Acyclic choosability of graphs with small maximum degree

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Abstract A proper vertex coloring of a graph $G = (V, E)$ is acyclic if G contains no bicolored cycle. A graph G is L-list colorable if for a given list assignment $L = \{L(v) : v \in V\}$, there exists a proper coloring c of G such that $c(v) \in L(v)$ for all $v \in V$. If G is L-list colorable for every list assignment with $|L(v)| \geq k$ for all $v \in V$, then G is said k-choosable. A graph is said to be acyclically k-choosable if the coloring obtained is acyclic. In this paper, we study the acyclic choosability of graphs with small maximum degree. In 1979, Burstein proved that every graph with maximum degree 4 admits a proper acyclic coloring using 5 colors [Bur79]. We prove that (a) every graph with maximum degree $\Delta = 3$ is acyclically 4-choosable and (b) every graph with maximum degree $\Delta = 4$ is acyclically 5-choosable. The proof of (b) uses a backtracking greedy algorithm and Burstein's theorem.

1 Introduction

Let G be a graph. Let $V(G)$ be its set of vertices and $E(G)$ be its set of edges. A proper vertex coloring of G is an assignment f of integers (or labels) to the vertices of G such that $f(u) \neq f(v)$ if the vertices u and v are adjacent in G. A *k-coloring* is a proper vertex coloring using k colors. A proper vertex coloring of a graph is *acyclic* if there is no bicolored cycle. The *acyclic chromatic number* of G, $\chi_a(G)$, is the smallest integer k such that G is acyclically k-colorable. Acyclic colorings were introduced by Grünbaum in [Grü73] and studied by Mitchem [Mit74], Albertson, Berman [AB77], and Kostochka [Kos76]. In 1979, Borodin proved Grünbaum's conjecture:

Theorem 1. *[Bor79] Every planar graph is acyclically 5-colorable.*

This bound is best possible: in 1973, Grünbaum gave an example of a 4-regular planar graph [Grü73] which is not acyclically colorable with four colors. Moreover, there exist bipartite 2-degenerate planar graphs which are not acyclically 4-colorable [KM76].

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Borodin, Kostochka and Woodall improved this bound for planar graphs with a given girth. We recall that the girth of a graph is the length of its shortest cycle.

Theorem 2. *[BKW99]*

1. *Every planar graph with girth at least 7 is acyclically 3-colorable.*
2. *Every planar graph with girth at least 5 is acyclically 4-colorable.*

In 1979, Burstein studied graphs with small maximum degree and proved :

Theorem 3. *[Bur79] Every graph with maximum degree 4 is acyclically 5-colorable.*

There are graphs with maximum degree 4 which need 5 colors, for example K_5.

A graph G is L-list colorable if for a given list assignment $L = \{L(v) : v \in V(G)\}$ there is a coloring c of the vertices such that $c(v) \in L(v)$ and $c(v) \neq c(u)$ if u and v are adjacent in G. If G is L-list colorable for every list assignment with $|L(v)| \geq k$ for all $v \in V(G)$, then G is said k-choosable. In [Voi93], Voigt proved that there are planar graphs which are not 4-choosable, and in [Tho94], Thomassen proved that every planar graph is 5-choosable. In this paper we focus on acyclic choosability of graphs. This is, for which value k, any list assignement L, with $|L(v)| \geq k$ for all $v \in V(G)$, allow an acyclic coloring of G. In [BFDFK$^+$02], the following theorem is proved and the next conjecture is given:

Theorem 4. *[BFDFK$^+$02] Every planar graph is acyclically 7-choosable.*

This means that for any given list assignment L, with $|L(v)| \geq 7$ for all $v \in V(G)$, there is an acyclic coloring c of G, such that it is possible to choose for each vertex v a color in $L(v)$. The *acyclic list chromatic number* of G, $\chi_a^l(G)$, is the smallest integer k such that G is acyclically k-choosable.

Conjecture 1. [BFDFK$^+$02] Every planar graph is acyclically 5-choosable.

In 2004, Montassier, Ochem and Raspaud studied the acyclic choosability of graphs with bounded maximum average degree. The maximum average degree, $Mad(G)$, of the graph G is defined as

$$Mad(G) = \max\{2|E(H)|/|V(H)|, H \subseteq G\}$$

Theorem 5. *[MOR04]*

1. *Every graph G with $Mad(G) < \frac{8}{3}$ is acyclically 3-choosable.*
2. *Every graph G with $Mad(G) < \frac{19}{6}$ is acyclically 4-choosable.*
3. *Every graph G with $Mad(G) < \frac{24}{7}$ is acyclically 5-choosable.*

In [MS04], the authors gave an upper bound on χ_a^l for the graphs with bounded degree :

Theorem 6. *[MS04] Let G be a graph with maximum degree Δ, then $\chi_a^l(G) \leq \lceil 50\Delta^{4/3} \rceil$.*

In this paper, we improve this bound for graphs with small maximum degree. The case with $\Delta \leq 2$ is trivial. Indeed, trees (including paths) are clearly 2-choosable, and cycles are 3-choosable. Our results are stated in the following theorems.

Theorem 7. *Let G be a graph with maximum degree $\Delta \leq 3$, then $\chi_a^l(G) \leq 4$.*

Theorem 8. *Let G be a graph with maximum degree $\Delta \leq 4$, then $\chi_a^l(G) \leq 5$.*

Note that Theorem 8 improves Burstein's result on maximum degree four graphs. In what follows, we call k-vertex a vertex of degree k. The next section is dedicated to the proof of Theorem 7. In Section 3, we prove Theorem 8.

2 Proof of Theorem 7

Let H be a counterexample to Theorem 7 with minimum order. Let $L = \{L(v) : v \in V(H)\}$ be a list assignment such that there exists no extracted acyclic coloring. Let c be a proper coloring of H, with $c(v) \in L(v)$ for all $v \in V(H)$, such that the number a of bicolored cycles is minimal. There is such coloring, since cubic graphs are 4-choosable. Let $\mathcal{C}$ be a bicolored cycle. We prove that we can recolor a part of $\mathcal{C}$ such that $\mathcal{C}$ is 3-colored and the total number of bicolored cycle is at most $a-1$. The coloring obtained contradicts the minimality of a, completing the proof.

Claim. The counterexample H does not contain 1-vertices nor 2-vertices, so H is 3-regular.

Proof.
1. Suppose that H contains a 1-vertex u adjacent to a vertex v. By minimality of H, the graph $H' = H \setminus \{u\}$ is acyclically 4-choosable. Let c be an acyclic coloring of H' such that $c(v) \in L(v)$ for all $v \in V(H')$. We extend this coloring to H by coloring u with any color in $L(u) \setminus \{c(v)\}$. Since u cannot be in a cycle, the coloring obtained is an acyclic coloring of H , contradicting the definition of H.
2. Suppose that H contains a 2-vertex v adjacent to two other vertices u and w. By minimality of H, the graph $H' = (V(H) \setminus \{v\}, E(H) \setminus \{uv, vw\} \cup \{uw\})$ is acyclically 4-choosable. There is an acyclic coloring c of H' which we can extend to H by coloring v with a color in $L(v) \setminus \{c(u), c(w)\}$. Indeed, v cannot be part of a bicolored cycle since $c(u) \neq c(w)$, u and v being adjacent in H'.

Assume w.l.o.g. that the cycle $\mathcal{C} = x_1x_2x_3 \ldots x_k$ with $k \geq 4$ is bicolored using the colors 1 and 2, with $c(x_1) = 1$. Each vertex x_i is adjacent to the vertices x_{i-1}, x_{i+1}, and y_i. Each vertex y_i is adjacent to x_i and to two other vertices z_i, t_i (see Figure 1). The vertices x_i, y_j, z_k, t_l are not necessarily distinct. We consider two cases according to the color of the vertex y_3 : first case, y_3 is colored with a color used in $\mathcal{C}$, so $c(y_3) = 2$; second case, y_3 is not colored 1 or 2, let $c(y_3) = 3$.

1. Suppose that $c(y_3) = 2$. We know that $1 \in L(x_3)$. The vertex x_3 cannot be colored 2 because its neighbours are colored 2. There is at most two other problematic colors, say 3 and 4, because they create bicolored cycles passing through y_2, x_2, x_3, y_3, and z_3 and through y_4, x_4, x_3, y_3, and t_3. So we consider that $L(x_3) = \{1, 2, 3, 4\}$, $c(y_2) = c(z_3) = 3$ and $c(y_4) = c(t_3) = 4$. In this case we have to modify $c(y_3)$. We color y_3 with a color in $L(y_3) \setminus \{2, 3, 4\}$. Then we finally color x_3 with 3 or 4.

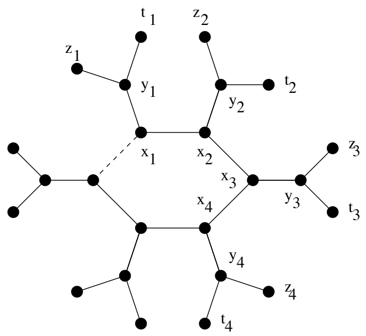

Figure1. .

2. Suppose that $c(y_3) = 3$. We know that $1 \in L(x_3)$. The vertex x_3 cannot be colored 2 or 3 because its neighbours are colored 2 or 3. There is at most one other problematic color, say 4, because it creates a bicolored cycle passing through y_2, x_2, x_3, x_4, and y_4. So we consider that $L(x_3) = \{1, 2, 3, 4\}$ and $c(y_2) = c(y_4) = 4$. If there was a color $b \in L(x_2) \backslash \{1, 2, 3, 4\}$, we could set $c(x_2) = b$ and $c(x_3) = 1$. So we consider that $L(x_2) = L(x_4) = \{1, 2, 3, 4\}$. In this case we can let $c(x_2) = c(x_4) = 3$ and $c(x_3) = 2$.

This completes the proof of Theorem 7.

3 Proof of Theorem 8

Let H be a counterexample to Theorem 8 with minimum order, and L a list assignment such that there is no extracted acyclic coloring. In the first subsection, we prove some structural properties of H, that will allow us to use an algorithm (presented in the second and in the third subsection) which gives an acyclic coloring of H from L, contradicting the definition of L.

3.1 Structural properties of *H*

Claim. The counterexample H is 4-regular.

Proof. 1. H does not contain any 1-vertices nor 2-vertices (see the first claim).

2. H does not contain any 3-vertices. Suppose that H contains a 3-vertex v adjacent to three vertices x, y, z with $d(x) \geq 3, d(y) \geq 3, d(z) \geq 3$. Let x_1, x_2, (x_3 if $d(x) = 4$) be the other neighbours of x (y_1, y_2, y_3 for y and z_1, z_2, z_3 for z). By minimality of H, the graph $H' = (V(H) \setminus \{v\}, E(H) \setminus \{vx, vy, vz\} \cup \{xy\})$ is acyclically 5-choosable. So, there is an acyclic coloring c of H'. If $c(x), c(y), c(z)$ are all distinct, it is easy to extend the coloring c to H by coloring v with a color different from $c(x), c(y), c(z)$. Hence suppose that w.l.o.g. $c(x) = c(z) = 1$ and

$c(y) = 2$. Observe that if $L(v) \neq \{1, 2, c(x_1), c(x_2), c(x_3)\}$, we can extend the coloring c to H. So, $L(v) = \{1, 2, c(x_1), c(x_2), c(x_3)\}$ and $\{c(x_1), c(x_2), c(x_3)\} = \{c(z_1), c(z_2), c(z_3)\} = \{3, 4, 5\}$. If $L(x) \neq \{1, 2, 3, 4, 5\}$, we are done : we color x with a color different from 1,2,3,4,5; the colors of x, y, z are all distinct and finally we color v. For the same reason, $L(z) = \{1, 2, 3, 4, 5\}$. In this case, we recolor x and z with 2 and we color v with 1.

Claim. All the cut vertices of H are incident to a bridge.

Proof. By contradiction, let v be a cut vertex with neighbours x_1, x_2, y_1 and y_2 such that x_1 and x_2 (resp. y_1 and y_2) are in the same connected component of $G\backslash\{v\}$. Let G' be the graph $G\backslash\{v\}$ with the edges x_1x_2 and y_1y_2. Since G' is smaller than G it has an acyclic coloring c where $c(x_1) \neq c(x_2)$ and $c(y_1) \neq c(y_2)$. If we consider this coloring in the graph G, the only uncolored vertex is v and since it is a cut vertex, whatever its assigned color we cannot create a bicolored cycle going through x_ivy_j. Furthermore, since $c(x_1) \neq c(x_2)$ (resp. $c(y_1) \neq c(y_2)$) we cannot create a bicolored cycle going through x_1vx_2 (resp. y_1vy_2). So v can be colored with a color in $L(v)\backslash\{c(x_1), c(x_2), c(y_1), c(y_2)\}$.

Claim. The graph H contains an edge uv such that :

1. $L(u) \neq L(v)$.
2. The edge uv is not a bridge.
3. The vertex u is not incident to a bridge.

Proof. 1. If $\forall v \in V(H), L(v) = \{1, 2, 3, 4, 5\}$, then by Burstein's Theorem, there exists an acyclic coloring c of H, which contradicts the definition of H. Hence, there exists an edge uv with $L(u) \neq L(v)$.

2. By contradiction, suppose that all the edges uv with $L(u) \neq L(v)$ are bridges. Let S be the set of such edges. The graph obtained from H by removing the edges of S is composed of several components $C_1, \ldots, C_k$ with $k \geq 2$ such that the lists of the vertices of a same component are all equal, i.e. $\forall v \in V(C_i), L(v) = L_i$. By minimality of H, the graph $H' = H \setminus C_1$ is acyclically 5-choosable; so, we can extract from the list assignment L an acyclic coloring c of H'. Independently, we can color C_1 with Burstein's Theorem (since $\forall v \in V(C_1), L(v_1) = L_1$). Finally, we can extend the coloring c of H' to H by permuting two colors of C_1 if necessary.
3. We prove that H contains at most one bridge. By contradiction. Suppose that H contains two bridges u_1u_2 and v_2v_3, which separate H into three components C_1, C_2, C_3, such that $u_1 \in C_1$, $u_2, v_2 \in C_2$, $v_3 \in C_3$, and $|C_1| \leq |C_3|$. By minimality of H, the graph H' induced by the vertices of C_2 and C_3 is acyclically 5-choosable. So, there exists an acyclic coloring $c_{H'}$ of H'. As well, the graph H'' induced by the vertices of C_1 admits an acyclic coloring $c_{H''}$ of H''. Now, if the colors assigned to the endpoints of u_1u_2 are distinct, we can extend the colorings $c_{H'}$ and $c_{H''}$ to H. If not, this implies that all the acyclic colorings $c_{H''}$ of C_1 verify $c_{H''}(u_1) = a = c_{H'}(u_2)$. So, the graph F composed by two copies of H'' and an edge linking the copies of u_1 is a counterexample and $|V(F)| = 2|V(C_1)| \leq |V(C_1)| + |V(C_3)| < |V(H)|$ which contradicts the definition of H.

From now, we suppose that we have an edge uv, which is not a bridge, with $L(u) \neq L(v)$ and such that u is not a cut vertex.

Claim. There is an order $x_1, x_2, \ldots, x_n$ on the vertices, such that x_1 and x_n are adjacent, $L(x_1) \neq L(x_n)$, and the vertices x_i, with $i < n$, have a neighbour x_j with $j > i$.

Proof. Since u is not a cut vertex, consider a spanning tree T of $H \setminus \{u\}$ rooted in v. Let $x_1 = u$ and order the others vertices from x_2 to x_n, according to a post order walk on T. Notice that $x_n = v$ and for $i < n$, each x_i has a father in T which is posterior in the order.

In the next subsections, we use this order to acyclically color the vertices of H. We will successively color $x_1, x_2, \ldots, x_n$. During this process, when we color x_i, we may change the color of x_j, for $1 < j < i < n$. Note that the color of x_1 remains unchanged until coloring x_n. At the beginning there is no constraints; so, let the color of x_1 be such that $c(x_1) \in L(x_1) \setminus L(x_n)$. In the next subsection we explain how to color the vertices x_i, for $i < n$. In the last subsection, we finally color x_n; that will complete the proof of Theorem 8.

3.2 The backtracking greedy algorithm : the coloring of $x_i, 1 < i < n$

At Step 1, we colored x_1 with a color a with $a \notin L(x_n)$. The following Claim allow us to color all the vertices until x_{n-1}.

Claim. Let c be a partial acyclic coloring of H on the vertices $\{x_1, \ldots, x_{i-1}\}$. Then, there exists a partial acyclic coloring c' of H on the vertices $\{x_1, \ldots, x_i\}$, $i < n$, which do not modify the color of x_1.

Proof. Let c be a partial acyclic coloring of H on the vertices $\{x_1, \ldots, x_{i-1}\}$. We would like to extend the coloring c to x_i. We know that x_i has at most three colored neighbours by the definition of the order. Let x_j, x_k, x_l be these vertices. We consider two cases following the adjacency of x_i to x_1. However, the analysis are almost the same.

1. The vertex x_i is adjacent to x_1, x_j, x_k. We recall that x_1 is adjacent to x_n which is not colored. Let x_1^1, x_1^2 be the other neighbours of x_1. Let x_j^1, x_j^2, x_j^3 be the other neighbours of x_j (x_k^1, x_k^2, x_k^3 for x_k). We consider the different cases following the coloring of x_j, x_k.
   - 1.1. The colors of x_j, x_k, x_1 are all distinct. We just let $c'(x_i) \in L(x_i) \setminus \{c(x_j), c(x_k), c(x_1)\}$.
   - 1.2. A color appears exactly twice on x_j, x_k, x_1.
     - 1.2.1. The color of x_1 appears twice. W.l.o.g., suppose that $c(x_1) = c(x_j) = a$ and $c(x_k) = 1$, $a \neq 1$. We just let $c'(x_i) \in L(x_i) \setminus \{1, a, c(x_1^1), c(x_1^2)\}$ (we recall that x_1 is adjacent to x_n which is not colored).

1.2.2. A color different from $c(x_1)$ appears twice. Let $c(x_j) = c(x_k) = 1$ and $c(x_1) = a$, $a \neq 1$. If $L(x_i) \neq \{1, a, c(x_j^1), c(x_j^2), c(x_j^3)\}$, we are done : we could color x_i with $c'(x_i) \in L(x_i) \setminus \{1, a, c(x_j^1), c(x_j^2), c(x_j^3)\}$. Hence, $L(x_i) = \{1, a, c(x_j^1), c(x_j^2), c(x_j^3)\}$ and $\{c(x_j^1), c(x_j^2), c(x_j^3)\} = \{c(x_k^1), c(x_k^2), c(x_k^3)\}$. Set $\{c(x_k^1), c(x_k^2), c(x_k^3)\} = \{2, 3, 4\}$. Now, we recolor x_j with a color different from $1, 2, 3, 4$ and we get case 1.1 or 1.2.1.

1.3. A color appears three times. So, suppose that $c(x_1) = c(x_j) = c(x_k) = a$. It is easy to see that if we cannot color x_i, this implies that all the neighbours of x_1 (resp. x_j, x_k) have distinct colors. So we recolor x_j with a color different from $a, c(x_j^1), c(x_j^2), c(x_j^3)$ and we get case 1.2.1.

2. The vertex x_i is not adjacent to x_1. Let x_j^1, x_j^2, x_j^3 be the other neighbours of x_j (x_k^1, x_k^2, x_k^3 for x_k and x_l^1, x_l^2, x_l^3 for x_l). Following the coloring of the vertices of x_j, x_k, x_l, we consider the different cases :

2.1. The colors of x_j, x_k, x_l are all distinct. We just color x_i with $(x_i) \in L(x_i) \setminus \{c(x_j), c(x_k), c(x_l)\}$.

2.2. A color appears exactly twice on x_j, x_k, x_l. W.l.o.g. We suppose that $c(x_j) = c(x_k) = 1$ and $c(x_l) = 2$. If $L(x_i) \neq \{1, 2, c(x_j^1), c(x_j^2), c(x_j^3)\}$, we are done : let $c'(x_i) \in L(x_i) \setminus \{1, 2, c(x_j^1), c(x_j^2), c(x_j^3)\}$. So, $L(x_i) = \{1, 2, c(x_j^1), c(x_j^2), c(x_j^3)\}$ and $\{c(x_j^1), c(x_j^2), c(x_j^3)\} = \{c(x_k^1), c(x_k^2), c(x_k^3)\}$; say $\{c(x_j^1), c(x_j^2), c(x_j^3)\} = \{3, 4, 5\}$. Now, if $L(x_j) \neq \{1, 2, 3, 4, 5\}$, we recolor x_j such that $c'(x_j) \in L(x_j) \setminus \{1, 2, 3, 4, 5\}$ and let $c'(x_i) \in L(x_i) \setminus \{c'(x_j), c'(x_k), c'(x_l)\}$. Consequently, $L(x_j) = \{1, 2, 3, 4, 5\}$ and for the same reason, $L(x_k) = \{1, 2, 3, 4, 5\}$. In this case, let $c'(x_j) = c'(x_k) = 2$, and $c'(x_i) = 1$.

2.3. A color appears three times on x_j, x_k, x_l. It is easy to observe that if we cannot color x_i, this implies that at least one vertex of x_j, x_k, x_l has a neighbourhood colored with three distinct colors. Hence we can recolor this vertex with a different color and get case 2.2.

3.3 The final step : the coloring of x_n

At this point, we have a partial acyclic coloring such that $c(x_1) = a$ with $a \notin L(x_n)$. Let x_1, u, v, w be the neighbourhood $N(x_n)$ of x_n. Let u_1, u_2, u_3 be the other neighbours of u (v_1, v_2, v_3 for v, and w_1, w_2, w_3 for w, and x_1^1, x_1^2, x_1^3 for x_1).

We show that we can extend the partial acyclic coloring to x_n by recoloring if necessary one or some vertices of $N(x_n)$. For this, we consider the different cases according to the coloring of $N(x_n)$:

1. The vertices of $N(x_n)$ have all distinct colors. In this case, it is easy to extend the coloring to x_n by coloring x_n with $c(x_n) \in L(x_n) \setminus \{c(u), c(v), c(w)\}$ (recall that $c(x_1) \notin L(x_n)$).

2. Exactly one color appears twice in $N(x_n)$:

2.1. Suppose that $c(u) = c(v) \neq a$, $c(w) \neq c(u)$, $c(w) \neq a$. If we can color x_n with a color different from $c(u), c(w), c(u_1), c(u_2), c(u_3)$ ($a \notin L(x_n)$), we are done. Hence, $L(x_n) = \{c(u), c(w), c(u_1), c(u_2), c(u_3)\}$; the colors of the u_i are distincts, the colors of the v_i are distinct and $\{c(u_1), c(u_2), c(u_3)\} =$

$\{c(v_1), c(v_2), c(v_3)\}$. Now, we color x_n with $c(u)$ and we recolor u and v with a proper color. The coloring obtained is acyclic.

2.2. The color of x_1, i.e. a, appears twice. Set $c(u) = c(x_1) = a$, $c(v) = b$, and $c(w) = c$ (a, b, c are distinct). If $L(x_n) \neq \{b, c, c(u_1), c(u_2), c(u_3)\}$, we can color x_n (with a color different from these of v, w, u_1, u_2, u_3) and the coloring obtained is an acyclic coloring. Otherwise, this implies that : the colors of the u_i are distinct ($i = 1, 2, 3$); the colors of the x_1^i are distinct; $S = \{c(u_1), c(u_2), c(u_3)\} = \{c(x_1^1), c(x_1^2), c(x_1^3)\}$, $a \notin S$, and $L(x_n) = \{c(u_1), c(u_2), c(u_3), b, c\}$. Now, we recolor u with a color different from $c(u_1)$, $c(u_2)$, $c(u_3)$, a. If this new color is equal to b or c, we have case 2.1, else, we have case 1.

3. Exactly two colors appear twice. W.l.o.g., set $c(u) = c(v) = 1$ and $c(w) = c(x_1) = a$.

First, we show that $L(x_n)$ contains necessarily the color 1. If $1 \notin L(x_n)$, say $L(x_n) = \{2, 3, 4, 5, 6\}$. If we cannot color x_n, this implies that there exists at least one of the vertices u, v, w, x_1 whose the neighbours have distinct colors; say u. So, we recolor u with a color different from $c(u_1), c(u_2), c(u_3), 1$. If this new color is a, then we can color x_n with a color of the neighbours of u according to the coloring of the neighbours of w and x_1, otherwise, we get case 2.2.

Hence, we suppose that $1 \in L(x_n)$ and set $L(x_n) = \{1, 2, 3, 4, 5\}$. If we cannot color x_n with $2, 3, 4, 5$; this implies that by coloring x_n with one of these colors, we will create a bicolored cycle. So, each of the colors $2, 3, 4, 5$ appears at least twice among the colors $c(u_i), c(v_i), c(w_i), c(x_1^i), i = 1, 2, 3$.

3.1. Suppose that three bicolored cycles can be created (using one of the colors of $L(x_n)$), going through u and v; this implies that the colors of the neighbours of u (resp. v) are distinct and $\{c(u_1), c(u_2), c(u_3)\} = \{c(v_1), c(v_2), c(v_3)\}$. Set, w.l.o.g. $c(u_1) = 2, c(u_2) = 3, c(u_3) = 4$ and $c(w_1) = c(x_1^1) = 5$ (if the color 5 does not appear in the neighbourhood of w and x_1, we can color x_n with 5). Now, if we can recolor u with a color different from $1, 2, 3, 4, a$, we get case 2.2. So $L(u) = \{1, 2, 3, 4, a\}$ and we set $c(u) = a$. Now, we can color x_n with one of the colors $2, 3, 4$ following the colors of the neighbourhood of w (we choose a color different from $c(w_2), c(w_3)$).

3.2. Suppose that two bicolored cycles going through u and v can be created by choosing a color of x_n in $\{2, 3, 4, 5\}$ and two bicolored cycles going through w and x_1 can be created by choosing a color of x_n in $\{2, 3, 4, 5\}$. So, we have : $c(u) = c(v) = 1, c(w) = c(x_1) = a$ and w.l.o.g $c(u_1) = c(v_1) = 2, c(u_2) = c(v_2) = 3, c(w_1) = c(x_1^1) = 4$ and $c(w_2) = c(x_1^2) = 5$.

Suppose $c(u_3) = b$ with $b \neq 2, b \neq 3$ (it may be that $a = b$). If we can recolor u with a color different from $1, 2, 3, a, b$, we get case 2.2. So, $a \neq b$, $L(u) = \{1, 2, 3, a, b\}$. If $c(w_3) \in \{4, 5\}$, let $c(u) = a$ and x_n be colored 2 or 3, according to the color of w_3. If $c(w_3) \notin \{4, 5\}$, similarly to u, we deduce that w can be colored 1. So, let $c(u) = a, c(w) = 1$ and color x_n with a color not in $\{1, a, b, c(w_3)\}$.

Hence, $c(u_3) \in \{2, 3\}, c(v_3) \in \{2, 3\}, c(w_3) \in \{4, 5\}, c(x_1^3) \in \{4, 5\}$. W.l.o.g. set $c(u_3) = 2$. Now, we will recolor u and/or v. If we can recolor u with a color different from $1, 2, 3$, we can color x_n with 2 or 3.

So we must study the coloring of the neighbourhood of u (at distance 2). Let u_1^1, u_1^2, u_1^3 be the other neighbours of u_1 (u_2^1, u_2^2, u_2^3 for u_2, and u_3^1, u_3^2, u_3^3 for u_3). We recall that at least one of u_1^1, u_1^2, u_1^3 or u_3^1, u_3^2, u_3^3 is colored by 1; say u_1^1 (as well, one of u_2^1, u_2^2, u_2^3 is colored by 1; say u_2^1). So, if we can recolor u with a color different from $1, 2, 3, c(u_1^2), c(u_1^3)$, we are done. Assume that $L(u) = \{1, 2, 3, b, c\}$ ($b \neq c, b \notin \{1, 2, 3\}, c \notin \{1, 2, 3\}$), $c(u_1^2) = c(u_3^1) = b$, $c(u_1^3) = c(u_3^2) = c$. Since $c(u_1^1), c(u_1^2), c(u_1^3)$ are distinct, let us recolor u_1. Assign to u_1 a color different from $1, 2, b, c$. If its new color is different from 3, we are done (we can then easily recolor u with a color different from 1). So, suppose that the new color of u_1 is 3. Hence, we cannot recolor u with a color in $\{1, 2, 3, b, c\}$, i.e. with b or c, if and only if $\{c(u_2^2), c(u_2^3)\} = \{b, c\}$. However, if $L(u_2) \neq \{1, 2, 3, b, c\}$, we can recolor u_2, then u. Finally, this implies that we have $L(u) = L(u_1) = L(u_2) = \{1, 2, 3, b, c\}$, $\{c(u_1^1), c(u_1^2), c(u_1^3)\} = \{c(u_2^1), c(u_2^2), c(u_3^3)\} = \{1, b, c\}$. In this case, we assign the color 2 to u_1 and u_2, the color 3 to u and the color 2 to x_n.

4. Suppose that a color appears exactly three times. It is easy to observe that if we cannot color x_n, this implies that the neighbours of at least one of the vertices u, v, w, x_1 have distinct colors, say u. so we can recolor u and get case 2.2.
5. A color appears four times : there is only one possibility, i.e. $c(u) = c(v) = c(w) = c(x_1) = a$. Since $a \notin L(x_n)$, if we cannot color x_n, this implies that the neighbours of at least one of the vertices u, v, w, x_1 have distinct colors, say u. So, we can recolor u and get case 4.

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