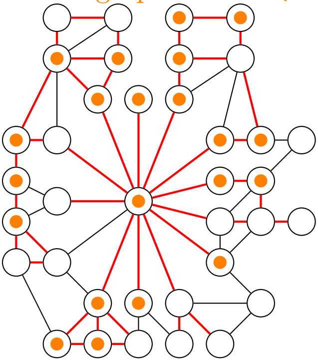
We might have $OPT(T) = nil$



(great figure by Édouard)

Maximum Independent Set (MIS)

Lemma [folklore]

A graph with n vertices and maximum degree d has at most $d^{2k}n$ connected induced subgraphs of $\leqslant k$ vertices

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$u, v \qquad\quad z$

Let T be a $CRIS_{\leqslant k}$ in G_i

How to compute $OPT(T)$?

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Let T be a $CRIS_{\leqslant k}$ in G_i

How to compute $OPT(T)$?

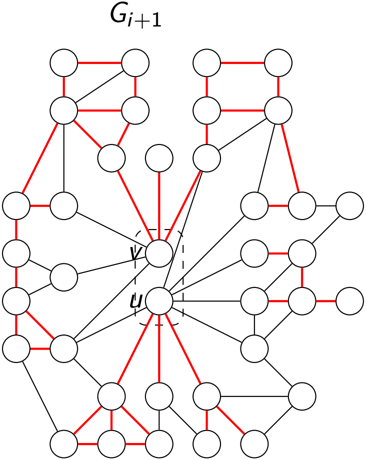
- if $z \notin T$, we take $OPT(T)$ from G_{i+1}
- if $z \in T$. How will OPT intersect $z(G)$?

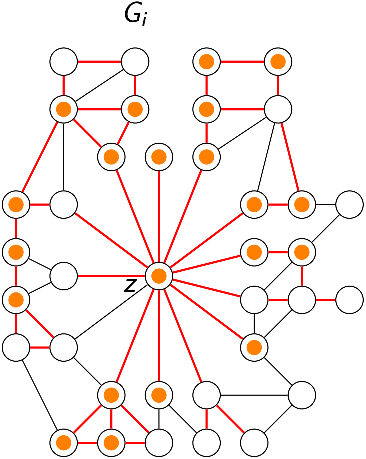
OPT intersects only $u(G)$	$\to T'_1 := T \setminus \{z\} \cup \{u\}$
OPT intersects only $v(G)$	$\to T'_2 := T \setminus \{z\} \cup \{v\}$
OPT intersects both $u(G)$, $v(G)$	$\to T'_3 := T \setminus \{z\} \cup \{u, v\}$

Construct a solution for each T'_ℓ and take the best as $OPT(T)$

Maximum Independent Set (MIS)

What is T'_ℓ in G_{i+1}?





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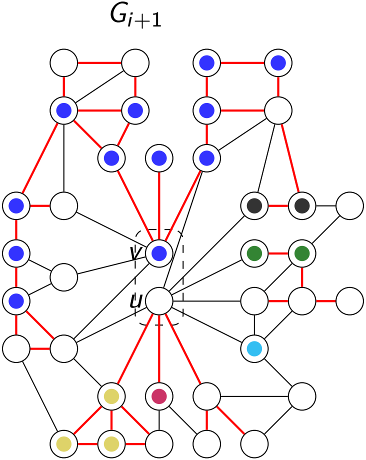

example: OPT intersects only $v(G)$

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$$T_1, \quad \dots, \quad T_q$$

which are all $CRIS_{\leqslant k}$ in G_{i+1} $\rightarrow$ take their $OPT(T_x)$

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- if:
  - there is a black edge between two T_x, T_y

  or

  - $OPT(T_x)$ is *nil* for some x

  $\rightarrow$ discard T'_ℓ

- otherwise: take $OPT(T_1) \cup \dots \cup OPT(T_q)$

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Then:

- if all T'_1, T'_1, T'_3 are discarded, OPT(T) gets *nil*
- otherwise: take the best

Maximum Independent Set (MIS)

Running time:

- n steps in the sequence
- at each step:
  - enumerate all $CRIS_{\leqslant k}$: $d^{2k} n$
  - look for a black edge between red c.c.: k^2

Theorem

Given $k \in \mathbb{N}$ and G on n vertices coming with a d-sequence, we can solve MIS in time $O(k^2 d^{2k} n^2)$

Same running time for:

- Maximum Clique
- Minimum Dominating Set
- r-Scattered Set

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Given $k \in \mathbb{N}$ and G on n vertices coming with a d-sequence, we can solve MIS in time ~~$O(k^2 d^{2k} n^2)$~~ $O(k^2 d^{2k} n) = 2^{O_d(k)} n$

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Maximum Independent Set (MIS)

Generalizations:

- weighted version in $2^{O_d(k \log k)} n$
- Induced Subgraph Isomorphism in $2^{O_d(k \log k)} n$
  (generalizes the result for H-minor free [Pilipczuk,Siebertz 2019])

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Lower bound (for MIS):

- given a $O(1)$-sequence, no $2^{o(n/\log n)} n^{O(1)}$ algorithm unless ETH
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Open questions:

- runs in poly-time in "number of connected red induced subgraphs"
  $\rightarrow$ graph classes admitting sequences with small number of such things?
  $\rightarrow$ does general graphs have contraction sequences with $O(c^n)$ such things for some $c < 2$?
  $\rightarrow$ what about other properties than "bounded red degree"?

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For any graph G of twin-width $\leqslant d$, we have $\chi(G) \leqslant (d+2)^{\omega(G)-1}$

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If a d-sequence is given, we can find such a coloring in polynomial-time.

Works by induction on $\omega(G)$. Let's prove the base case $\omega(G) = 2$, that is:

Given a triangle-free graph G and a d-sequence of it, one can find in polynomial-time a $(d+2)$-coloring of G.

Coloring (χ-boundedness)

Consider the d-sequence backward:

$$G_1 \rightarrow G_2 \rightarrow \quad \dots \quad \rightarrow G_i \rightarrow G_{i+1} \quad \dots \quad \rightarrow G_n$$

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Observation 2 (for triangle-free graphs only)

In the triangle-free case:

if z is incident to a black edge, **then** $z(G)$ is an independent set

Coloring (χ-boundedness)

$G_1 \to G_2 \to \quad \dots \quad \to G_i \to G_{i+1} \quad \dots \quad \to G_n$

"proper coloring" = with respect to $E_i \cup R_i$

- Assume G_i is properly $(d+2)$-colored
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  How do we color v?

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This is a proper $(d+2)$-coloring of G_{i+1}. Proof:

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  - otherwise, z had only $\leqslant d$ (red) neighbors, so v has $\leqslant d+1$ black/red neighbors

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We have just seen

K_3-free graphs coming with a d-sequence can be $(d+2)$-colored in polynomial-time.

Generalization to K_t-free graphs, by induction on t:

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- now Observation 2 becomes: *if z is incident to a black edge, then $z(G)$ is K_{t-1}-free*

  $\rightarrow$ we get by induction a coloring of $z(G)$ with $(d+2)^{t-3}$ colors

  ...

Coloring (χ-boundedness)

Related work/open question:

- provides an "elementary" proof of "bounded rank-width classes are χ-bounded" [Dvořák, Kráľ, 2012]
- bounded clique-width classes are **polynomially** χ-bounded [Bonamy,Pilipczuk, 2020]

$\rightarrow$ are bounded twin-width graphs polynomially χ-bounded?

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Minimum Dominating Set

Versatile tree of d-contractions [Tww II]

Up to a small degradation on the twin-width value d of a graph:

- at each step of the sequence: there exist $\frac{|V(G_i)|}{s}$ disjoint pairs of vertices that we can contract
- all trigraphs of the tree have red degree $\leqslant d'$

$\rightarrow$ can be computed in poly-time (given a d-sequence)

$\rightarrow$ s and d' are functions of d only

Minimum Dominating Set

Linear program:

$$
\begin{aligned}
& \text{minimize } \textstyle\sum_{x\in V} w(x) & \\
\text{s.t.} \quad & \textstyle\sum_{y\in N[x]} w(x) \geqslant 1 & \text{for all } x \in V \\
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Let $\gamma^*(G)$ be the optimal value of the LP

let w^* be its associated solution

We will prove the following:

Given an s-versatile tree of d-contractions, one can compute in polynomial-time a dominating set D of size $\leqslant 2s(d+1)\gamma^*(G)$

Minimum Dominating Set

Using the s-versatile tree of d-contractions, we construct a d-sequence

Contraction Rule

At each step, choose a pair (u, v) such that $w^*(u(G))$, $w^*(v(G)) < \frac{1}{2(d+1)}$

stop the sequence when there is no such pair

$$G_n \rightarrow G_{n-1} \rightarrow \quad \dots \quad \rightarrow G_{stuck}$$

Let n_{stuck} be the number of vertices in G_{stuck}

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Proof:

- in G_{stuck}, there are $\geqslant \frac{n_{stuck}}{s}$ disjoint pairs of d-contractions

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  Contraction rule $\Rightarrow$ at least $\frac{n_{stuck}}{s}$ parts have weight $\geqslant \frac{1}{2(d+1)}$
- $\sum_{u \in V(G_{stuck})} w^*(u(G)) = \gamma^*(G)$

Minimum Dominating Set

End of the algorithm:

Pick one arbitrary vertex from each $u \in V(G_{stuck})$ $\rightarrow$ solution D

- $|D| \leqslant 2s(d+1)\gamma^*(G)$ by Obs 1

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  Proof: let $u \in V(G_{stuck})$, show that $u(G)$ is dominated

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  - if u is incident to a black edge: done

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- $|D| \leqslant 2s(d+1)\gamma^*(G)$ by Obs 1
- D is a dominating set of G
  Proof: let $u \in V(G_{stuck})$, show that $u(G)$ is dominated
  - if u is incident to a black edge: done
  - otherwise: only $\leqslant d$ red neighbors
    for $y \in u(G)$, let $v_1, \ldots, v_q$ be the bags with at least one edge with y
    Claim: one of $u(G), v_1(G), \ldots, v_q(G)$ is a singleton:

$$w^*(u) + \sum_{i=1}^{q} w^*(v_i) \geqslant 1$$

One of them must have weight $\geqslant \frac{1}{d+1}$
$\rightarrow$ must be a singleton by our Contraction Rule

Minimum Dominating Set

Related work/open questions:

- There is a PTAS in minor-closed classes [Cabello, Gajser, 2015]

  OPEN:

  - c-approximation in bounded tww graphs (c independent of tww)?
  - PTAS in bounded tww graphs?

Minimum Dominating Set

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Maximum Independent Set

- any c-approximation implies a PTAS in bounded tww graphs (iterated lexicographic product preserves tww)
- PTAS?

Questions?