On the L$(p, 1)$-labelling of graphs

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Abstract

The L(p, q)-labelling of graphs, is a graph theoretic framework introduced by Griggs and Yeh (7) to model the *channel assignement problem*. In this paper we improve the best known upper bound for the L$(p, 1)$-labelling of graphs with given maximum degree. We show that for any integer $p \geq 2$, any graph G with maximum degree Δ admits an L$(p, 1)$-labelling such that the labels range from 0 to $\Delta^2 + (p-1)\Delta - 2$.

Key words: Channel assignement problem, L(p, q)-labelling

1 Introduction

Let G be a connected graph with maximum degree Δ. For a set of vertices $S \subset V(G)$, the graph $G \backslash S$ is the graph induced by $V(G) \backslash S$. The distance $d(u, v)$ between two vertices u and v is the number of edges in the shortest path from u to v. We say that v is a *d-neighbor* of u if $d(u, v) = d$. We generally use the common term *neighbor* instead of 1-neighbor. Let $N_d(v)$ be the set of d-neighbors of v. An *$L(\alpha_1, \alpha_2, \ldots, \alpha_k)$-labelling* of a graph G is a function $l : V(G) \to [0, \lambda]$ such that for any pair of vertices u and v if $d(u, v) = d \leq k$ then $|l(u) - l(v)| \geq \alpha_d$. The problem is to find an L$(\alpha_1, \alpha_2, \ldots, \alpha_k)$-labelling of G that minimizes λ. We denote $\lambda_{\alpha_1, \alpha_2, \ldots, \alpha_k}(G)$ the minimum value of λ. For a sequence of non-negative integers $S = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, we will use the notation $\lambda_S(G)$ instead of $\lambda_{\alpha_1, \alpha_2, \ldots, \alpha_k}(G)$.

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L($\alpha_1, \ldots, \alpha_k$)-labellings arise from the *channel assignement problem*. The channel assignement problem is to assign a channel to each radio transmitter so that close transmitters do not interfer and such that we use the minimum span of frequency. Roberts proposed to assign channels such that "close" transmitters receive different channels and "very close" transmitters receive channels that are at least two channels apart. This is an L(2,1)-labelling of a graph G where the vertices are the transmitters, the "very close" transmitters are adjacent vertices and the "close" transmitters are vertices at distance 2 in G. Since the constraints between transmitters disminish with the distance, the L($\alpha_1, \alpha_2, \ldots, \alpha_k$)-labelling of graph is interesting for this problem when the sequence $\alpha_1, \alpha_2, \ldots, \alpha_k$ is decreasing. Many work has been done on L(2,1)-labelling since the first paper of Griggs and Yeh (7). Many papers deal with bounding $\lambda_{\alpha_1,\alpha_2}$ for some graph families (1; 4; 5; 8; 9; 11; 14; 15; 16) or given some graph invariants such as $\chi(G)$, $\omega(G)$ or Δ (2; 3; 10; 12). In their paper (7), Griggs and Yeh proved that $\lambda_{2,1}(G) \leq \Delta^2+2\Delta$ and proposed the following conjecture.

Conjecture 1 *For any graph G with maximum degree $\Delta \geq 2$, $\lambda_{2,1}(G) \leq \Delta^2$.*

Actually they proved it for $\Delta = 2$ and for graphs of diameter at most 2. They also proved that determining $\lambda_{2,1}(G)$ is NP-complete. The conjecture is still open for $\Delta \geq 3$ and for various families of graphs. In (9), Kang proved it for Hamiltonian cubic graphs. The results in (1; 8; 14) prove the conjecture for planar graphs with maximum degree $\Delta \neq 3$.

In (2) the authors gave an algorithm for the L(2,1)-labelling and improved the upper bound of $\lambda_{2,1}$ to $\Delta^2 + \Delta$. In (3), with the same algorithm they obtained that $\lambda_{p,1}(G) \leq \Delta^2 + (p-1)\Delta$. Let $\sigma(S, \Delta)$ be the function defined for any sequence $S = (\alpha_1, \ldots, \alpha_k)$ by $\sigma(S, \Delta) = \sum_{i=1}^{k} \alpha_i \Delta(\Delta-1)^{i-1}$. With the algorithm used in (2; 3), we can extend their result as follow:

Proposition 2 *For any sequence of non-negative integers $S = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, with $k \geq 1$, and any graph G with maximum degree Δ, we have that $\lambda_S(G) \leq \sigma(S, \Delta)$.*

This is not the best known bound. In (10), Král and Škrekovski had a result on the list channel assignement problem. As a corollary of their result we have that:

Theorem 3 (Král and Škrekovski) *For any sequence of non-negative integers $S = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, with $k \geq 2$ and $\alpha_1 > \alpha_2$, and any graph G with maximum degree $\Delta \geq 3$, we have that $\lambda_S(G) \leq \sigma(S, \Delta) - 1$.*

We slightly improved this bound for some specific sequences S.

Theorem 4 *For any sequence $S = (\alpha_1, \ldots, \alpha_k)$ such that $k \geq 2$, $\alpha_1 \geq 2$, $\alpha_k = 1$ and $1 \leq \alpha_i < \alpha_1$ for $1 < i < k$, and for any connected graph G with maximum degree $\Delta \geq 3$, there is an ordering of the vertices, $v_0, v_1, \ldots, v_n$ and an $L(\alpha_1, \ldots, \alpha_k)$-labelling l of G such that:*

(1) $l(v_0) \leq \sigma(S, \Delta) - 1$,
(2) $l(v_j) \leq \sigma(S, \Delta) - j$ *for* $1 \leq j < k$, *and*
(3) $l(v_j) \leq \sigma(S, \Delta) - k$ *for* $k \leq j$.

This implies that just a constant number of vertices, k, may be labelled more than $\sigma(S, \Delta) - k$. We have a stronger result for $k = 2$.

Theorem 5 *For any sequence $S = (p, 1)$ with $p \geq 2$ and any graph G with maximum degree $\Delta \geq 3$, we have that $\lambda_{p,1}(G) \leq \sigma(S, \Delta) - 2 = \Delta^2 + (p-1)\Delta - 2$.*

So, for the L(2,1)-labelling we obtain that $\lambda_{2,1}(G) \leq \Delta^2 + \Delta - 2$ and we get a little closer to Conjecture 1. To prove Theorem 4 and Theorem 5 we need the following structural lemma.

Lemma 6 *Every graph G with maximum degree $\Delta \geq 3$ has either:*

(i) *a vertex v with degree less than Δ,*
(ii) *a cycle of length three,*
(iii) *two cycles of length four passing through the same vertex v,*
(iv) *a vertex v with three neighbors u, x and y, such that there is a cycle of length four passing through the edge uv and such that the graph $G \backslash \{x, y\}$ is connected, or*
(v) *a vertex u with two adjacent vertices v and w such that the graph $G \backslash X$ is connected, where X is the set $(N_1(v) \bigcup N_1(u)) \backslash \{w\}$.*

In Section 2, we extend the labelling algorithm presented in (2) and its analysis implies Proposition 2. In Section 3, we slightly modify this algorithm and we prove Theorem 4. In Section 4, we prove Theorem 5 using Lemma 6. Finally, we prove Lemma 6 in Section 5.

2 The basic algorithm

The algorithm presented in (2) performs an L$(2, 1)$-labelling of a a graph G with maximum degree Δ. The analysis of the algorithm gives the following bound, $\lambda_{2,1}(G) \leq \Delta^2 + \Delta$. Here we present an extended version of this algorithm that performs an L$(\alpha_1, \ldots, \alpha_k)$-labelling, for any sequence $(\alpha_1, \ldots, \alpha_k)$. The analysis of this algorithm establishes Proposition 2. Let $v_0, \ldots, v_n$ be an ordering of the vertices in $V(G)$.

Algorithm 1
$i = 0$;
WHILE *there are unlabelled vertices* **DO**
  FOR $v_j = v_n$ **TO** v_0 **DO**
    IF v_j *is unlabelled* **AND** v_j *can be labelled* i **THEN**
      Let v_j *be labelled* i*;*
  $i = i + 1$;

In this algorithm a vertex v_j "can be labelled i" if it has no d-neighbor already labelled x with $i - \alpha_d < x < i + \alpha_d$. Let us denote $l(v)$ the label the algorithm assigns to the vertex v.

Claim 7 *The fact that a vertex v is not labelled i is not due to a d-neighbor u whose label verifies $i < l(u) < i + \alpha_d$.*

Indeed, when the algorithm "proposed" v to be labelled i, the vertex u was still unlabelled (since $l(u) > i$). So, a vertex u can only "forbid" its d-neighbor v to be labelled $l(u)$, $l(u) + 1, \ldots, l(u) + \alpha_d - 1$.

Claim 8 *According to the order on the vertices used by the algorithm, let v_p and v_q be two vertices of G such that $p < q$. The fact that v_q is not labelled $l(v_p)$ is not due to v_p.*

Indeed, when the algorithm "proposed" v_q to be labelled $l(v_p)$, the vertex v_p was still unlabelled (since $p < q$).

Definition 9 *Denote $F(u, v)$, the set of labels which have been forbiden by u to v during the execution of the algorithm. Let $F(v) = \bigcup_{u \in V(G)} F(u, v)$ be the set of all the labels that have been forbiden to v.*

By Claim 7 and Claim 8, we know the elements in $F(u, v)$.

Remark 10 *Given two vertices v_p and v_q with $d(v_p, v_q) = d$, we have either:*

- $F(v_p, v_q) = \emptyset$, *if* $d > k$, $\alpha_d = 0$ *or* $l(v_q) \le l(v_p)$,
- $F(v_p, v_q) = \{l(v_p) + 1, \ldots, l(v_p) + \alpha_d - 1\}$, *if* $d \le k$, $\alpha_d > 0$, $l(v_q) > l(v_p)$ *and* $p < q$, *or*
- $F(v_p, v_q) = \{l(v_p), l(v_p) + 1, \ldots, l(v_p) + \alpha_d - 1\}$, *if* $d \le k$, $\alpha_d > 0$, $l(v_q) > l(v_p)$ *and* $p > q$.

This implies that $|F(v_p, v_q)| = 0$ when $d > k$ and that $|F(v_p, v_q)| \le \alpha_d$ either.

Claim 11 *The set $F(v)$ equals the interval $[0, \ldots, l(v) - 1]$, so $l(v) = |F(v)|$.*

Indeed, it is clear that (1) the algorithm labels a vertex v with the first value not in $F(v)$ and that (2) hence v is labelled there is no more value forbiden to v.

Finally, the set $F(v)$ being a union of possibly disjoint sets we have $|F(v)| \leq \sum_{u \in V(G)} |F(u,v)|$. In a graph of maximum degree Δ, one can easily see by induction on i that there are at most $\Delta(\Delta-1)^{i-1}$ vertices in $N_i(v)$. Since for any vertex u with $d(u,v) = d$ we have $|F(u,v)| \leq \alpha_d$ (with $\alpha_d = 0$ for $d > k$), we obtain that $l(v) = |F(v)| \leq \sum_{i=1}^{k} \alpha_i \Delta(\Delta-1)^{i-1}$.

3 The improved algorithm and proof of Theorem 4

To improve the bound we have in Proposition 2, we have to be more carefull on the order the algorithm considers the vertices. Indeed, according to the second point of Remark 10, if for a given vertex v_q there are x vertices v_p such that $d(v_p, v_q) = d \leq k$, $\alpha_d > 0$ and $p < q$, then $|F(v_p, v_q)| \leq \alpha_d - 1$ and $l(v_q) = |F(v_q)| \leq \sum_{u \in V(G)} |F(u, v_q)| \leq \sigma(S, \Delta) - x$. It would be interesting if the algorithm could use an order on the vertices, $v_0, \ldots, v_n$, such that many vertices v_q have some d-neighbors v_p such that $d(v_p, v_q) = d \leq k$, $\alpha_d > 0$ and $p < q$. Note that in any order the vertex v_0 has no such d-neighbors.

Definition 12 *Given a tree T rooted in a vertex r, a root-to-leaves order on the vertices of T is an order $v_0, v_1, \ldots, v_n$ such that $v_0 = r$ and such that for any $x \in [0 \ldots n]$ the subgraph of T induced by $\{v_0, v_1, \ldots, v_x\}$ is connected (i.e. is a tree).*

There are various possible root-to-leaves orders for a given tree. Note that in a root-to-leaves order any vertex $v \in V(T)$ appears after its "ancestors" in T. The following lemma gives interesting properties of those orders.

Lemma 13 *Given a connected graph G, consider any spanning tree T of G rooted in any vertex $r \in V(G)$. Let $v_0, \ldots, v_n$ be a root-to-leaves ordering of the vertices in T. For any integer $t \geq 0$, we have that :*

(i) $v_0 = r$.
(ii) For any integers i and j such that $i < j < t$ we have $d(v_i, v_j) \leq t$.
(iii) For any integer j such that $j \geq t$, there are at least t vertices v_i such that $i < j$ and $d(v_i, v_j) \leq t$.

PROOF. (i) holds by definition of root-to-leaves orders. Since the graph $T[v_0, \ldots, v_{t-1}]$, the subgraph of T induced by the vertices $v_0, \ldots, v_{t-1}$, is a tree with t vertices, its diameter is at most $t-1$. So (ii) clearly holds. For (iii), since the graph $T[v_0, \ldots, v_j]$ is a tree, we consider two cases. If all the vertices are at distance at most t from v_j in this subtree, there are j vertices (from v_0 to v_{j-1}) at distance at most t from v_j and since $j \geq t$ (iii) holds. If there is a vertex at distance $t+1$ from v_j in this subtree, the t vertices of the path linking v_j to this vertex are at distance at most t from v_j, so (iii) holds.

Given any spanning tree T of a connected graph G rooted in any vertex $r \in V(G)$, let $v_0, \ldots, v_n$ be any root-to-leaves ordering of the vertices in T. Now assume that Algorithm 1 performs an L($\alpha_1, \ldots, \alpha_k$)-labelling of G using this order of the vertices. Lemma 13 (with $t = k$) and Remark 10 imply that the points (2) and (3) of Theorem 4 hold:

(2) For any vertex v_j with $1 \leq j < k$, there are j vertices v_i (from v_0 to v_{j-1}) such that $i < j$ and $d(v_i, v_j) \leq k$. Since $\alpha_l \geq 1$ for all $l \leq k$, Remark 10 implies that the algorithm labels v_j at most $\sigma(\Delta, S) - j$.

(3) For any vertex v_j with $j \geq k$, there are k vertices v_i such that $i < j$ and $d(v_i, v_j) \leq k$. Since $\alpha_l \geq 1$ for all $l \leq k$, Remark 10 implies that the algorithm labels v_j at most $\sigma(\Delta, S) - k$.

We prove now that appropriately choosing T, r and the root-to-leaves order, the point (1) of Theorem 4 also holds. The following structural lemma is easily deduced from Lemma 6 or from Lemma 1.15 in (13).

Lemma 14 *Every graph G with maximum degree $\Delta \geq 3$ has either:*

(a) a vertex v with degree less than Δ,
(b) a cycle of length $l \leq 4$, or
(c) a vertex v with two neighbors x and y such that the graph $G \backslash \{x, y\}$ is connected.

We consider three cases according to which case of Lemma 14 the graph G corresponds.

Case (a): If there is a vertex of degree less than Δ, let the root r be this vertex. Then, consider any spanning tree T of G and any root-to-leaves ordering of T. In this case, since there are at most $\Delta - 1$ vertices in $N_1(v_0)$, $|F(v_0)|$ is bounded by $\sigma(S, \Delta) - \alpha_1$. Since $\alpha_1 \geq 2$, we have that $l(v_0) < \sigma(S, \Delta) - 2$ and (1) holds.

Case (b): If there is a cycle of length $l \leq 4$, let the root r be any vertex of this cycle. Then, consider any spanning tree T of G and any root-to-leaves ordering of T. In this case, since there are at most $\Delta(\Delta - 1) - 1$ vertices in $N_2(v_0)$, $|F(v_0)|$ is bounded by $\sigma(S, \Delta) - \alpha_2$. Since $\alpha_2 \geq 1$, we have that $l(v_0) \leq \sigma(S, \Delta) - 1$ and (1) holds.

Case (c): If there is a vertex with two neigbors x and y such that the graph $G \backslash \{x, y\}$ is connected, let the root r be this vertex. Let T' be any spanning tree of the connected graph $G \backslash \{x, y\}$. Let T be the tree $T' \cup \{rx, ry\}$. Since T' is a spanning tree of $G \backslash \{x, y\}$, it is clear that T is a spanning tree of G.

Since x and y are leaves in T, there is a root-to-leaves ordering of T such that $v_0 = r$ (by definition), $v_{n-1} = x$ and $v_n = y$. Note that v_n is the first vertex considered by the algorithm (the loop goes from v_n to v_0) when $i = 0$. At this moment all the vertices are unlabelled, so the vertex v_n is necessarily labelled 0. Since v_n and v_{n-1} have a common neighbor, v_0, we have $d(v_n, v_{n-1}) \leq 2$. If $d(v_n, v_{n-1}) = 1$, G has a cycle of length three, (v_0, v_n, v_{n-1}), and this case was proved in Case (b). So, let $d(v_n, v_{n-1}) = 2$. This implies (since $l(v_n) = 0$) that v_{n-1} cannot be labelled less than α_2. Let us consider two cases:

(1) If $l(v_{n-1}) = \alpha_2$, since $\alpha_1 > \alpha_2$, the value α_2 is in both $F(v_{n-1}, v_0)$ and $F(v_n, v_0)$. This implies that $|F(v_{n-1}, v_0) \cup F(v_n, v_0)| \leq 2\alpha_1 - 1$, and so that $l(v_0) = |F(v_0)|$ is bounded by $\sigma(S, \Delta) - 1$. So (1) holds.

(2) If $l(v_{n-1}) > \alpha_2$, since $F(v_n, v_{n-1}) = \{0, \ldots, \alpha_2 - 1\}$, there is a vertex $v_t \neq v_n$ such that $\alpha_2 \in F(v_t, v_{n-1})$. This vertex v_t is such that $d(v_t, v_{n-1}) = d \leq k$ and $\alpha_2 < l(v_t) + \alpha_d$. Furthermore, since v_{n-1} was the first unlabelled vertex "offered" to be labelled α_2 (v_n was already labelled 0), we have $l(v_t) < \alpha_2$. If $v_t = v_0$, since $l(v_t) < \alpha_2 \leq \sigma(S, \Delta) - 1$, we are done, so let $v_t \neq v_0$. Since $\alpha_2 \in F(v_t, v_{n-1}) = \{l(v_t), \ldots, l(v_t) + \alpha_d - 1\}$, $l(v_t) < \alpha_2$ and $\alpha_k = 1$, we have that $d < k$. This implies that $d(v_t, v_0) = d' \leq d + 1 \leq k$ and that the value $l(v_t)$ is in both $F(v_t, v_0)$ and $F(v_n, v_0)$. This implies that $|F(v_t, v_0) \cup F(v_n, v_0)| \leq \alpha_{d'} + \alpha_1 - 1$ and so that $l(v_0) = |F(v_0)|$ is bounded by $\sigma(S, \Delta) - 1$. So (1) holds.

4 Proof of Theorem 5

We prove Theorem 5 for a sequence $S = (p, 1)$, with $p \geq 2$, and a connected graph G (if G is disconnected we consider each of its connected components). Let $v_0, \ldots, v_n$ be any root-to-leaves ordering of any spanning tree T of G rooted in any vertex $r \in V(G)$. We have seen in the previous section that, using this order on the vertices of G, Algorithm 1 does a L$(p, 1)$-labelling of G such that the vertices v_i, with $i \geq 2$, are labelled at most $\sigma(S, \Delta) - 2$. Furthermore, with such order on the vertices we have that $|F(v_0, v_1)| \leq p - 1$. This means that the set $F(v_0)$ (resp. $F(v_1)$) has at most $\sigma(S, \Delta)$ (resp. $\sigma(S, \Delta) - 1$) elements, and that we should "save" two (resp. one) elements. We prove that, appropriately choosing T, r and the root-to-leaves ordering, we can bound $l(v_0) = |F(v_0)|$ and $l(v_1) = |F(v_1)|$ by $\sigma(S, \Delta) - 2$. We consider distinct cases according to which case of Lemma 6 the graph G corresponds.

Case (i): If there is a vertex of degree less than Δ, let the root r be this vertex. Then, consider any spanning tree T of G and any root-to-leaves ordering of T. Since $v_0 = r$ has at most $\Delta - 1$ neighbors and $(\Delta - 1)^2$ vertices at distance 2, we bound $|F(v_0)|$ by $(\Delta - 1)^2 + p(\Delta - 1)$ which is less than $\Delta^2 + (p - 1)\Delta - 2$. The vertex v_1 has at most Δ neighbors, including v_0, and at most $\Delta(\Delta - 1) - 1$

vertices at distance 2. With the fact that $|F(v_0, v_1)| \leq p - 1$, we have that $|F(v_1)| \leq \Delta(\Delta - 1) - 1 + p(\Delta - 1) + p - 1$, which equals $\Delta^2 + (p - 1)\Delta - 2$.

Case (ii): If there is a cycle of length three passing through the edge uv, consider a spanning tree T rooted in v that uses the edge uv. Then let this tree be rooted in v ($v_0 = v$) and consider a root-to-leaves ordering of T such that $v_1 = u$. Since the vertices in a cycle of length three have at most $\Delta(\Delta - 1) - 2$ vertices at distance 2, we can bound $|F(v_0)|$ and $|F(v_1)|$ by $\Delta^2 + (p - 1)\Delta - 2$.

Case (iii): If there are two cycles of length four passing through the same vertex v, let u be a neighbor of v in one of these cycles. Consider a spanning tree T rooted in v that uses the edge uv. Then consider a root-to-leaves ordering of T such that $v_0 = v$ and $v_1 = u$. Since v_0 has at most $\Delta(\Delta - 1) - 2$ vertices at distance 2, we can bound $|F(v_0)|$ by $\Delta^2 + (p - 1)\Delta - 2$. The vertex v_1 has at most $\Delta(\Delta - 1) - 1$ vertices at distance 2. With the fact that $|F(v_0, v_1)| \leq p - 1$, we have that $|F(v_1)|$ is bounded by $\Delta^2 + (p - 1)\Delta - 2$.

Case (iv): If there is a cycle of length four passing through an edge uv and two vertices x and $y \in N_1(v) \backslash \{u\}$ such that $G \backslash \{x, y\}$ is connected, let T' be any spanning tree of $G \backslash \{x, y\}$. Let T be the tree $T' \cup \{vx, vy\}$ rooted in v. Since T' is a spanning tree of $G \backslash \{x, y\}$, it is clear that T is a spanning tree of G. Since x and y are leaves in T, let $v_0, \ldots, v_n$ be a root-to-leaves ordering of T that finishes with x and y (i.e. $v_{n-1} = x$ and $v_n = y$).

The vertex v_1 has at most $\Delta(\Delta - 1) - 1$ vertices at distance 2. With the fact that $|F(v_0, v_1)| \leq p - 1$, we have that $|F(v_1)|$ is bounded by $\Delta^2 + (p - 1)\Delta - 2$.

Note that v_n is the first vertex considered by the algorithm (the loop goes from v_n to v_0) when $i = 0$. At this moment all the vertices are unlabelled, so the vertex v_n is labelled 0. Since v_n and v_{n-1} have a common neighbor, v_0, we have $d(v_n, v_{n-1}) \leq 2$. If $d(v_n, v_{n-1}) = 1$, G has a cycle of length three, (v_0, v_n, v_{n-1}), and this case was proved in Case (ii). So, let $d(v_n, v_{n-1}) = 2$. This implies (since $l(v_n) = 0$) that v_{n-1} cannot be labelled 0. We consider two cases according to $l(v_{n-1})$:

(1) If $l(v_{n-1}) = 1$, since $p \geq 2$, the value 1 is in both $F(v_{n-1}, v_0)$ and $F(v_n, v_0)$. This implies that $|F(v_{n-1}, v_0) \cup F(v_n, v_0)| \leq 2p - 1$. With the fact that v_0 has at most $\Delta(\Delta - 1) - 1$ vertices at distance 2, we have that $|F(v_0)|$ is bounded by $\Delta^2 + (p - 1)\Delta - 2$.

(2) If $l(v_{n-1}) > 1$, there is a vertex $v_t \in N_1(v_{n-1})$ labelled 0. Indeed, since $F(v_n, v_{n-1}) = \{0\}$, there is a vertex $v_t \neq v_n$ such that $1 \in F(v_t, v_{n-1})$. Furthermore, since v_{n-1} was the first unlabelled vertex "offered" to be labelled 1 (v_n was already labelled 0), we have $l(v_t) = 0$ and $d(v_t, v_{n-1}) = 1$. If $v_t = v_0$, since $0 \leq \sigma(S, \Delta) - 2$, we are done, so let $v_t \neq v_0$. Since v_0 and v_t are adjacent to v_{n-1} and since there is no cycle (v_0, v_t, v_{n-1}) (we would be in Case (ii)), we have $d(v_0, v_t) = 2$. This implies that the value 0 is in

both $F(v_t, v_0)$ and $F(v_n, v_0)$ and so that $|F(v_t, v_0) \cup F(v_n, v_0)| \leq 1+p-1$. With the fact that v_0 has at most $\Delta(\Delta-1)-1$ vertices at distance 2, we have that $|F(v_0)|$ is bounded by $\Delta^2 + (p-1)\Delta - 2$.

Case (v): If there is a vertex u with two neighbors v and w such that, for $X = N_1(v) \bigcup N_1(u) \backslash \{w\}$, the graph $G \backslash X$ is connected, let T' be any spanning tree of $G \backslash X$. Note that the vertex v, the neighbors of v (including u) and the neighbors of u except w are not in $G \backslash X$. So let T be the tree rooted in v which is the union of T', all the edges incident to u and all the edges incident to v. Since T' is a spanning tree of $G \backslash X$, it is clear that T is a spanning tree of G such that the neighbors of u and v, except u, v and w, are leaves. This implies that there are root-to-leaves orderings of T that finish with the vertices in $L = N_1(v) \cup N_1(u) \backslash \{u, v, w\}$. In these orderings, since the subgraphs of T induced by $\{v_0, v_1\}$ or $\{v_0, v_1, v_2\}$ are connected, since $N_1(v) \setminus L = \{u\}$ and since $N_1(u) \setminus L = \{v, w\}$, we have that $v_0 = v$, $v_1 = u$ and $v_2 = w$. So, let $v_0, \ldots, v_n$ be a root-to-leaves ordering of T such that $v_0 = v$, $v_1 = u$, $v_2 = w$, $N_1(v_0) = \{v_1, v_{n-\Delta+2}, \ldots, v_n\}$ and $N_1(v_1) = \{v_0, v_2, v_{n-2\Delta+4}, \ldots, v_{n-\Delta+1}\}$. We consider two subcases according to the maximum degree Δ of the graph G.

Case (v) with $\Delta \geq 4$: For v_1, let us consider the labels the algorithm assigns to two neighbors of v_1, $v_{n-\Delta}$ and $v_{n-\Delta+1}$. Since $d(v_{n-\Delta}, v_{n-\Delta+1}) \leq 2$ we have $l(v_{n-\Delta}) \neq l(v_{n-\Delta+1})$. Let a and b be such that $\{a, b\} = \{n-\Delta, n-\Delta+1\}$ and $l(v_a) < l(v_b)$. We consider two cases according to $l(v_b)$:

(1) If $l(v_b) < l(v_a) + p$ then the value $l(v_b)$ belongs to both $F(v_b, v_1)$ and $F(v_a, v_1)$, and we have $|F(v_b, v_1) \cup F(v_a, v_1)| \leq 2p-1$. With the fact that $|F(v_0, v_1)| \leq p-1$, we have that $|F(v_1)|$ is bounded by $\Delta^2 + (p-1)\Delta - 2$.

(2) If $l(v_b) \geq l(v_a) + p$, we wonder why v_b has not been labelled $l(v_a) + p - 1$ when the algorithm proposed it this value. There are two possible reasons. The vertex v_b had either (1) a neighbor v_x such that $l(v_a) \leq l(v_x) \leq l(v_a) + p - 1$, or (2) a 2-neighbor v_y labelled $l(v_a) + p - 1$ and such that $y > b$. In the first case, v_x would be at distance 2 from v_1 (if there was a cycle (v_1, v_b, v_x) we would be in Case (ii)) and the value $l(v_x)$ would be in both $F(v_x, v_1)$ and $F(v_a, v_1)$. In the second case, since $y > b$ and $y \neq a$ (by $l(v_y) = l(v_a) + p - 1$), the vertex v_y is a neighbor of v_0 (indeed $y > n-\Delta+1$) and a 2-neighbor of v_1. So, the value $l(v_a)+p-1$ would be in both $F(v_y, v_1)$ and $F(v_a, v_1)$. In both cases, (1) or (2), with the fact that $|F(v_0, v_1)| \leq p-1$, we have that $|F(v_1)|$ is bounded by $\Delta^2 + (p-1)\Delta - 2$.

For v_0, let us consider the labels the algorithm assigns to v_n, v_{n-1} and v_{n-2}. Since v_n is the first vertex the algorithm proposes the value 0, it is labelled 0. These three vertices are all at distance 2 from the others (if there was a cycle of length three we would be in Case (ii)), so they have different labels. Let a and b be such that $\{a, b\} = \{n-1, n-2\}$ and $0 = l(v_n) < l(v_a) < l(v_b)$. We consider three cases according to $l(v_a)$ and $l(v_b)$:

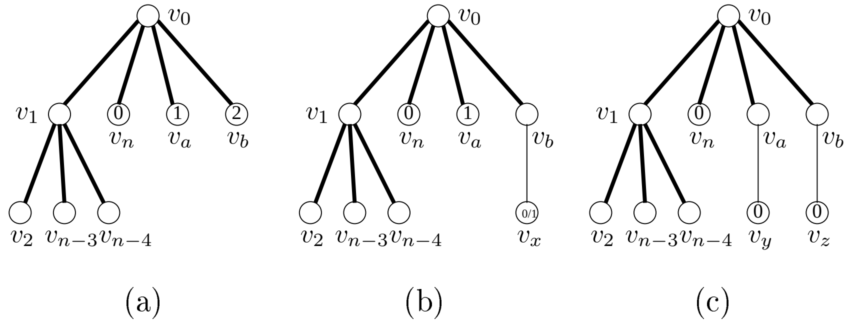

Figure 1. The vertex v_0 in the case (v) with $\Delta = 4$.

(1) If $l(v_a) = 1$ and $l(v_b) = 2$ (see Figure 1.(a)), the values 1 and 2 are each forbiden twice to v_0. Formally we have $1 \in F(v_n, v_0) \bigcap F(v_a, v_0)$ and $2 \in F(v_a, v_0) \bigcap F(v_b, v_0)$. This implies that $|F(v_0)| \leq \sigma(S, \Delta) - 2$.

(2) If $l(v_a) = 1$ and $l(v_b) > 2$ (see Figure 1.(b)), there is a vertex $v_x \in N_1(v_b)$ labelled 0 or 1. Indeed, since $F(v_n, v_b) = \{0\}$ and $F(v_a, v_b) = \{1\}$, there is a vertex v_x, with $v_x \neq v_n$ and $v_x \neq v_a$, such that $2 \in F(v_x, v_b)$. Furthermore, since v_b was the first unlabelled vertex "offered" to be labelled 2 (v_n and v_a were already labelled), we have $l(v_x) \in \{0, 1\}$ and $d(v_x, v_b) = 1$. If $v_x = v_0$, since $1 \leq \sigma(S, \Delta) - 2$, we are done, so let $v_x \neq v_0$. The vertex v_x is at distance 2 from v_0 (if there was a cycle (v_0, v_b, v_x) we would be in Case (ii)), so we have $1 \in F(v_n, v_0) \bigcap F(v_a, v_0)$ and $l(v_x) \in F(v_n, v_0) \bigcap F(v_x, v_0)$. This implies that $|F(v_0)| \leq \sigma(S, \Delta) - 2$.

(3) If $l(v_a) > 1$ (see Figure 1.(c)), the vertices v_a and v_b are not labelled 1 ($l(v_a) < l(v_b)$) there are two vertices, $v_y \in N_1(v_a)$ and $v_z \in N_1(v_b)$, labelled 0. Indeed, since $F(v_n, v_a) = \{0\}$ (resp. $F(v_n, v_b) = \{0\}$), there is a vertex $v_y \neq v_n$ (resp. $v_z \neq v_n$), such that $1 \in F(v_y, v_b)$ (resp. $1 \in F(v_z, v_b)$). Furthermore, since v_a and v_b were the first unlabelled vertices "offered" to be labelled 1 (v_n was already labelled), we have $l(v_y) = l(v_z) = 0$ and $d(v_y, v_a) = d(v_z, v_b) = 1$. If $v_0 = v_y$ or v_z, since $0 \leq \sigma(S, \Delta) - 2$, we are done, so let $v_0 \neq v_y$ and v_z. If $v_y = v_z$, there is a cycle (v_0, v_a, v_y, v_b) and we would be in Case (iv), so let $v_y \neq v_z$. The vertex v_y (resp. v_z) is at distance 2 from v_0 (if there was a cycle (v_0, v_a, v_y) or (v_0, v_b, v_z) we would be in Case (ii)), so we have $0 \in F(v_n, v_0) \bigcap F(v_y, v_0) \bigcap F(v_z, v_0)$. This implies that $|F(v_0)| \leq \sigma(S, \Delta) - 2$.

Case (v) with $\Delta = 3$: When $\Delta = 3$, we have $N_1(v_0) = \{v_1, v_n, v_{n-1}\}$, $N_1(v_1) = \{v_0, v_2, v_{n-2}\}$ and $X = \{v_0, v_1, v_{n-2}, v_{n-1}, v_n\}$. In this case we have to be more precise on the structure of G around v_0 and v_1. Let us consider that we are in none of the cases (i), (ii), (iii) and (iv). Since we are not in configuration (iv) $d(v_n, v_{n-2}) \geq 2$ and $d(v_{n-1}, v_{n-2}) \geq 2$.

First we consider that one of the vertices v_n or v_{n-1} is at distance at least 3 from v_{n-2}. Note that since v_n and v_{n-1} are both leaves in T, by permuting them in the root-to-leaves order we still have a root-to-leaves order. So, w.l.o.g.

let v_n be such that $d(v_n, v_{n-2}) \geq 3$. The order of the vertices implies that both v_n and v_{n-2} are labelled 0. Indeed, when the algorithm proposes the label 0, v_n accept it, then v_{n-1} reject it (since $d(v_n, v_{n-1}) = 2$) and then v_{n-2} accept it (since $d(v_n, v_{n-2}) \geq 3$). So we have $0 \in F(v_n, v_0) \bigcap F(v_{n-2}, v_0)$ and $0 \in F(v_n, v_1) \bigcap F(v_{n-2}, v_1)$. If $l(v_{n-1}) = 1$ we have $1 \in F(v_n, v_0) \bigcap F(v_{n-1}, v_0)$ and so, both $|F(v_0)|$ and $|F(v_1)|$ are bounded by $\Delta^2 + (p-1)\Delta - 2$. If $l(v_{n-1}) > 1$, there is a vertex $v_x \in N_1(v_{n-1})$ labelled 0. Indeed, since $F(v_n, v_{n-1}) = \{0\}$, there is a vertex $v_x \neq v_n$ such that $1 \in F(v_x, v_{n-1})$. Furthermore, since v_{n-1} was the first unlabelled vertex "offered" to be labelled 1 (v_n was already labelled), we have $l(v_x) = 0$ and $d(v_x, v_{n-1}) = 1$. The vertex v_x is at distance 2 from v_0 (if there was a cycle (v_0, v_{n-1}, v_x) we would be in Case (ii)), so we have $0 \in F(v_x, v_0) \bigcap F(v_n, v_0) \bigcap F(v_{n-2}, v_0)$. With the fact that $|F(v_0, v_1)| \leq p - 1$, we have that both $|F(v_0)|$ and $|F(v_1)|$ are bounded by $\Delta^2 + (p-1)\Delta - 2$.

Now we consider that $d(v_n, v_{n-2}) = d(v_{n-1}, v_{n-2}) = 2$. Let v_x (resp. v_y) be the vertex adjacent to v_n and v_{n-2} (resp. v_{n-1} and v_{n-2}). The vertices v_x and v_y are distinct because if there was a vertex with neighbors v_n, v_{n-1} and v_{n-2} the graph $G \backslash X$ would be disconnected, which is impossible by definition of Case (v). By construction of T, the edges $v_0 v_n$, $v_0 v_{n-1}$ and $v_1 v_{n-2}$ are the only edges in T, adjacent to v_n, v_{n-1} or v_{n-2}. So the edges $v_n v_x$, $v_{n-2} v_x$, $v_{n-1} v_y$ and $v_{n-2} v_y$ are not in T, and the vertices v_x and v_y having just one adjacent edge in T are leaves of T. This implies that the root-to-leaves order can also verify $v_{n-3} = v_x$ and $v_{n-4} = v_y$. We know that $d(v_n, v_{n-4}) > 1$ and $d(v_{n-1}, v_{n-3}) > 1$, else $G \backslash X$ would be disconnected. We consider different cases according to $d(v_n, v_{n-4})$ and $d(v_{n-1}, v_{n-3})$:

- If one of these distances is greater than 2 (see Figure 2.(a)), w.l.o.g. consider that $d(v_n, v_{n-4}) > 2$ (we could exchange v_n and v_{n-3} with v_{n-1} and v_{n-4} in the root-to-leaves ordering of T). During its first iteration (when $i = 0$) the algorithm labels v_n with 0. Since $d(v_n, v_{n-1}) = 2$, $d(v_n, v_{n-2}) = 2$ and $d(v_n, v_{n-3}) = 1$ the vertices v_{n-1}, v_{n-2} and v_{n-3} are not labelled 0. Then, since $d(v_n, v_{n-4}) > 2$, the algorithm labels v_{n-4} with 0 and we have $0 \in F(v_n, v_0) \bigcap F(v_{n-4}, v_0)$ and $0 \in F(v_n, v_1) \bigcap F(v_{n-4}, v_1)$. Since the vertices v_{n-1}, v_{n-2} and v_{n-3} are adjacent to v_n or v_{n-4}, their labels are greater than $p - 1$. We consider two case according to $l(v_{n-1})$:
  - If $l(v_{n-1}) = p$ then, since $d(v_{n-1}, v_1) = d(v_{n-1}, v_{n-2}) = 2$, we have that $l(v_1) \neq p$ and $l(v_{n-2}) > p$. If $l(v_{n-2}) = p + 1$, we have $p + 1 \in F(v_{n-1}, v_0) \bigcap F(v_{n-2}, v_0)$. If $l(v_{n-2}) > p + 1$, since v_{n-2} was the first unlabelled vertex offered to be labelled $p+1$, it implies that either the vertex v_{n-3} is labelled p, or the vertex v_1 is labelled $l(v_1) \leq p$. In the first case we would have $p \in F(v_{n-1}, v_0) \bigcap F(v_{n-3}, v_0)$. In the other case we would have either $l(v_1) \in F(v_1, v_0) \bigcap F(v_n, v_0)$ (if $l(v_1) < p$) or $p \in F(v_1, v_0) \bigcap F(v_{n-1}, v_0)$ (if $l(v_1) = p$).
  - If $l(v_{n-1}) > p$ it is because the unique vertex $v_z \in N_1(v_{n-1}) \backslash \{v_0, v_{n-4}\}$ is labelled less than p. In this case we have $l(v_z) \in F(v_z, v_0) \bigcap F(v_n, v_0)$.

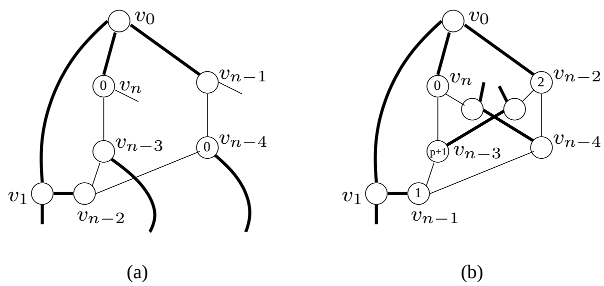

Figure 2. Case (v) with $\Delta = 3$ and $d(v_n, v_{n-2}) = d(v_{n-1}, v_{n-2}) = 2$.

Whatever the subcase, with the fact that $|F(v_0, v_1)| \leq p - 1$, we have that $|F(v_0)|$ and $|F(v_1)|$ are bounded by $\Delta^2 + (p-1)\Delta - 2$.

- If these two distances equal 2, $d(v_n, v_{n-4}) = d(v_{n-1}, v_{n-3}) = 2$, we have to slightly modify the order on the vertices by permuting v_{n-1} with v_{n-2} (see Figure 2.(b)). Since these two vertices are leaves in T, the order obtained still corresponds to a root-to-leaves ordering of T. With this order on the vertices, the algorithm labels the vertices v_n, v_{n-1}, v_{n-2} and v_{n-3}, respectively 0, 1, 2 and $p+1$. Indeed:
  - The first unlabelled vertex "proposed" to be labelled 0 is v_n and so $l(v_n) = 0$. This implies that none of the vertices v_{n-1}, v_{n-2}, v_{n-3} and none of their neighbors (except v_n) are labelled 0.
  - The first unlabelled vertex "proposed" to be labelled 1 is v_{n-1} and since none of its neighbors is labelled 0, we have $l(v_{n-1}) = 1$. This implies that none of the vertices v_{n-2}, v_{n-3} and none of their neighbors (except v_{n-1}) are labelled 1.
  - The first unlabelled vertex "proposed" to be labelled 2 is v_{n-2} and since none of its neighbors is labelled 0 or 1, we have $l(v_{n-2}) = 2$. This implies that the neighbor of v_{n-3} distinct from v_n and v_{n-1} cannot be labelled less than $p+2$.
  - The vertex v_{n-3} cannot be labelled less than $p+1$ (since $l(v_{n-1}) = 1$). Furthermore, none of its neighbors is labelled $l \in \{2, \ldots, p\}$. So, since v_{n-3} is the first unlabelled vertex "proposed" to be labelled $p+1$, we have $l(v_{n-3}) = p+1$.

  This implies that $1 \in F(v_n, v_0) \bigcap F(v_{n-1}, v_0)$, $2 \in F(v_{n-1}, v_1) \bigcap F(v_{n-2}, v_1)$ and $p+1 \in F(v_{n-2}, v_0) \bigcap F(v_{n-3}, v_0)$. With the fact that $|F(v_0, v_1)| \leq p-1$, we have that $|F(v_0)|$ and $|F(v_1)|$ are bounded by $\Delta^2 + (p-1)\Delta - 2$.

This conclude the proof of Theorem 5.

5 Proof of Lemma 6

Let G be a graph with maximum degree $\Delta \geq 3$. We prove the lemma by showing that if G has none of the configurations (i), (ii), (iii) and (iv), then

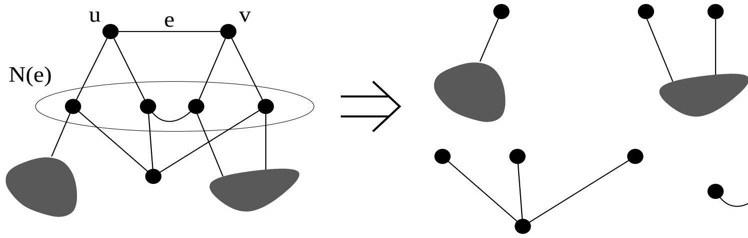

Figure 3. e-bags.

it contains configuration (v).

Definition 15 *Given an edge $e = uv \in E(G)$, the set of neighbors of e is $N(e) = (N_1(u) \bigcup N_1(v)) \setminus \{u, v\}$. Given $e = uv \in E(G)$ an e-bag B is a maximal subgraph of $G \setminus \{u, v\}$ such that, for any pair of vertices x and $y \in V(B)$, there is a path from x to y without internal vertices in $N(e)$ (see Figure 3).*

Note that two different e-bags can only share vertices of $N(e)$, else their union would be a bigger e-bag, contradicting their maximality. Given an e-bag B, let $L(B) = V(B) \bigcap N(e)$ be the set of vertices linking B to the rest of the graph. The others vertices of B form the set of inner vertices of B, $I(B) = V(B) \setminus L(B)$. Given a set $Y \subseteq N(uv) \cup \{u, v\}$, the graph $G \setminus Y$ is disconnected if there is an e-bag B with $L(B) \subseteq Y$ and $|I(B)| > 0$.

Remark 16 *An edge $e \in E(G)$ corresponds to the edge uv of configuration (v) iff there is a vertex $w \in N(e)$ contained by all the e-bags.*

We can found this edge uv of configuration (v) by doing the following process:

(1) Consider two non-incident edges e and $f \in E(G)$.
(2) Verify if e corresponds to the edge uv of the configuration (v).
(3) If not, let B_0 be the e-bag containing f. Since e does not correspond to the edge uv, there are e-bags B_i, with $i > 0$, such that $L(B_0) \setminus L(B_i) \neq \emptyset$ (else with $e = uv$ and any $w \in L(B_0)$ we would have configuration (v)). Let $\mathcal{B}$ be the set of all these e-bags. Let B_1 be an e-bag of $\mathcal{B}$ that minimizes $|L(B_1)|$ and (if there are various e-bags B_i minimizing $|L(B_i)|$) then maximizes $|I(B_1)|$. Finally since $|I(B_1)| \geq 2$ (c.f. Lemma 17), let e be an edge of B_1 with its two ends in $I(B_1)$ and go to step (2).

We can prove that this process terminates because each time we change e, the size of $I(B_0)$ increases. Indeed, since none of the vertices in $L(B_0) \setminus L(B_1)$ has a neighbor in B_1, all the vertices of $B_0 \setminus L(B_1)$ (i.e. $I(B_0) \cup (L(B_0) \setminus L(B_1))$) are in $I(B_0)$ in the next step. So if the following lemma holds, Lemma 6 holds.

Lemma 17 *If a graph G does not contain configurations (i), (ii), (iii) and (iv), and if a given edge $e = ab \in E(G)$ does not correspond to the edge uv of configuration (v) then the e-bag B_1 (defined before) is such that $|I(B_1)| \geq 2$.*

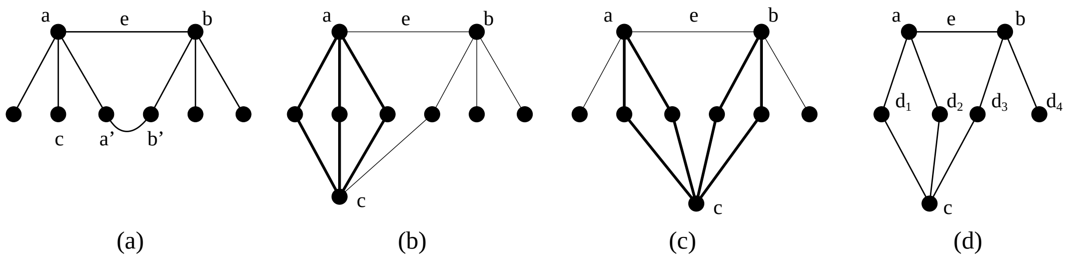

Figure 4. Cases with $|I(B_1)| = 0$ and $|I(B_1)| = 1$.

PROOF. If $|I(B_1)| = 0$, let $e' = a'b'$ be its unique edge and note that a' and b' belong to $N(e)$. This implies that $|L(B_1)| = 2$ and that any e-bag $B_i \in \mathcal{B}$ has either $|L(B_i)| \geq 3$ or $|L(B_i)| = 2$ and $|I(B_i)| = 0$. If a' and b' are both neighbors of a (resp. b) there is a cycle of length three and we are in configuration (ii), so let $a' \in N_1(a)$ and $b' \in N_1(b)$ (see Figure 4.(a)). Then we consider any vertex $c \in L(B_0) \setminus L(B_1)$ (so $c \neq a'$ and b'). W.l.o.g. let $c \in N_1(a)$. Since $\{a', b, c\} \subseteq N_1(a)$ and (a, b, b', a') is cycle, if $G \setminus \{c, a'\}$ is connected, we are in configuration (iv). So let $G \setminus \{c, a'\}$ be disconnected. This implies that there is a vertex $d \in V(G) \setminus \{c, a'\}$ such that all the paths from d to a pass through c or a'. The e-bag B_i containing d is such that $L(B_i) \subseteq \{c, a'\}$ and $d \in I(B_i)$. Since $b' \in L(B_0) \setminus L(B_i)$, we have $B_i \in \mathcal{B}$. With the fact that $|L(B_i)| \leq 2$ and $|I(B_i)| \geq 1$, this contradicts the definition of B_0 and we have $|I(B_1)| \geq 1$.

If $|I(B_1)| = 1$, let c be the unique vertex in $I(B_1)$. Since $\deg(c) = |L(B_1)| = \Delta \geq 3$, any e-bag $B_i \in \mathcal{B}$ has either $|L(B_i)| > \Delta$ or $|L(B_i)| = \Delta$ and $|I(B_i)| = 1$ (when $|I(B_i)| = 0$ we have $|L(B_i)| = 2 < \Delta$). If $\Delta \geq 4$ there are at least two cycles of length four passing through c, so we are in configuration (iii) (see Figure 4.(b) and Figure 4.(c)). For $\Delta = 3$ (see Figure 4.(d)), let $N_1(a) = \{b, d_1, d_2\}$ and $N_1(b) = \{a, d_3, d_4\}$. W.l.o.g. let $N_1(c) = L(B_1) = \{d_1, d_2, d_3\}$. Since $B_1 \in \mathcal{B}$, we have $L(B_0) \setminus L(B_1) \neq \emptyset$ and so $d_4 \in L(B_0)$. Since (a, d_1, c, d_2) is a cycle, if the graph $G \setminus \{d_2, d_3\}$ is connected we are in configuration (iv), so let $G \setminus \{d_2, d_3\}$ be disconnected. This implies, that there is a vertex z such that all the paths from z to a pass through d_2 or d_3. The e-bag B_i containing z is such that $L(B_i) \subseteq \{d_2, d_3\}$. Since $d_4 \in L(B_0) \setminus L(B_i)$, we have $B_i \in \mathcal{B}$. With the fact that $|L(B_i)| \leq 2$, this contradicts the minimality of $|L(B_1)| = \Delta$ (since $\Delta > 2$). So we have $|I(B_1)| \geq 2$ and this completes the proof of Lemma 17.

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