# Segment representation of planar graphs

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joint work with
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Introduction

Planar graphs as L-intersections

Planar graphs as PURE-3-DIR
- TC-schemes
- Every 3-colorable triangulation has a TC-scheme
- How TC-schemes give 3-DIR representations

Planar graphs not in PURE-4-DIR

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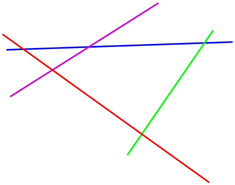
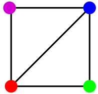
# SEG and PURE-$k$-DIR graphs





**SEG** $=$ { intersection graphs of segments in $\mathbb{R}^2$ }

**PURE**-$k$-**DIR** $=$ { - same - with at most $k$ different slopes used and parallel segments don't intersect }

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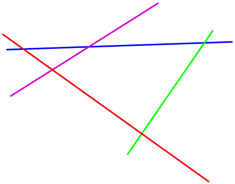

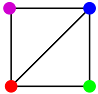

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**Theorem** (Chalopin and G. '09) (G., Isenmann and Pennarun '18)

Planar graphs are intersection graphs of segments.

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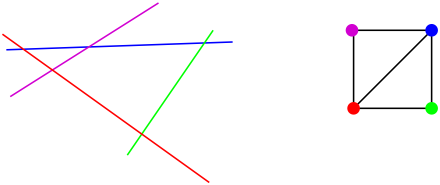

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**Theorem** (G., Isenmann and Pennarun '18)

Planar graphs are intersection graphs of L's.

By (Middendorf et al. '92), L's can be turned into segments.

# Planar graphs in $k$-DIR

**Open Problem** (de Fraysseix & Ossona de Mendez '07)

Is every planar graph $G$ in (PURE-)$\chi(G)$-DIR ?

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- True for $\chi = 2$ (Hartman et al. '91, de Fraysseix et al. '94)
- For $\chi = 3$, known for:
  - Outerplanar graphs (Scheinerman '89)
  - Triangle free planar graphs (de Castro et al. '02)
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**Theorem** G. 2019 & 2020

Yes for $\chi(G) = 3$, and no for $\chi(G) = 4$.

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# From L to segments

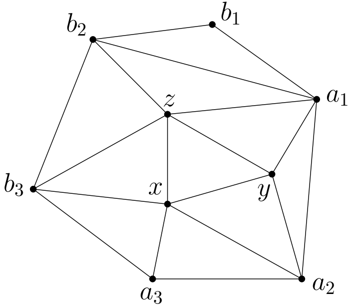
# 2-sided near-triangulations

*Near-triangulation*: every inner face is a triangle

### Definition

A near-triangulation is *2-sided*:

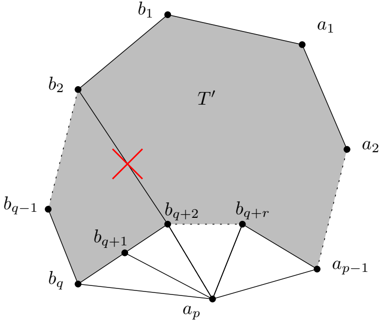
- if every $\triangle$ is a face
- if the outerface is a cycle $(a_1, a_2, \ldots, a_p, b_q, \ldots, b_1)$
- if $a_1, a_2, \ldots, a_p$ and $b_1, \ldots, b_q$ are induced paths (no chord $a_i a_j$ or $b_i b_j$)

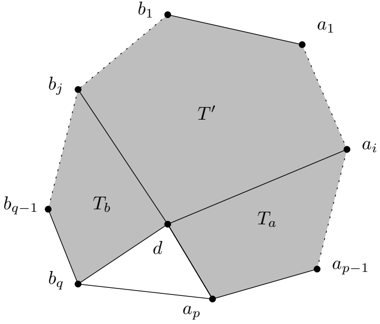


# 2-sided near-triangulations

Three decomposition operations:

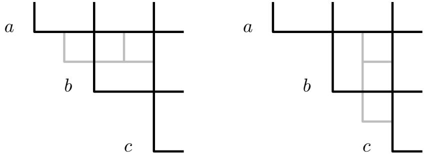
- *$a_p$-removal* ($p > 1$, a fan around $a_p$ not adjacent to $b_j$) $\rightarrow$ rename the neighbors of $a_p$ into $b_j$ vertices $\rightarrow T'$
- *$b_q$-removal* (symmetric)
- *cutting* ($p, q > 1$, $d$ common neighbor of $a_p$ and $b_q$) $\rightarrow$ $T' + T_a + T_b$



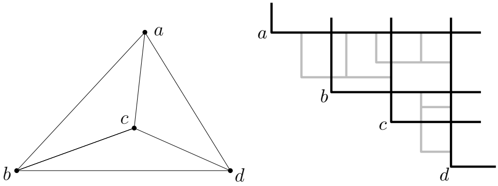



# Intersection representation

*Anchor* for an inner face *abc*:



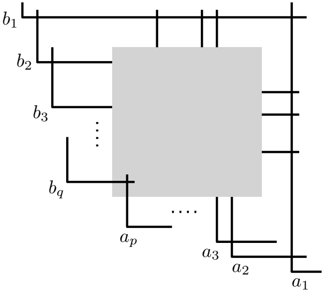

*Full representation* : intersection representation with ∟'s
+ anchors for each inner face



# Intersection representation

**Theorem**

Every 2-sided near-triangulation has a full representation like this $\longrightarrow$



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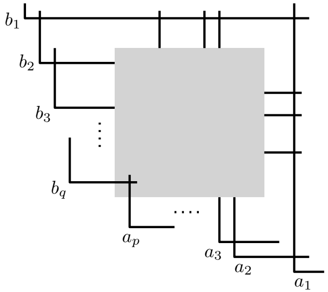
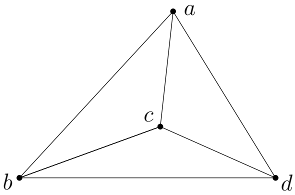
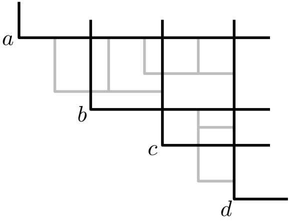
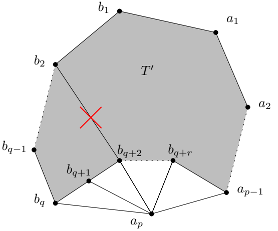


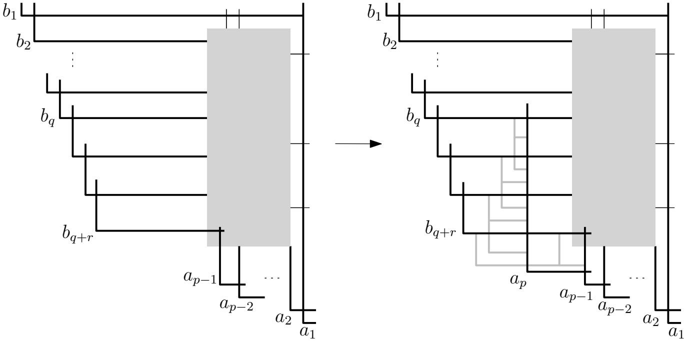



# Intersection representation

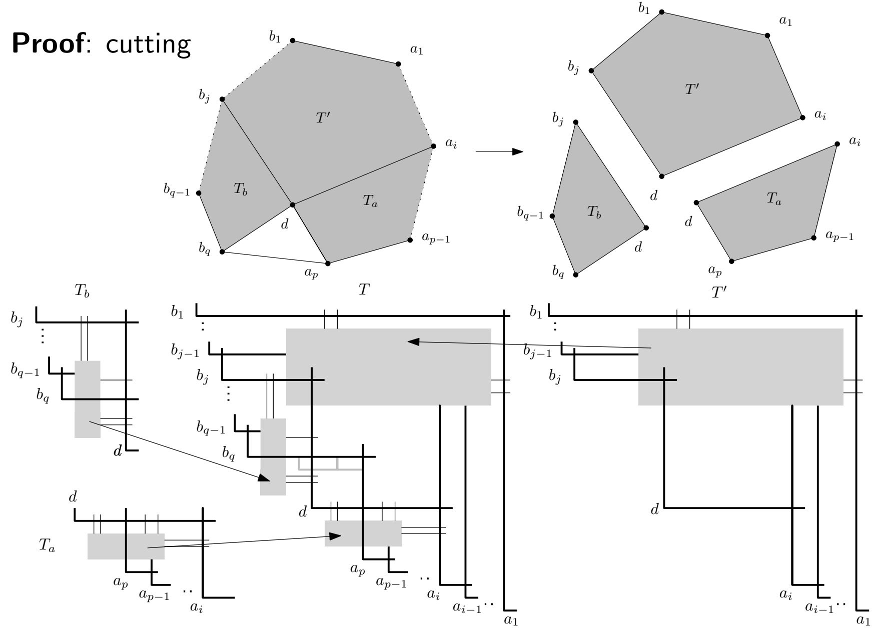
**Proof**: $a_p$-removal



# Intersection representation

**Proof**: cutting



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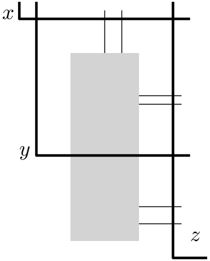
**Theorem**

Every triangulation $T$ with outer vertices $x, y, z$ has a full representation like this $\longrightarrow$



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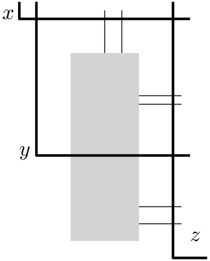


## Theorem

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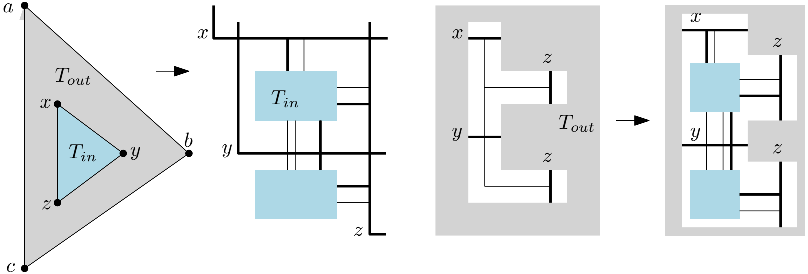
**Proof**

If $T$ has no separating $\triangle$ $\rightarrow$ preceding Theorem.

If $T$ has a separating $\triangle$ $\rightarrow$ $T_{in}$ and $T_{out}$



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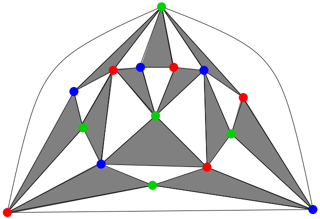
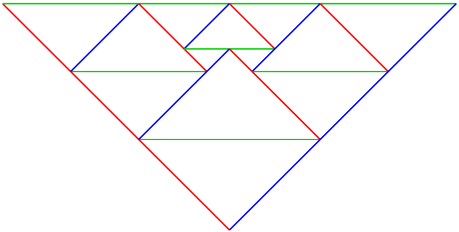
## Definition

A **TC-scheme** of an Eulerian triangulation $T$ is a 3-slope contact representation of $T$, with degeneracies.





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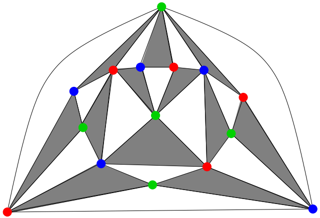
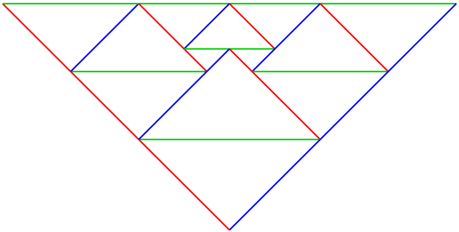
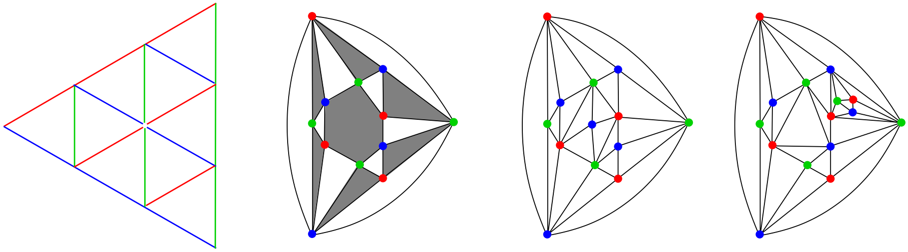




- Every face of a TC-scheme is a triangle. Each side is contained in a segment of the representation.
- Two parallel segments intersect on at most one point, their endpoint.

# TC-scheme with degeneracies



Scheme S $\quad\mathcal{M}(S)\quad T_1\quad T_2$

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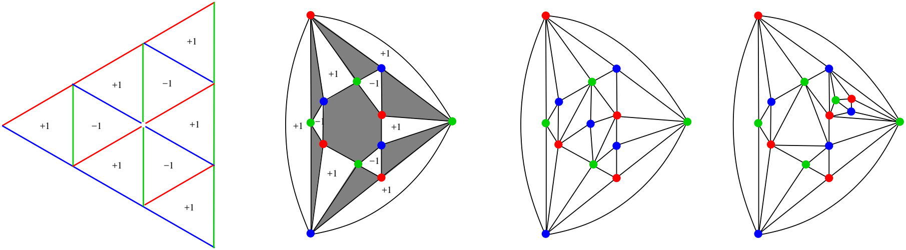


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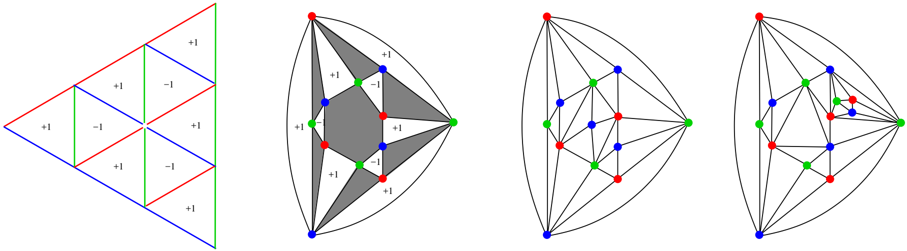


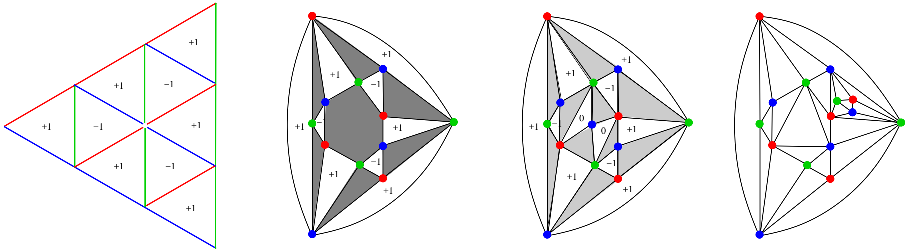

Scheme S $\qquad \mathcal{M}(S) \qquad T_1 \qquad T_2$

For a white face $F_i$, $f_i$ denotes the size of the corr. triangle in $S$.

## Remark

- For an inner vertex $v$: $\sum_{F_i \ni v} f_i = 0$
- For an outer vertex $v$: $\sum_{F_i \ni v} f_i = 3$

# TC-scheme with degeneracies



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# Linear system of necessary (& sufficient*) conditions

## $\mathcal{L}$ linear system with $n-2$ variables $f_i$

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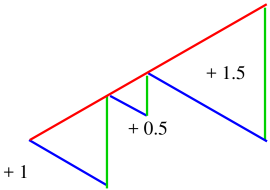


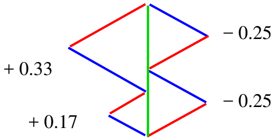

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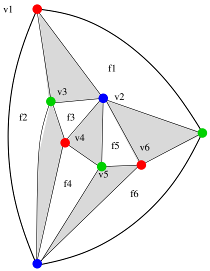
$n-2$ equations suffice, and $\mathcal{L}$ has a solution iff:




$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \times \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

for some vector $(f_1, \ldots, f_6)$

# $\exists$ solution because $\det(M) \neq 0$

$$det(M) = \sum_{\sigma} sign(\sigma) \prod_{i} M_{i,\sigma(i)}$$

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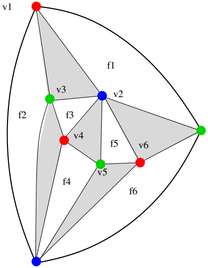
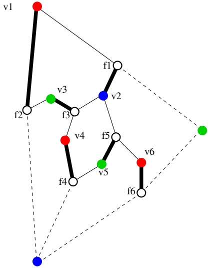



$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$



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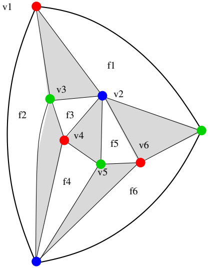


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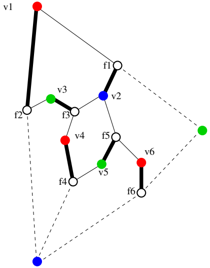


- ∃ matching in $Hex(M)$.
  Thus, some elts in the sum $det(M)$ are $+1$ or $-1$.

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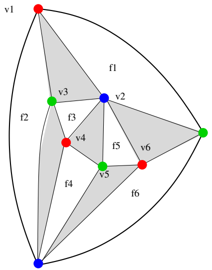
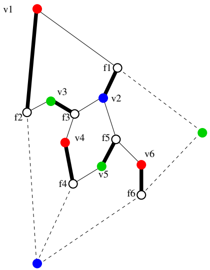



| | | | | | |
|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 1 | 0 |
| 0 | 1 | 1 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 0 | 0 |
| 0 | 0 | 0 | 1 | 1 | 0 |
| 0 | 0 | 0 | 0 | 1 | 1 |




- $\exists$ matching in $Hex(M)$.
  Thus, some elts in the sum $det(M)$ are $+1$ or $-1$.
- Perf. matchings in $Hex(M)$ are connected by $C_6$-flips.
  Thus, all the permutations have the same sign.

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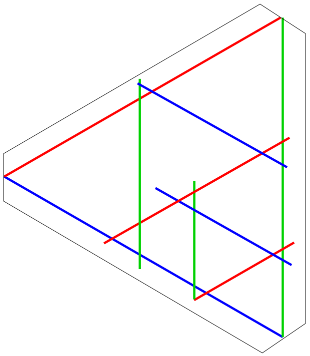
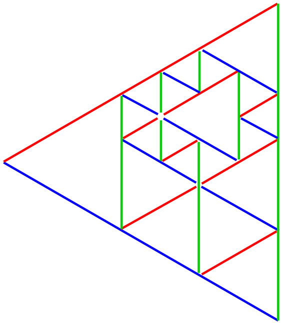
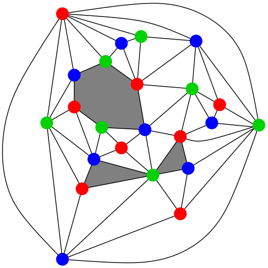
# Induction on smaller triangulations

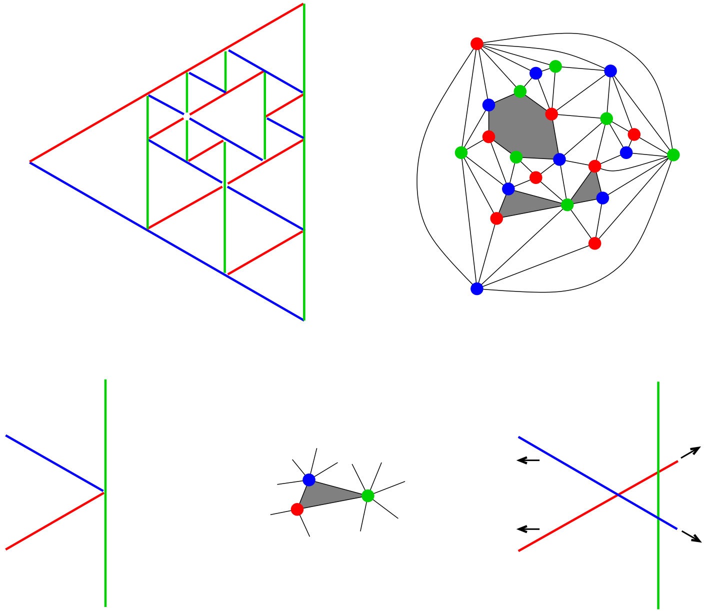
## Induction Hypothesis

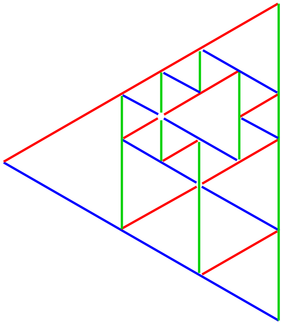
3-colorable triangulations with $< n$ vertices have a representation like this:

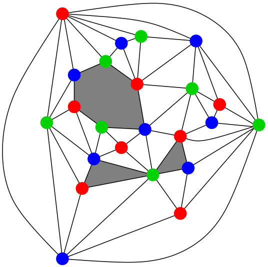


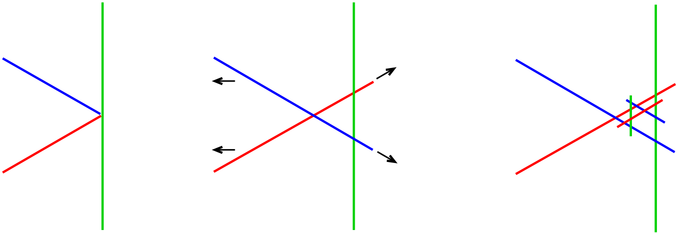


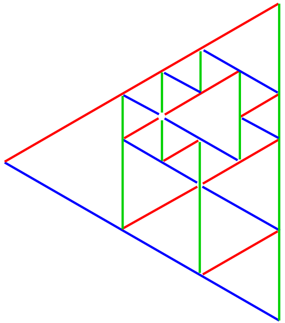


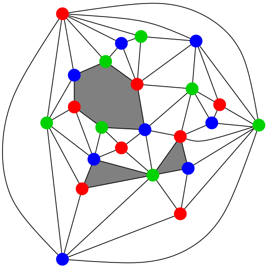


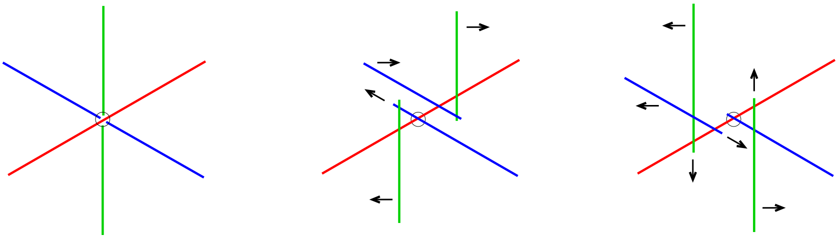


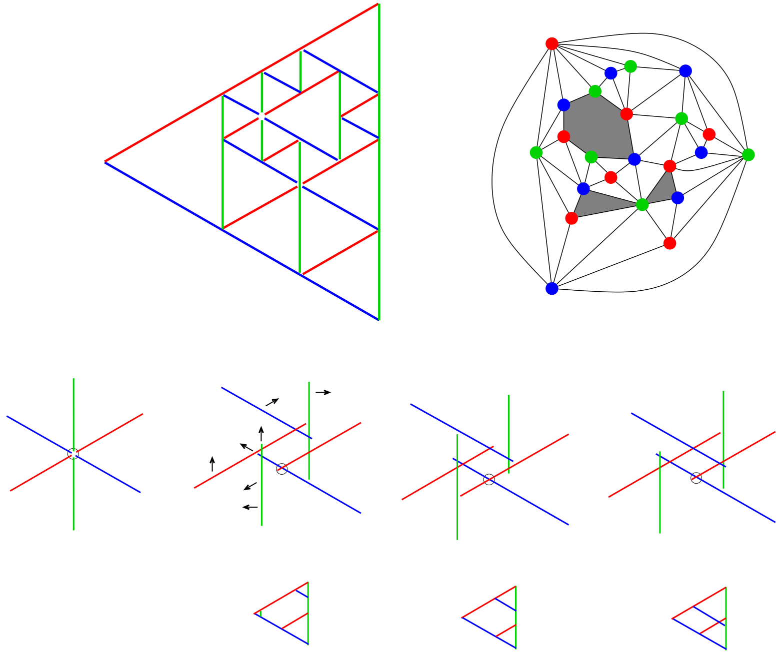












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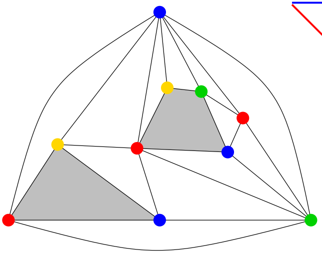
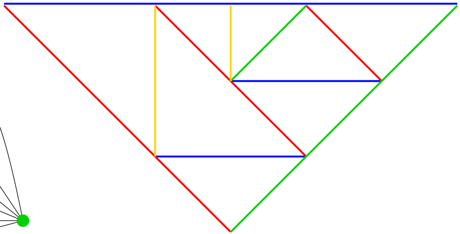
TC-schemes
Every 3-colorable triangulation has a TC-scheme
How TC-schemes give 3-DIR representations

Planar graphs not in PURE-4-DIR

# Planar Graphs in 4-DIR ?

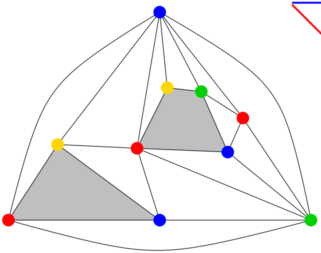


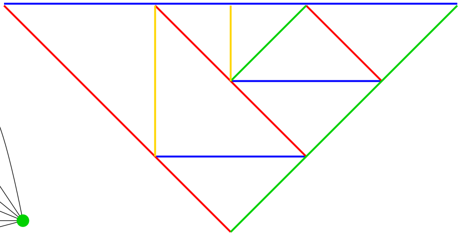


## Remark

4-colored triangulations may have a TC-scheme

# Planar Graphs in 4-DIR ?

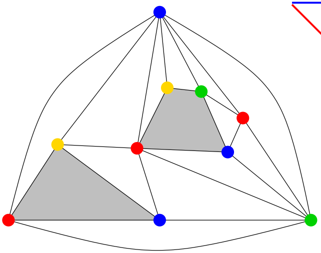


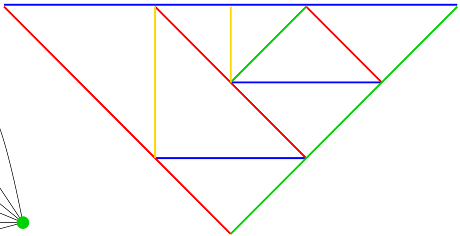


## Remark

4-colored triangulations may have a TC-scheme, but some configurations cannot be solved.

# Planar Graphs in 4-DIR ?





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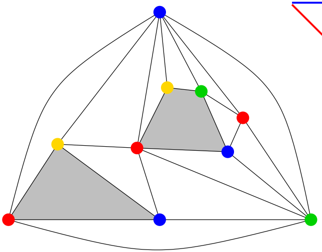
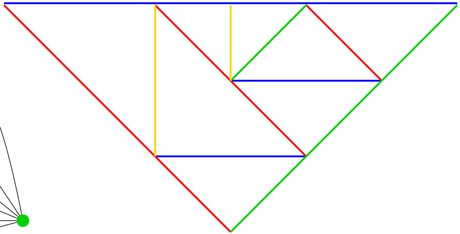
## Conjecture of mine

Every triangulation has a 4-coloring without induced 4-cycle colored 1, 2, 3 and 4, clockwisely.

# Planar Graphs in 4-DIR ?





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4-colored triangulations may have a TC-scheme, but some configurations cannot be solved.

## Conjecture of mine — False by (Kardos and Narboni '19)

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# Planar Graphs in 4-DIR ?

**Conjecture of Máčajová et al.**

Every signed planar graph has a vertex $\{-a, +a, -b, +b\}$-coloring $c$ such that

- for any positive edge $uv$, $c(u) \neq c(v)$, and
- for any negative edge $uv$, $c(u) \neq -c(v)$.

# Planar Graphs in 4-DIR ?

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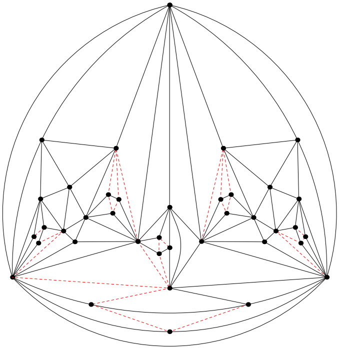

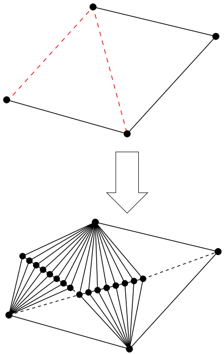

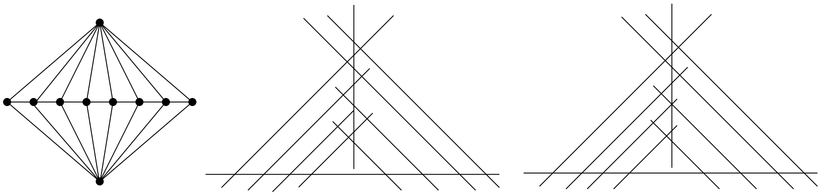

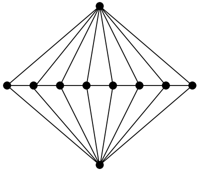

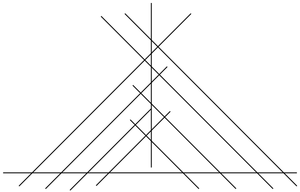

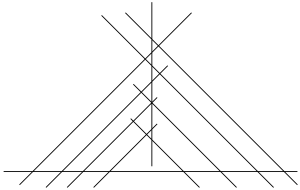

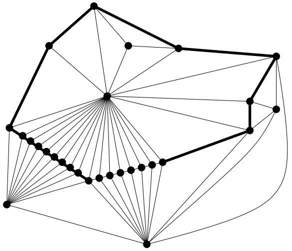

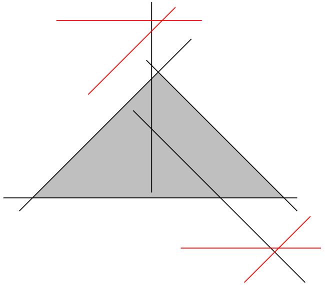

# Thank you!