Caterpillar arboricity of planar graphs

D. Gonçalves

LaBRI, U.M.R. 5800, Université Bordeaux I
351, cours de la liberation 33405 Talence Cedex, France.
goncalve@labri.fr

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Abstract

We solve a conjecture of Y. Roditty, B. Shoham and R. Yuster [2, 8] on the caterpillar arboricity of planar graphs. We prove that for every planar graph $G = (V, E)$, the edge set E can be partitioned into four subsets $(E_i)_{1\leq i\leq 4}$ in such a way that $G[E_i]$, for $1 \leq i \leq 4$, is a forest of caterpillars. We also provide a linear-time algorithm which constructs for a given planar graph G, four forests of caterpillars covering the edges of G.

1 Introduction

The *arboricity* $a(G)$ of a graph G is defined as the minimum number of forests needed to cover all the edges of G. In a similar way, the *linear arboricity* $la(G)$ (resp. *star arboricity* $sa(G)$, *caterpillar arboricity* $ca(G)$) of G is the smallest number of forests needed to cover all the edges of G such that each connected component of each forest is a path (resp. a star, a caterpillar). A *star* is a tree in which all the edges are incident to the same vertex. A *caterpillar* is a tree such that when removing all the vertices of degree one and all the edges containing them, it yields a path.

Since stars are caterpillars (resp. caterpillars are trees), a decomposition of a graph G into k forests of stars (resp. caterpillars) can be considered as a decomposition of G into k forests of caterpillars (resp. trees). So, for any graph G :

$$a(G) \leq ca(G) \leq sa(G) \tag{1}$$

Gyárfás and West [4] proved that the lower bound is tight in the case of complete bipartite graphs. Actually, they showed that $ca(K_{m,n}) = a(K_{m,n}) = \lceil mn/(m + n - 1)\rceil$. Here, we show that the upper bound of this inequality is also tight. To achieve that, we exhibit a family of graphs for which the equality holds. A *k-tree* is either a k-clique or a graph T with a vertex v of degree k, such that the neighbors of v form a k-clique and such that $T\setminus\{v\}$ is a k-tree. A partial k-tree is a subgraph of a k-tree.

Theorem 1 *For any $k > 0$, there is a partial k-tree T_k such that $sa(T_k) = ca(T_k) = k+1$.*

Nash-Williams [7] proved that $a(G) = \max_{H\subseteq G}\lceil |E_H|/(|V_H| - 1)\rceil$. Since a subgraph of a planar graph is planar and since planar graphs with n vertices have at most $3n - 6$ edges, planar graphs have arboricity at most three. The *acyclic chromatic number* of a graph G, is the minimal number of colors needed to color the graph G properly and such that the graph induced by the vertices colored a or b, for any pair of color a and b, is acyclic. Hakimi, Mitchem and Schmeichel [5] proved that the star arboricity of a graph

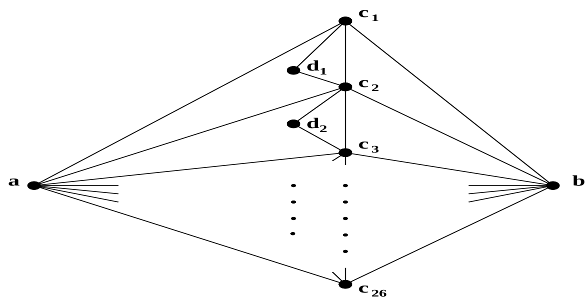

Figure 1: The graph H.

is upperbounded by its acyclic chromatic number. Borodin [1] showed that planar graphs have acyclic chromatic number at most five. This implies that planar graphs have star arboricity at most five. This upper bound is tight since there are planar graphs with star arboricity at least five. We are interested in determining such upper bound for the caterpillar arboricity of planar graphs. For any planar graph G, with arboricity 3 , we have $3 = a(G) \leq ca(G) \leq sa(G) \leq 5$, so this upper bound is either 3, 4 or 5. But the graph H, in Figure 1, reduces the possible values to 4 and 5.

Theorem 2 $ca(H) > 3$.

The main result of this paper is the following theorem :

Theorem 3 *For any planar graph G, we have $ca(G) \leq 4$.*

In Section 2, we provide definitions of the main terms and prove Theorems 1 and 2. In Section 3 we focus on planar graphs and prove Theorem 3. Section 4 is dedicated to the description and the analysis of a linear-time algorithm which founds, for any planar graph G on the input, four forests of caterpillars covering the edges of G. In Section 5, we provide applications of this result in monotone paths and for the track number of planar graphs.

2 Definitions

In a caterpillar, the set of *feet vertices* is a stable set of vertices of degree one, such that by deleting all of them we obtain a path with at least one vertex. The other vertices are the *spine vertices*. Note that a spine vertex can also have degree one and that there are different subsets of vertices which could be the set of the feet vertices. For instance K_2 can have none or one foot vertex (see Figure 2). Since there are no edges linking two feet vertices, the edges of a caterpillar are either *feet edges* if they link a foot vertex and a spine vertex, or *spine edges* if they link two spine vertices. Consider the graph $G = (V, E)$ and let $(E_i)_{1\leq i\leq k}$ be a partition of E such that for all i the subgraph $G[E_i]$ is a forest of caterpillars. The partially oriented graph $\overrightarrow{G} = (V, \overrightarrow{E})$ is the graph G where some edges have been oriented. This partial orientation allows us to distinguish the feet vertices (resp. feet edges) from the spine vertices (resp. spine edges) in $G[E_i]$. The edges of $\overrightarrow{G}$ are denoted $\{u, v\}$, if they are unoriented, and (u, v), if they are oriented from u to v. If $\{u, v\} \in E_i$ is a spine edge in $G[E_i]$ then the edge remains unoriented in $\overrightarrow{G}$, and we have $\{u, v\} \in \overrightarrow{E}$. If $\{u, v\} \in E_i$ is a foot edge in $G[E_i]$ where v is the foot vertex, then the edge $\{u, v\}$ is oriented from u to v in $\overrightarrow{G}$, and we have $(u, v) \in \overrightarrow{E}$. Remark that a vertex can be a foot vertex (or spine vertex) in one of the forests of caterpillars (*e.g.* $G[E_1]$), and have any status, foot or spine vertex in the other forests of caterpillars (*e.g.* $G[E_i]$ with $i > 1$).

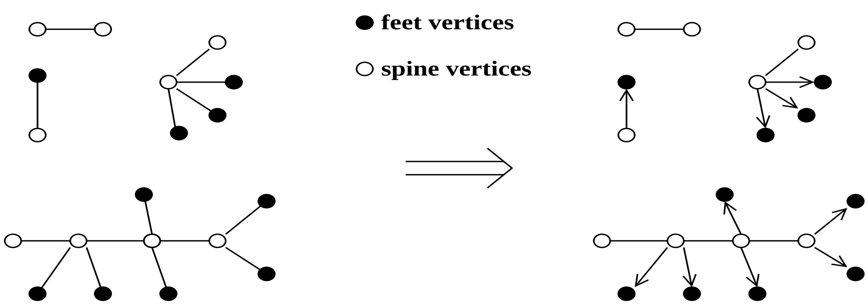

Figure 2: Partial orientation of the edges in a forest of caterpillars.

If $(E_i)_{1\leq i\leq k}$ is a k-partition of the edge set of G, such that for each i, $1 \leq i \leq k$, $G[E_i]$ is a forest of caterpillars and $\overrightarrow{G}$ is the resulting partial orientation of G, then the two following rules hold.

- The *feet preserving rule* ($\mathcal{R}_{fp}$) ensures that the feet vertices of the ith forest of caterpillars have degree one in $G[E_i]$. This is that if $(u, v) \in \overrightarrow{E_i}$, then for any neighbor w of v in G we have either $w = u$ or $\{v, w\} \notin E_i$.
- The *spine preserving rule* ($\mathcal{R}_{sp}$) ensures that after removing all the feet vertices in $G[E_i]$ we obtain a forest of paths. To achieve that, the spine edges of E_i have to form a forest of paths, this is an acyclic graph with maximum degree two.

The reverse is true as well, if a k-partition $(E_i)_{1\leq i\leq k}$ of E and a partial orientation $\overrightarrow{G}$ of G respect these two rules, this implies that for every i, $0 < i \leq k$, $G[E_i]$ is a forest of caterpillars. Remark that these rules imply that in each $\overrightarrow{G[E_i]}$ a vertex v is incident to at most two edges which are not oriented from v to the other end of the edge. So considering the k subgraphs $\overrightarrow{G[E_i]}$, in $\overrightarrow{G}$ a vertex v is incident to at most $2k$ edges which are not oriented from v to the other end of the edge.

Proof of Theorem 1. The case $k = 1$ is obvious, so let us consider that $k > 1$. Partial k-trees have acyclic chromatic number at most $k+1$. Since the star arboricity of a graph is bounded by its acyclic chromatic number [5] we obtain that partial k-trees have star arboricity at most $k + 1$. Since $ca(G) \leq sa(G)$ we have to exhibit a partial k-tree T_k with $ca(T_k) > k$.

We construct T_k, starting with a k-clique with vertices $x_1, ..., x_k$ and adding new vertices. We denote by $(a_1, ..., a_l) \longrightarrow b$ the adding of a new vertex b linked to l vertices $a_1, ..., a_l$ of a k-clique ($l \leq k$). First we add the vertex y_1 by $(x_1, ..., x_k) \longrightarrow y_1$ and the vertices y_i by $(x_1, ..., x_{k-1}, y_{i-1}) \longrightarrow y_i$ for $1 < i \leq 4k(k-1)+2$. Then we add the vertices z_i by $(y_{i-1}, y_i) \longrightarrow z_i$ for $1 < i \leq 4k(k-1)+2$. This graph is a partial k-tree, since we construct it by adding vertices linked to a set of vertices included in some k-clique.

If $ca(T_k) \leq k$, T_k would have a k-partition of its edges and a partial orientation following the rules $\mathcal{R}_{fp}$ and $\mathcal{R}_{sp}$. If the edges of T_k are partitionned into k subsets, any vertex v would have at most $2k$ incident edges not oriented from v to the other end. Let us consider the edges linking a vertex x_j for $1 \leq j \leq k-1$ with a vertex y_i for $0 \leq i \leq 4k(k-1)+2$. Any x_j has at most $2k$ such edges that are not oriented from x_j to the other end. Since we consider $k-1$ vertices x_j, there are at most $2k(k-1)$ edges linking these x_j's to the y_i's, that are not oriented from x_j to y_i. This implies that at most $2k(k-1)$ vertices y_i have an incident edge $\{x_j, y_i\}$, for $0 \leq j \leq k-1$, which is not oriented from x_j to y_i. Since there are $4k(k-1)+2$ vertices y_i, there are at least $2k(k-1)+2$ vertices y_i such that all the edges $\{x_j, y_i\}$, for $1 \leq j \leq k-1$, are oriented from x_j to y_i. We say that these vertices are saturated vertices. Since there are at least $2k(k-1)+2$ saturated vertices over the $4k(k-1)+2$ vertices y_i, there is a value i such that y_i and y_{i+1} are two saturated vertices. By the feet preserving rule $\mathcal{R}_{fp}$, the $k-1$ edges linking the x_j's to a saturated vertex v have to be in $k-1$ distinct subsets of the k-partition and all the other edges incident to v have to be in the remaining subset. Given that y_i(resp. y_{i+1}) is a saturated vertex the edges $\{y_i, y_{i+1}\}$ and $\{y_i, z_{i+1}\}$(resp. $\{y_i, y_{i+1}\}$ and $\{y_i, z_{i+1}\}$) have to be in the same subset of the partition. This implies that all the edges of the cycle (y_i, y_{i+1}, z_{i+1}) are in the same subset of the partition, contradicting the fact that each subset induces a forest of caterpillars.

□

Proof of Theorem 2. Let the graph $H = (V, E)$ be the graph depicted in Figure 1. If $ca(H) \leq 3$, then there is a partial orientation $\overrightarrow{H} = (V, \overrightarrow{E})$ and a 3-partition $(E_i)_{1\leq i\leq 3}$ following the rules $\mathcal{R}_{fp}$ and $\mathcal{R}_{sp}$. Since the edges of H are partitioned into 3 subsets, there are at most 6 edges in $\overrightarrow{H}$ incident to the vertex a (resp. vertex b) that are not oriented from a (resp. b) to the other end of the edge. So, there are at most 12 vertices c_j such that at least one of the edges $\{a, c_j\}$ or $\{b, c_j\}$ is not oriented to c_j in $\overrightarrow{H}$. This implies that at least 14 vertices c_j over 26 have both edges $\{a, c_j\}$ and $\{b, c_j\}$ oriented to c_j in $\overrightarrow{H}$. This implies that there are two consecutive such vertices. Let j be such that (a, c_j),(a, c_{j+1}),(b, c_j) and $(b, c_{j+1}) \in \overrightarrow{E}$. Given the feet preserving rule $\mathcal{R}_{fp}$, the edges $\{a, c_j\}$ and $\{b, c_j\}$ (resp. $\{a, c_{j+1}\}$ and $\{b, c_{j+1}\}$), have to be in different subsets of the partition, and all the other edges incident to c_j (resp. c_{j+1}), including $\{c_j, c_{j+1}\}$ and $\{c_j, d_j\}$ (resp. $\{c_j, c_{j+1}\}$ and $\{c_{j+1}, d_j\}$), have to be in the same remaining subset. So all the edges of the cycle (c_j, c_{j+1}, d_j) belong to the same subset of the partition, contradicting the fact that each subset of the partition induces a forest of caterpillars.

□

3 Planar graphs

A *triangulation* G is a planar graph that has only triangular faces. Every planar graph is the subgraph of some triangulation. Since a subgraph of a forest of caterpillars is also a forest of caterpillars, we can restrict our work to triangulations. Our purpose here is to prove that planar graphs have caterpillar arboricity at most four. We do that by constructing a 4-partition of the edges and a partial orientation of G that follows $\mathcal{R}_{fp}$ and $\mathcal{R}_{sp}$. To construct them for G we first do it for three subgraphs of G and then extend them to G. To allow this extension, the partial orientations and the partitions of these subgraphs have to follow two more rules. The following is the first one.

- The *no saturated vertex rule* ($\mathcal{R}_{nsv}$) imposes that each vertex v of G is a foot vertex in at most two $G[E_i]$, $1 \leq i \leq 4$. So each vertex v has at most two incident edges oriented towards v.

Let $G = (V, E)$ be a triangulation, with the vertices u, v and w on its outer boundary. For an edge $\{u, v\}$ on the outer boundary we define its *partner vertex*. Let the sequence x_1, x_2, ..., $x_t = w$ be the common neighbors of u and v ordered so that all the vertices x_j are inside the cycle (u, v, x_i) for $j \leq i$ (see Figure 3). The partner vertex of $\{u, v\}$ is the vertex x_{t-1}. In a triangulation G with at least four vertices, for any edge $\{u, v\}$ the vertices u and v have at least two common neighbors, so the partner vertex is well defined.

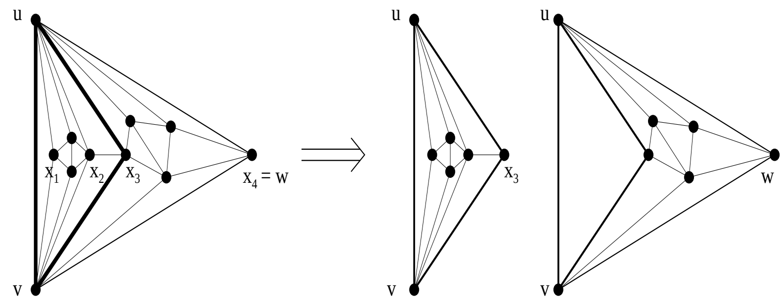

Figure 3: The partner vertex x_3 and a separating 3-cycle (u, v, x_3).

- The *external configuration rule*, $\mathcal{R}_{ec}(u, v, w, a, b, c)$, is defined for u, v and w being the vertices on the outer boundary, and $\{a, b, c\}$ being a set of size three included in $\{1, 2, 3, 4\}$. Let z be the partner vertex of the edge $\{u, v\}$. This rule imposes that all the edges incident to u are in the same subset of the partition, E_a. The edge $\{u, z\}$ has to be a spine edge (unoriented) and any other edge incident to u has to be oriented from u to the other end. All the edges incident to v, except $\{u, v\}$, are oriented from v to the other end and belong to E_b. All the edges incident to w, except for $\{u, w\}$ and $\{v, w\}$, are oriented from w to the other end and belong to E_c.

In a triangulation, a *separating* 3*-cycle* is a cycle of length three which does not delimit a face, this is with at least one vertex in its interior and at least one other in its exterior. A triangulation can be splited around a separating 3-cycle into two smaller triangulations as in Figure 3 with the cycle (u, v, x_3). We are going to prove the following theorem which implies Theorem 3.

Theorem 4 *Given an embedded triangulation G with vertices u, v and w on its outer boundary and any* 3*-set $\{a, b, c\} \subset \{1, 2, 3, 4\}$, there is a 4-partition of its edges and a partial orientation $\overrightarrow{G}$, following the four rules $\mathcal{R}_{fp}$, $\mathcal{R}_{sp}$, $\mathcal{R}_{nsv}$ and $\mathcal{R}_{ec}(u, v, w, a, b, c)$.*

Proof. Without loss of generality we consider that $a = 1$, $b = 2$ and $c = 3$. The proof of Theorem 4 works by induction on the number of vertices. It clearly holds for K_3, so let G have at least four vertices, and denote by z the partner vertex of the edge $\{u, v\}$.

We distinguish another vertex of the graph. Consider the sequence of the neighbors of u ordered as in the planar embedding, going from w to v. Denote by y the first vertex of this sequence being also a neighbor of z. We know that $y \neq v$ since u and z have at least two common neighbors. On the other hand note that the vertex y may be equal to w. If the cycle (u, v, z) (resp. (u, z, y)) is a separating 3-cycle, denote by G_l (resp. G_r) the triangulation in its interior. The triangulation which is in the exterior of both cycles is G_e (see Figure 4). In G_e the vertices u and v (resp. u and z) have only two common neighbors, z and w (resp. v and y). So in G_e the partner vertex of $\{u, v\}$ is still z. We construct G_m from G_e by deleting three edges, $\{v, z\}$, $\{u, z\}$ and $\{y, z\}$, and then merging u and v into the same vertex u'. In Figure 5 we can see that since u and z had just v and y as common neighbors the graph G_m is a well defined triangulation, without multiple edges. This graph has the vertices u', v and w on its outer boundary and has less vertices than G_e.

The graphs G_l, G_r and G_m having less vertices than G, we can apply the induction hypothesis. In the next paragraph we extend G_m's partition and partial orientation to G_e and in the last paragraph we show how to merge G_l, G_r and G_e's partitions and partial orientations.

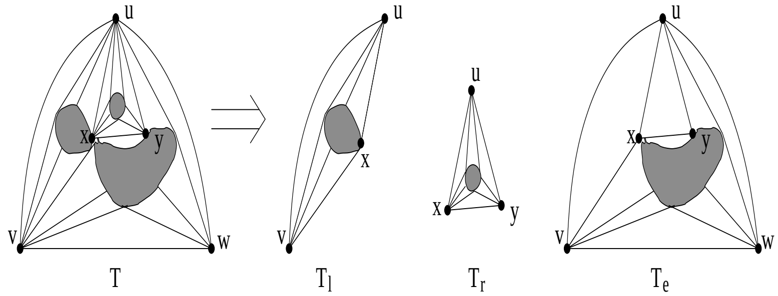

Figure 4: The graph G splited into G_l, G_r and G_e.

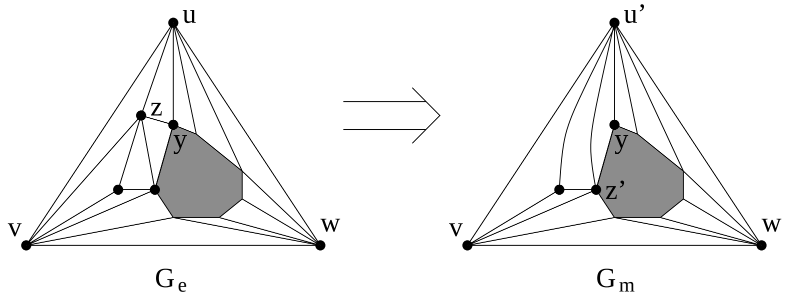

Figure 5: The graphs G_e and G_m.

From G_m to G_e : By the induction hypothesis, the graph G_m has a partial orientation $\overrightarrow{G_m}$ and a 4-partition $(E_m^i)_{1 \leq i \leq 4}$ of its edges, following the four rules $\mathcal{R}_{fp}$, $\mathcal{R}_{sp}$, $\mathcal{R}_{nsv}$ and $\mathcal{R}_{ec}(u', v, w, 1, 2, 3)$. We denote by z' the partner vertex of the edge $\{u', v\}$ in the graph G_m. Note that in G_e, the vertex z' is a neighbor of z and not a neighbor of u, else the cycle (u, v, z') would contain z, contradicting z's definition. We construct a partial orientation $\overrightarrow{G_e}$ of G_e, and a 4-partition $(E_e^i)_{1 \leq i \leq 4}$ of its edges that follow the four rules $\mathcal{R}_{fp}$, $\mathcal{R}_{sp}$, $\mathcal{R}_{nsv}$ and $\mathcal{R}_{ec}(u, v, w, 1, 2, 3)$, proceeding as follows :

- All the edges different from $\{u, z\}$, $\{v, z\}$ and $\{y, z\}$ are oriented as in $\overrightarrow{G_m}$ and belong to the same subset of the partition. So $E_m^i \subseteq E_e^i$ for all i.
- The edge $\{u, z\}$ is unoriented and belongs to E_e^1.
- The edge $\{v, z\}$ is oriented from v to z and belongs to E_e^2.
- Since the vertex y is distinct from z', and according to the rule $\mathcal{R}_{ec}(u', v, w, 1, 2, 3)$ in G_m, the edge $\{u, y\}$ is oriented from u to y and belongs to E_m^1. So, y is a foot vertex in $G_m[E_m^1]$. Now, according to the rule $\mathcal{R}_{nsv}$, y is a spine vertex in $G_m[E_m^i]$ for at least two values $i \in \{2, 3, 4\}$. So it is the case for a value $i \neq 2$, let say without loss of generality that it is the case for $i = 3$. Note that when $y = w$, since G_m follow the rule $\mathcal{R}_{ec}(u', v, w, 1, 2, 3)$ it is actually the case for $i = 3$. The edge $\{y, z\}$ is oriented from y to z and belongs to E_e^3.

Remark that all the vertices of G_e, except for u, v, z and y, have their incident edges oriented and partitioned in the same way as in G_m. So, all these vertices respect the rules $\mathcal{R}_{fp}$, $\mathcal{R}_{nsv}$ and $\mathcal{R}_{ec}(u, v, w, 1, 2, 3)$. In Figure 6 we note that u and v have their incident edges as expected by $\mathcal{R}_{ec}(u, v, w, 1, 2, 3)$ and they consequently also follow $\mathcal{R}_{fp}$ and $\mathcal{R}_{nsv}$. The vertex z is not in the outer boundary and has one incoming edge in E_e^2 and one in E_e^3, two spine edges in E_e^1 and the rest of its incident edges are in E_e^1 and oriented to the other end. So it follows these three rules. Since y is a spine vertex in $G_m[E_m^3]$ and we just

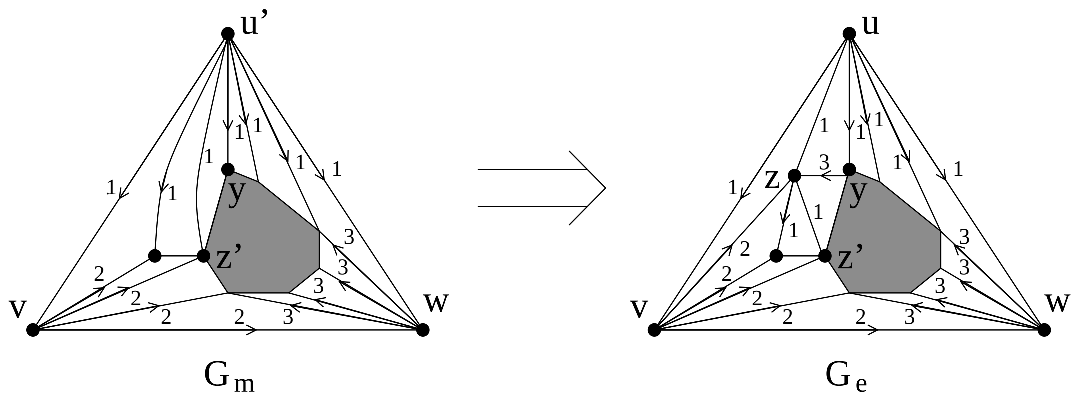

Figure 6: Partial orientation and 4-partition of G_e.

add it an incident edge oriented from y to the other end and belonging to E_e^3, it follows $\mathcal{R}_{fp}$ and $\mathcal{R}_{nsv}$. In the case that $y = w$ it also follows $\mathcal{R}_{ec}(u, v, w, 1, 2, 3)$.

We now deal with $\mathcal{R}_{sp}$. The vertices having their incident edges oriented and partitioned in the same way as in G_m have at most two incident spine edges in the same E_e^i. According to our construction, the vertices v and y have no new incident spine edge, and there are respectively one and two spine edges in the neighborhood of u and z. So the spine edges in each E_e^i induce a graph with maximal degree at most two. We now prove that there is no cycle of spine edges all being in the same subset of the partition. If there was such a cycle, not present in G_m, it should pass through the only new spine edge, $\{u, z\}$. But since u has just one incident spine edge, $\{u, z\}$ cannot be part of a cycle. So there is no such cycle in G_e.

Merging G_e, G_l and G_r : None of the edges incident to z in G_e are in E_e^4, so we consider that z is a spine vertex of $G_e[E_e^4]$. By the induction hypothesis, let $\overrightarrow{G_l}$ (resp. $\overrightarrow{G_r}$) be the partial orientation of G_l (resp. G_r), and $(E_l^i)_{1\leq i\leq 4}$ (resp. $(E_r^i)_{1\leq i\leq 4}$) be the 4-partition of its edges such that the rules $\mathcal{R}_{fp}$, $\mathcal{R}_{sp}$, $\mathcal{R}_{nsv}$ and $\mathcal{R}_{ec}(z, u, v, 4, 1, 2)$ (resp. $\mathcal{R}_{ec}(z, u, y, 4, 1, 3)$) are respected. Now we define a partial orientation $\overrightarrow{G}$ of G and a 4-partition $(E_i)_{1\leq i\leq 4}$ of its edges, such that the four rules $\mathcal{R}_{fp}$, $\mathcal{R}_{sp}$, $\mathcal{R}_{nsv}$ and $\mathcal{R}_{ec}(u, v, w, 1, 2, 3)$ are respected (see Figure 7). If an edge e of G belongs to G_e, then e is oriented as in $\overrightarrow{G_e}$ and remains in the same subset of the partition, this means that $E_e^i \subseteq E_i$ for all i. All the remaining edges are either in G_l or in G_r. We orient them as in $\overrightarrow{G_l}$ or $\overrightarrow{G_r}$ and maintain them in the same subset of the partition, $(E_l^i \cup E_r^i)\backslash E(G_e) \subseteq E_i$.

Now we prove that $\overrightarrow{G}$ and $(E_i)_{1\leq i\leq 4}$ also follow the four rules. We first verify that $\mathcal{R}_{fp}$, $\mathcal{R}_{nsv}$ and $\mathcal{R}_{ec}(u, v, w, 1, 2, 3)$ are followed in the neighborhood of every vertex.

- For all the vertices, except u, v, z and y, the set of their ajacent edges is the same as in G_l, G_r or G_e. If a vertex x is different from u, v, z or y, all its incident edges have exactly the same orientation and belong to the same subset of the 4-partition as in G_l, G_r or G_e. Since these three rules where followed in the neighborhood of these vertices in G_l, G_r or G_e, it is still the case in G.

Since G_l follow the rule $\mathcal{R}_{ec}(z, u, v, 4, 1, 2)$ and G_r the rule $\mathcal{R}_{ec}(z, u, y, 4, 1, 3)$, the edges we add around u, v, z and y, passing from G_e to G, produce no violation of these rules :

- For u, the difference between the set of its incident edges in G_e and in G is just that we add more edges belonging to E_1 and oriented from u to the other end. Since the partner vertex of $\{u, v\}$ in G_e and in G, is z in both cases, it is correct that $\{u, z\}$ remains a spine edge of E_1. So $\mathcal{R}_{fp}$, $\mathcal{R}_{nsv}$ and $\mathcal{R}_{ec}(u, v, w, 1, 2, 3)$ are followed.

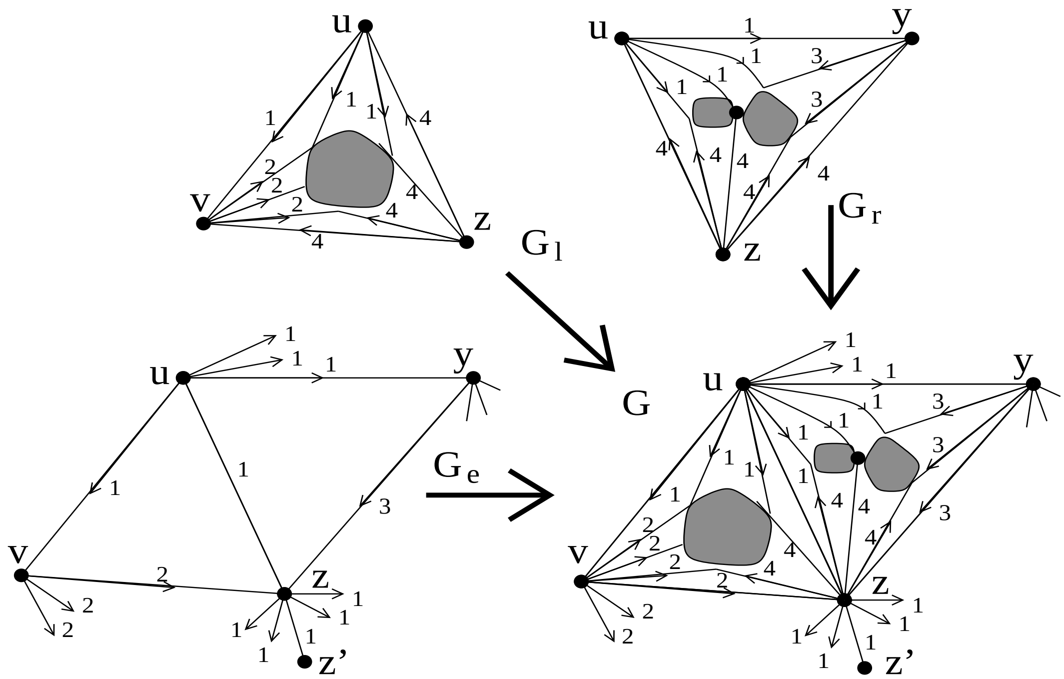

Figure 7: The merging of G_e, G_l and G_r.

- For v, we just add more edges oriented from v to the other end, and belonging to E_2. Since v is a spine vertex of $G_e[E_e^2]$ it produces no conflict with $\mathcal{R}_{fp}$ and $\mathcal{R}_{nsv}$. Furthermore v still follows $\mathcal{R}_{ec}(u,v,w,1,2,3)$.
- For z, we add two spine edges, one from G_l and one from G_r, and many feet edges oriented from z to the other end. All these new edges belong to E_4. Since there was no edges incident to z in E_e^4, these new edges create no conflict with the rules.
- For y, we just add more edges oriented from y to the other end, and belonging to E_3. Since y is a spine vertex of $G_e[E_e^3]$ it produces no conflict with $\mathcal{R}_{fp}$ and $\mathcal{R}_{nsv}$. Furthermore if $y = w$, then w still follows $\mathcal{R}_{ec}(u,v,w,1,2,3)$.

We now deal with the rule $\mathcal{R}_{sp}$. Since this rule is followed in G_e, G_l and G_r, there could be three incident spine edges in the same subset only in the neighborhood of the vertices u, v, y or z. But we add no spine edges to u, v and y, when we reconstruct the graph G from G_e. In the neighborhood of z we add only two spine edges belonging to E_4 but there was none previously. So the spine edges of each subset induce a graph of degree at most two. A cycle of spine edges included in a subset E_i should pass by the edges of at least two of the graphs G_e, G_l and G_r. This cycle should enter into G_l or G_r by a vertex on their outer boundary, and go out by another one. This is impossible since the vertices u, v and y have no incident spine edges in G_l and G_r. So our construction follows the four rules and we proved the theorem.

□

This is a constructive proof based on a planar embedding of G. In the next section we describe a linear-time algorithm that divides a planar graph into four forests of caterpillars.

4 The algorithm

Given a planar graph it is linear to find its planar embedding [6]. Then we can add edges until having a triangulation. This is made in linear time since a planar graph with n vertices has at most $3n-6$ edges. The algorithm proceeds as in the proof, partially orienting the triangulation and assigning each edge a subset in such a way that the four rules $\mathcal{R}_{fp}$, $\mathcal{R}_{sp}$, $\mathcal{R}_{nsv}$ and $\mathcal{R}_{ec}(u, v, w, 1, 2, 3)$ are respected.

To represent this embedded triangulation G we use the set of its vertices, the set of its edges and the vertices u, v, w and z (u, v and w being the vertices in its outer boundary and z the partner vertex of $\{u, v\}$). For each vertex of G we have the cyclic list of its incident edges ordered as in the embedding, two pointer respectively pointing u and v if u or v are its neighbors and two integers i_1 and i_2 indicating that this vertex is a spine vertex in the ${i_1}^{th}$ and in the ${i_2}^{th}$ subset of the partition. For each edge we have its end vertices, its orientation and the subset of the partition it belongs to.

As in the proof we construct the graphs G_m, G_l and G_r, and we recusively apply the algorithm for them. To do this we visit z's neighbors in the clockwise order starting from u. The last vertex before v being a neighbor of u (having a pointer to u) is the vertex y and the first vertex being a neighbor of v (having a pointer to v) is the vertex z', the partner vertex of $\{u', v\}$ in G_m. Since the vertex u will represent the vertex u' of G_m, for all z's neighbors strictly between y and v we set one of the pointers (one which is not pointing v) to u. Then in u's list of neighbors we cut the part strictly between y and v and we replace it by the part of z's list of neighbors strictly between y and v. For the vertex y (resp. v) we cut in its list of neighbors the part going from z included to u excluded (resp. from u excluded to z included). Now, with z' being the new partner vertex of $\{u, v\}$ we can apply recursively the algorithm to G_m in order to have a partition and a partial orientation that follow the rules $\mathcal{R}_{fp}$, $\mathcal{R}_{sp}$, $\mathcal{R}_{nsv}$ and $\mathcal{R}_{ec}(u, v, w, 1, 2, 3)$. Then we extend this result to G_e setting that the edges $\{v, z\}$ and $\{u, z\}$ in G are respectively a foot edge of E_2 and a spine edge of E_1. For the edge $\{y, z\}$, according to the value of i_1 and i_2 for the vertex y, we set that it is a foot edge of E_3 or E_4, let say E_3. Then we can set that the vertex z is a spine edge in E_1 and E_4. Now we want to apply the algorithm to G_l so that we follow the rule $\mathcal{R}_{ec}(z, u, v, 4, 1, 2)$, so this time z, u and v have the neighbor list corresponding to G_l and we set one of the pointers of z's neighbors to z (the pointer that does not point to u, but that may point v). Then we can recursively apply the algorithm and do the same for the graph G_r following $\mathcal{R}_{ec}(z, u, y, 4, 1, 3)$.

Without considering the recusrsive calls this algorithm does $O(\deg_G(z))$ operations. Since the vertex z is no more an inner vertex (a vertex not in the outer boundary) in G_m, G_l or G_r, the complexity of the whole algorithm is bounded by $O(\sum \deg(v))$, where the sum is over all the inner vertices. In planar graphs this sum is bounded by $O(n)$ where n is the number of vertices of the graph.

5 Applications

An *edge-ordered graph* is a pair (G, f), where $G = (V, E)$ is a graph and f is a bijective function, $f : E \to \{1, 2, \ldots, |E|\}$. The mapping f is called an *edge ordering* of G. A *monotone path of length k* in (G, f) is a simple path $P_k : v_0, v_1, \ldots, v_k$ in G such that $f(\{v_i, v_{i+1}\}) < f(\{v_{i+1}, v_{i+2}\})$ for $i = 0, 1, \ldots, k-2$. To bound the length of monotone paths, we bound the length of monotone trials, these being paths not necessarily simple. Given a graph G denote by $\alpha'(G)$ the minimum of the maximum length of a monotone trial over all edge orderings of G. In [8] the authors show that for a partition of $E = \cup_{i=1}^{l} E_i$, we have $\alpha'(G) \leq \sum_{i=1}^{l} \alpha'(G(V, E_i))$. For any tree T and any caterpillar C we have $\alpha'(T) \leq 3$ and $\alpha'(C) \leq 2$. This implies that $\alpha'(G) \leq 3 \times a(G)$ and that $\alpha'(G) \leq 2 \times ca(G)$. Given that planar graphs have arboricity at most three, for any planar graph G, $\alpha'(G)$ is bounded

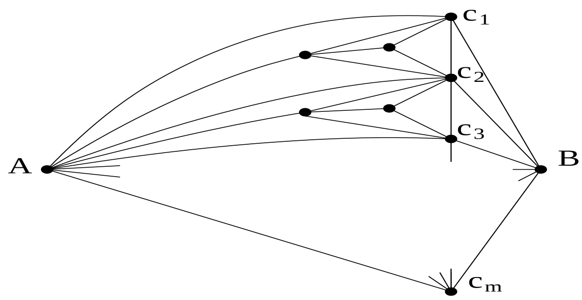

Figure 8: The counterexample.

by $3 \times 3 = 9$. Roditty's conjecture about the caterpillar arboricity of planar graphs was raised in [8] to improve this bound to $2 \times 4 = 8$. There is still a gap, since we know no planar graphs having a monotone path of length at least 6 for all edge orderings.

Corollary 1 *For a planar graph G, $\alpha'(G) \leq 8$.*

We tried to decrease this bound by partitioning the set of edges of planar graphs into a forest and two forests of caterpillars. This could have decreased the upper bound to seven but the graph depicted in Figure 8 does not admit such partition for m sufficiently large.

An *intersection representation* f of a graph G is an assignement of sets to the vertices so that vertices are adjacent if and only if the corresponding sets intersect. The *interval number* $i(G)$ is the minimum i such that G has an intersection representation in which, each set is a union of at most i intervals on the real line. The graphs with interval number one are the *interval graphs*. Scheinerman and West [9] proved that planar graphs have interval number at most three. A more restrictive intersection model is obtained by using sets that consist of an interval from each of t parallel lines. Such a representation of G is a *t-track representation*, and the *track number* $t(G)$ is the minimum t such that G has a t-track representation. An equivalent definition of the track number of G is the minimum number of interval graphs whose union is G. Since caterpillars are interval graphs, we have the following result.

Corollary 2 *For a planar graph G, $t(G) \leq 4$.*

Recently in [3] the authors proved that this bound is tight exhibiting a bipartite planar graphs with track number at least 4.

6 Conclusion

These results raise new questions. Is it possible with a similar argument to obtain a linear-time algorithm partitioning the edges of a planar graph into five forests of stars ? We also wonder on how tight is our result. Could we partition the edges of planar graphs in a more restrictive way ? For example into three forests of caterpillars and one forest of stars. In [3] the authors give a first result, they show some planar graphs that have no edge partition into four forests of bistars, trees of diameter at most three.

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