# Edge partition of planar graphs into two outerplanar graphs [1]

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**Abstract**

An *outerplanar graph* is a planar graph that can be embedded in the plane without crossing edges, in such a way that all the vertices are on the outer-boundary. We prove that every planar graph $G = (V, E)$ has a bipartition of its edge set $E = A \cup B$ such that the graphs induced by these subsets, $G[A]$ and $G[B]$, are outerplanar. This proves a conjecture of Chartrand, Geller, and Hedetniemi (*J. Combin. Theory Ser. B*, 10 (1971) 12–41).

*Key words:* planar graphs, edge-partition, outerplanar graphs, hamiltonian cycle

## 1 Introduction

Much work has been done in partitioning the edge sets of graphs such that each subset induces a subgraph of a certain form. See for example the concepts of chromatic index, arboricity, thickness, or track number. In this vein, Chartrand, Geller, and Hedetniemi ([3] and Problem 6.3 in [12]) made the famous $[m, n]$-conjecture. They defined the graphs with property $P_m$ as the graphs containing no subdivision of $K_{m+1}$ or $K_{\lceil m/2 \rceil+1,\lfloor m/2 \rfloor+1}$. Observe that the graphs with property $P_4$ (resp. $P_3$) are the planar graphs (resp. outerplanar graphs). The $[m, n]$-conjecture was that any graph with property $P_m$ has an edge partition into $m-n+1$ graphs with property $P_n$, for $m \geq n \geq 2$. This conjecture is false in general. In [10], it is disproved for any $n$ and $m > cn^2$, for some constant $c$. In this paper (that is the extended version of [7]) we prove a special case of the conjecture, the case where $n = 3$ and $m = 4$. In other

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words, we prove that every planar graph has an edge bipartition into outerplanar graphs. There have been various results toward this case of the conjecture. Colbourn and El-Mallah [6] gave a first partial result showing that every planar graph has an edge bipartition into partial 3-trees. Then Kedlaya [13] and Ding et al. [5] proved that a bipartition into partial 2-trees exists. Another result [5] is that every planar graph has an edge partition into two outerplanar graphs and a vee-forest (*i.e.* a forest in which each connected component contains at most three vertices). A proof of this case of the $[m, n]$-conjecture was already claimed in [11] but finally appeared to be incorrect.

A simple case of planar graphs that can be divided into two outerplanar graphs are the hamiltonian planar graphs (*i.e.* containing a cycle going through every vertex). In this case, the first outerplanar graph is constructed with the edges of a hamiltonian cycle together with the edges in the interior of this cycle, and the second one with the edges of this hamiltonian cycle together with the edges in the exterior of this cycle. There is a lot of flexibility in this construction since the edges of the hamiltonian cycle are in both subgraphs. We say that a bipartition of an embedded planar graph (*i.e.* a plane graph) is *hamiltonian* if there is a hamiltonian cycle $C$ such that all the edges strictly inside $C$ are in the same subset and all the edges strictly outside $C$ are in the other subset. Whitney [23] proved that 4-connected triangulations are hamiltonian and Tutte [21] generalized this result to 4-connected planar graphs. So we know that the conjecture holds for 4-connected planar graphs. Note that with a hamiltonian partition, the graph inside the hamiltonian cycle is outerplanarly embedded. This means that given an embedding of the planar graph, the embedding it induces for this subgraph is such that all the vertices are on its outer-boundary. An interesting result of Kedlaya [13] is that there exists a planar graphs $G$ such that whatever its embedding, and whatever the bipartition of $G$ into outerplanar graphs we consider, none of the outerplanar subgraphs are outerplanarly embedded. This implies for example that there are planar graphs with no edge partition into an outerplanar graph and a forest (forests being always outerplanarly embedded).

A *triangulation* is a plane graph in which all the faces are triangles. Since every planar graph is a subgraph of a triangulation and since every subgraph of an outerplanar graph is outerplanar, we restrict our work to triangulations. A graph $G$ is *chordal* if every cycle of length $l \geq 4$ has a *chord*, which is an edge linking two non-consecutive vertices of the cycle. Let $S$ be the graph with a cycle $(x_1, y_1, x_2, y_2, x_3, y_3)$ and chords $y_1y_2$, $y_1y_3$ and $y_2y_3$ (see Figure 1). A graph is *$S$-free* if it does not contain any subgraph isomorphic to $S$. The main result of the paper is the following theorem.

**Theorem 1** *Every triangulation $T$ has an edge bipartition into chordal outerplanar graphs (e.g. COGs). Furthermore, if $T$ is 4-connected there is such a bipartition that is hamiltonian and for which the two COGs are $S$-free.*

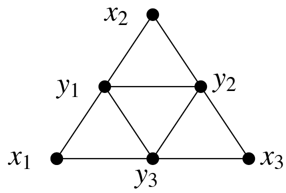


Fig. 1. The graph $S$.

In Section 2 we give another proof of the fact that 4-connected triangulations are hamiltonian. The technique used is inspired on the original proof of Whitney [23] and may yield to the same hamiltonian cycle. This section is necessary for considering the special case of 4-connected triangulations in Theorem 1. In Section 3 we give some properties of outerplanar graphs. In Section 4 we study edge partitions of 4-connected triangulations. This study allows us to prove Theorem 1 in Section 5. Then we finally discuss some perspectives.

## 2 Hamiltonian cycle

A *near-triangulation* is a plane graph in which all the inner faces are triangular (but not necessarily the outer-face). In a near-triangulation $T$, a *separating 3-cycle* $C$ is a cycle of length three with at least one vertex inside $C$ and one vertex outside $C$. A *W-triangulation* is a 2-connected near-triangulation without separating 3-cycles. Note that the 4-connected triangulations being triangulations without any separating 3-cycle, they are W-triangulations. The W-triangulations being 2-connected, they have no articulation vertex (a vertex whose removal increases the number of connected components). Hence, the outer-boundary of a W-triangulation is a cycle. A chord of this cycle is also called a *chord* of $T$. The following lemma tells us in which case the subgraph of a W-triangulation is also a W-triangulation.

**Lemma 2** *Let $T$ be a W-triangulation and $C$ a cycle of $T$. The subgraph of $T$ inside $C$ (i.e. the graph induced by the edges on $C$ and the edges inside $C$) is a W-triangulation.*

**PROOF.** Let the near-triangulation $T'$ be the subgraph of $T$ delimited by $C$. By definition of a W-triangulation, $T$ has no separating 3-cycle, hence $T'$ has no separating 3-cycle. So we just have to show that $T'$ is 2-connected, this is that it has no articulation vertex.
For any vertex $v$ of $T'$, since $T'$ is a near-triangulation, at most one of the faces incident to $v$ is not triangular, the outer-face. Furthermore, the outer-face being delimited by a cycle, the vertex $v$ appears at most once on the outer-boundary. So the neighborhood of $v$ induces a connected graph and thus $T' \backslash v$ is connected. Hence $T'$ has no articulation vertex and it is a W-triangulation.

**Definition 3** *A W-triangulation $T$ is* 3-bounded *if its outer-boundary is divided into three paths, $(a_1, \dots, a_p)$, $(b_1, \dots, b_q)$, and $(c_1, \dots, c_r)$ verifying the following conditions:*

- *The ends of the paths are such that $a_1 = c_r$, $b_1 = a_p$ and $c_1 = b_q$.*
- *The paths are non-trivial, this is $p > 1$, $q > 1$ and $r > 1$.*
- *The W-triangulation $T$ has no chord $a_ia_j$ (resp. $b_ib_j$ or $c_ic_j$) with $1 \le i < p$ and $i+1 < j \le p$ (resp. $1 \le i < q$ and $i+1 < j \le q$, or $1 \le i < r$ and $i+1 < j \le r$).*

*Given such a 3-bounded W-triangulation $T$, $(a_1, \dots, a_p)$-$(b_1, \dots, b_q)$-$(c_1, \dots, c_r)$ is a* 3-boundary *of $T$ (see Figure 2).*

In a 3-boundary the order and the orientation of the paths matters. Indeed, $(a_1, \dots, a_p)$-$(b_1, \dots, b_q)$-$(c_1, \dots, c_r)$, $(b_1, \dots, b_q)$-$(c_1, \dots, c_r)$-$(a_1, \dots, a_p)$, and $(a_p, \dots, a_1)$-$(c_r, \dots, c_1)$-$(b_q, \dots, b_1)$ are distinct 3-boundaries. A *hamiltonian path* $P$ of a graph $G$ is a path of $G$ with vertex set $V(P) = V(G)$.

**Property 4** *For any 3-bounded W-triangulation $T$ and any 3-boundary $(a_1, \dots, a_p)$-$(b_1, \dots, b_q)$-$(c_1, \dots, c_r)$ of $T$, there is a hamiltonian path $P$ in $T$ from $c_1$ to $b_1$ passing through the edge $a_1a_2$ (see Figure 2).*

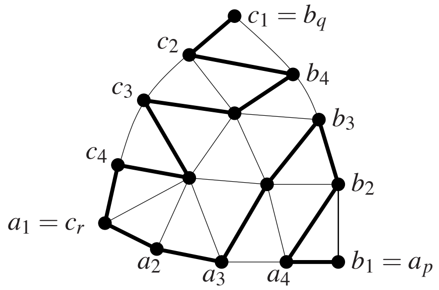


Fig. 2. The 3-boundary of $T$ and the path $P$ of Property 4.

Note that $P$ successively goes through $c_1$, $a_1$, $a_2$, and then $b_1$. Property 4 applies to 4-connected triangulations. Indeed, a 4-connected triangulation $T$ with outer-boundary $abc$ is a W-triangulation 3-bounded by $(a, b)$-$(b, c)$-$(c, a)$. So if this property holds for $T$, there is a hamiltonian path $P$ from $c$ to $b$. Adding the edge $bc$ to this path $P$ we obtain a hamiltonian cycle.

We now define the notion of *adjacent path* of a W-triangulation with respect to a 3-boundary. Let $T \ne K_3$ be a W-triangulation 3-bounded by $(a_1, \dots, a_p)$-$(b_1, \dots, b_q)$-$(c_1, \dots, c_r)$, without chord $a_ib_j$, with $1 \le i \le p$ and $1 \le j \le q$, and without chord $a_ic_j$, with $1 \le i \le p$ and $1 \le j \le r$. The W-triangulation $T$ having at least 4 vertices and having no separating 3-cycle, the vertices $b_1$ and $b_2$ have exactly one common neighbor in $V(T) \setminus \{a_1\}$, denoted $d_1$. Let $V_a \subsetneq V(T)$ be the set of vertices of $T$ adjacent to a vertex $a_i$ with $i > 1$, excluding the vertices $a_i$ with $i > 1$ and the vertex $b_2$. The graph $T$ being a W-triangulation, the neighbors of $a_i$ in $V_a$, with $1 < i \leq p$, induce a connected graph. Furthermore, the vertices $a_i$ and $a_{i+1}$ have a common neighbor in $V_a$, hence the set $V_a$ induces a connected graph. This set contains the vertices $a_1$ and $d_1$, which respectively are the neighbors of $a_2$ and $a_p$. Denote $(d_1, d_2, \ldots, d_s, a_1)$ the shortest path linking $d_1$ and $a_1$ in the graph $T[V_a]$ (see Figure 3). This path is the *adjacent path* of $T$ for the 3-boundary $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$ and it verifies the following 3 points:

- There is no edge $d_i d_j$, with $1 \leq i < s$ and $i + 1 < j \leq s$, and no edge $a_1 d_i$, with $1 \leq i < s$. Indeed, if such edge existed, the path $(d_1, d_2, \ldots, d_s, a_1)$ would not be the shortest path linking $d_1$ and $a_1$ in $T[V_a]$.
- The W-triangulation $T$ having no chord $a_i b_j$ or $a_i c_j$, the set $V_a$ does not contain any vertex $b_i$ or $c_j$, except $c_r = a_1$. Hence the vertices $d_i$, with $1 \leq i \leq s$, are not vertices $b_j$ or $c_k$, with $1 \leq j \leq q$ and $1 \leq k \leq r$.
- Since $d_1 \neq a_1$ this path has length at least 1.

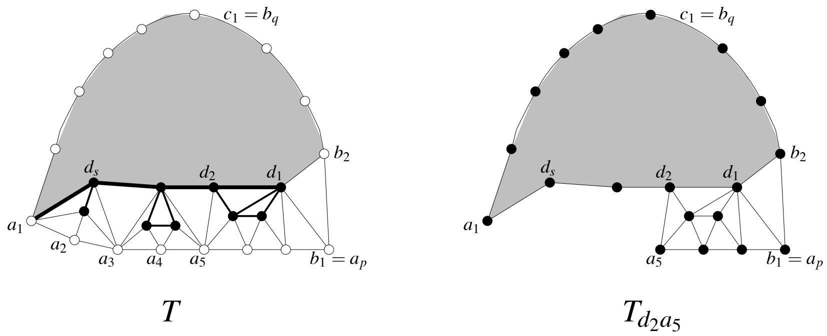


Fig. 3. The adjacent path of $T$ and the graph $T_{d_2 a_5}$.

Given a W-triangulation $T$, with 3-boundary $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$, without chord $a_i b_j$ or $a_i c_j$, consider the adjacent path $(d_1, d_2, \ldots, d_s, a_1)$. For any edge $d_x a_y \in E(T)$, with $1 \leq x \leq s$ and $1 < y \leq p$, we define $T_{d_x a_y}$ as the graph contained inside the cycle $C = (d_s, \ldots, d_x, a_y, \ldots, a_p, b_2, \ldots, b_q, c_2, \ldots, c_r)$ in $T$ (see Figure 3). Since the vertices $d_i$ are distinct from the vertices $a_j$, $b_j$ or $c_j$, $C$ is a cycle and $T_{d_x a_y}$ is a W-triangulation (*c.f.* Lemma 2).

The following property is needed to prove Property 4.

**Property 5** *Let $T$ be a W-triangulation 3-bounded by $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$, without chord $a_i b_j$ or $a_i c_j$ and with adjacent path $(d_1, d_2, \ldots, d_s, a_1)$. For any edge $d_x a_y \in E(T)$, with $1 \leq x \leq s$ and $1 < y \leq p$, there are two disjoint paths $P$ and $Q$ in $T_{d_x a_y}$, one from $c_1$ to $a_1$ and one from $a_y$ to $b_1$, such that each vertex of $T_{d_x a_y}$ is contained either in $P$ or in $Q$ (see Figure 4).*

We prove these two properties by doing a crossed induction.

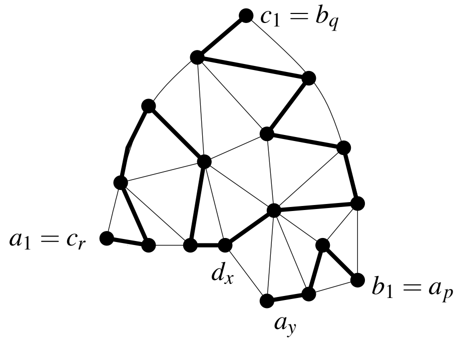


Fig. 4. Property 5.

**PROOF of Property 4 and Property 5.** We prove, by induction on $m \geq 3$, that the following two statements hold:

- Property 4 holds if $T$ has at most $m$ edges.
- Property 5 holds if $T_{d_x a_y}$ has at most $m$ edges.

The initial case, $m = 3$, is easy to prove since there is only one W-triangulation having at most 3 edges, $K_3$. For Property 4 we have to consider all the possible 3-boundaries of $K_3$. Since they are all equivalent, we denote $a_1$, $b_1$, and $c_1$ the vertices of $K_3$ and we consider the 3-boundary $(a_1, b_1)$-$(b_1, c_1)$-$(c_1, a_1)$. In this case, the path $P = (c_1, a_1, b_1)$ clearly verifies Property 4. For Property 5, since a W-triangulation $T_{d_x a_y}$ has at least 4 vertices, $a_1$, $b_1$, $c_1$, and $d_1$, we have $T_{d_x a_y} \neq K_3$ and there is no W-triangulation $T_{d_x a_y}$ with at most 3 edges. So by vacuity, Property 5 holds for the W-triangulation $T_{d_x a_y}$ with at most 3 edges.

The induction step applies to both Property 4 and Property 5. This means that we prove Property 4 (resp. Property 5) for the W-triangulations $T$ (resp. $T_{d_x a_y}$) with $m$ edges using both Property 4 and Property 5 on W-triangulations with less than $m$ edges. We first prove the induction for Property 4.

**Case 1: Proof of Property 4 for a W-triangulation $T$ with $m$ edges.** Let $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$ be the 3-boundary of $T$. We consider various cases according to the existence of a chord $a_i b_j$ or $a_i c_j$ in $T$. We successively consider the case where there is a chord $a_1 b_j$, with $1 < j < q$, the case where there is a chord $a_i b_j$, with $1 < i < p$ and $1 < j \leq q$, and the case where there is a chord $a_i c_j$, with $1 < i \leq p$ and $1 < j < r$. We then conclude with the case where there is no chord $a_i b_j$, with $1 \leq i \leq p$ and $1 \leq j \leq q$ (by definition of a 3-boundary there is no chord $a_1 b_q$, $a_i b_1$ or $a_p b_j$), and no chord $a_i c_j$, with $1 \leq i \leq p$ and $1 \leq j \leq r$ (by definition of a 3-boundary there is no chord $a_p c_1$, $a_i c_r$ or $a_1 c_j$).

**Case 1.1: There is a chord $a_1 b_i$, with $1 < i < q$ (see Figure 5).** Let $T_1$ (resp. $T_2$) be the W-triangulation (*c.f.* Lemma 2), subgraph of $T$, in-

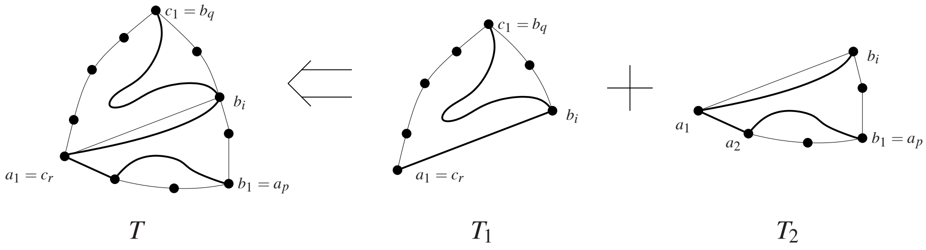


Fig. 5. Case 1.1: chord $a_1b_i$.

side the cycle $(b_i, \ldots, b_q, c_2, \ldots, c_r)$ (resp. $(a_1, \ldots, a_p, b_2, \ldots, b_i)$). It is clear that $V(T) = V(T_1) \cup V(T_2)$ and $V(T_1) \cap V(T_2) = \{a_1, b_i\}$. Since there is no chord $a_xa_y$, $b_xb_y$ or $c_xc_y$ for any $x$ and $y$, $(b_ic_r)$-$(c_r, \ldots, c_1)$-$(b_q, \ldots, b_i)$ (resp. $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_i)$-$(b_ia_1)$) is a 3-boundary of $T_1$ (resp. $T_2$). Since $a_1a_2 \notin E(T_1)$ (resp. $c_1c_2 \notin E(T_2)$), the W-triangulation $T_1$ (resp. $T_2$) has less edges than $T$, so Property 4 holds for $T_1$ (resp. $T_2$) with the mentioned 3-boundary. Let $P_1$ (resp. $P_2$) be a hamiltonian path of $T_1$ (resp. $T_2$) going from $c_1$ to $a_1$ (resp. from $b_i$ to $b_1$) and passing through the edge $b_ia_1$ (resp. $a_1a_2$).

Since $a_1$ is an end of $P_1$, this path clearly ends with the edge $b_ia_1$. Let $P_1' = P_1 \setminus \{a_1\}$. This path goes from $c_1$ to $b_i$ and passes through all the vertices in $V(T_1)$ except $a_1$. Now let $P = P_1' \cup P_2$, this is the graph with vertex set $V(P) = V(P_1') \cup V(P_2)$ and with edge set $E(P) = E(P_1') \cup E(P_2)$. Since the unique common vertex of $P_1'$ and $P_2$, $b_i$, is an end of both $P_1'$ and $P_2$, the graph $P$ is a path from $c_1$ to $b_1$. Furthermore, this path passes through all the vertices in $V(T)$ since $V(P_1') \cup V(P_2) = (V(T_1) \setminus \{a_1\}) \cup V(T_2) = V(T)$. Finally since $a_1a_2 \in E(P_2) \subset E(P)$ the path $P$ fulfills Property 4.

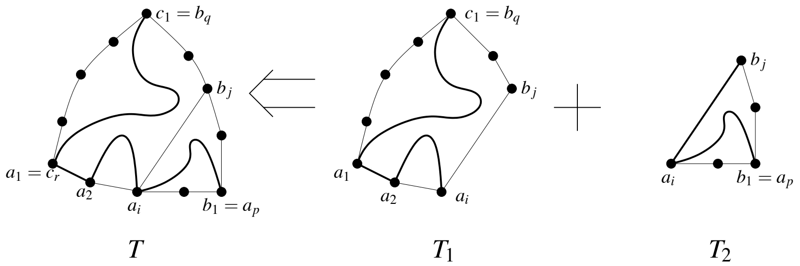


Fig. 6. Case 1.2: chord $a_ib_j$.

**Case 1.2: There is a chord $a_ib_j$, with $1 < i < p$ and $1 < j \leq q$ (see Figure 6).** If there are several chords $a_ib_j$ consider one that maximizes $j$ (*i.e.* such that there is no edge $a_ib_k$ with $j < k \leq q$). Let $T_1$ (resp. $T_2$) be the W-triangulation (*c.f.* Lemma 2), subgraph of $T$, inside the cycle $(a_2, \ldots, a_i, b_j, \ldots, b_q, c_2, \ldots, c_r)$ (resp. $(a_i, \ldots, a_p, b_2, \ldots, b_j)$). It is clear that $V(T) = V(T_1) \cup V(T_2)$ and $V(T_1) \cap V(T_2) = \{a_i, b_j\}$. Since there is no chord $a_xa_y$, $b_xb_y$, $c_xc_y$ or $a_ib_k$ with $k > j$, $(a_1, \ldots, a_i)$-$(a_i, b_j, \ldots, b_q)$-$(c_1, \ldots, c_r)$ (resp. $(a_i, b_j)$-$(b_j, \ldots, b_1)$-$(a_p, \ldots, a_i)$) is a 3-boundary of $T_1$ (resp. $T_2$). Since $b_1b_2 \notin E(T_1)$ (resp. $a_1a_2 \notin E(T_2)$), the W-triangulation $T_1$ (resp. $T_2$) has less edges than $T$ so Property 4 holds for $T_1$ (resp. $T_2$) with the mentioned 3-boundary. Let $P_1$ (resp. $P_2$) be a hamiltonian path of $T_1$ (resp. $T_2$) going from $c_1$ to $a_i$ (resp. from $b_1$ to $b_j$) and passing through the edge $a_1a_2$ (resp. $a_ib_j$).

Since $b_j$ is an end of $P_2$, this path clearly ends with the edge $a_ib_j$. Let $P_2' = P_2 \setminus \{b_j\}$. This path goes from $b_1$ to $a_i$ and passes through all the vertices in $V(T_2)$ except $b_j$. Now let $P = P_1 \cup P_2'$. Since the unique common vertex of $P_1$ and $P_2'$, $a_i$, is an end of both $P_1$ and $P_2'$, the graph $P$ is a path from $c_1$ to $b_1$. Furthermore, this path passes through all the vertices in $V(T)$ since $V(P_1) \cup V(P_2') = V(T_1) \cup (V(T_2) \setminus \{b_j\}) = V(T)$. Finally since $a_1a_2 \in E(P_1) \subset E(P)$ the path $P$ fulfills Property 4.

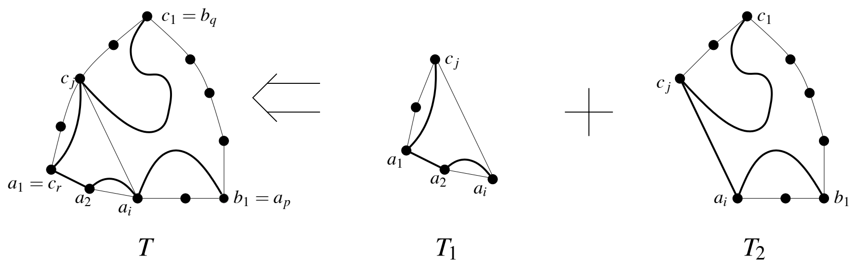


Fig. 7. Case 1.3: chord $a_ic_j$.

**Case 1.3: There is a chord $a_ic_j$, with $1 < i \leq p$ and $1 < j < r$ (see Figure 7).** If there are several chords $a_ic_j$ consider one that maximizes $i$ (*i.e.* such that there is no edge $a_kc_j$ with $i < k \leq p$). Let $T_1$ (resp. $T_2$) be the W-triangulation (*c.f.* Lemma 2), subgraph of $T$, inside the cycle $(a_2, \ldots, a_i, c_j, \ldots, c_r)$ (resp. $(a_i, \ldots, a_p, b_2, \ldots, b_q, c_2, \ldots, c_j)$). It is clear that $V(T) = V(T_1) \cup V(T_2)$ and $V(T_1) \cap V(T_2) = \{a_i, c_j\}$. Since there is no chord $a_xa_y$, $b_xb_y$, $c_xc_y$ or $a_kc_j$ with $k > i$, $(a_1, \ldots, a_i)$-$(a_i, c_j)$-$(c_j, \ldots, c_r)$ (resp. $(c_j, a_i, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_j)$) is a 3-boundary of $T_1$ (resp. $T_2$). Since $b_1b_2 \notin E(T_1)$ (resp. $a_1a_2 \notin E(T_2)$), the W-triangulation $T_1$ (resp. $T_2$) has less edges than $T$, so Property 4 holds for $T_1$ (resp. $T_2$) with the mentioned 3-boundary. Let $P_1$ (resp. $P_2$) be a hamiltonian path of $T_1$ (resp. $T_2$) going from $c_j$ to $a_i$ (resp. from $c_1$ to $b_1$) and passing through the edge $a_1a_2$ (resp. $c_ja_i$).

Let $P_2' = P_2 \setminus \{c_ja_i\}$. This graph is a union of two vertex disjoint paths, one from $c_1$ to $c_j$ and one from $a_i$ to $b_1$. Now let $P = P_1 \cup P_2'$. Since the common vertices of $P_1$ and $P_2'$, $a_i$ and $c_j$, are ends of $P_1$, and are ends in distinct components of $P_2'$, the graph $P$ is a path from $c_1$ to $b_1$. Furthermore, this path passes through all the vertices in $V(T)$ since $V(P_1) \cup V(P_2') = V(T_1) \cup V(T_2) = V(T)$. Finally since $a_1a_2 \in E(P_1) \subset E(P)$ the path $P$ fulfills Property 4.

**Case 1.4: There is no chord $a_ib_j$ or $a_ic_j$.** In this case we consider the adjacent path $(d_1, \ldots, d_s, a_1)$ (see Figure 3) of $T$ with respect to the 3-boundary $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$. Let $d_sa_y \in E(T)$ be the edge with $1 < y \leq p$ such that $y$ is minimum. There is such an edge since the vertex $d_s$ is, by definition, adjacent to a vertex $a_y$ with $y > 1$. The W-triangulation $T_{d_s a_y}$ has less edges than $T$ ($a_1a_2 \notin E(T_{d_s a_y})$), so Property 5 holds for $T_{d_s a_y}$. Let $P'$ and $Q'$ be the paths of $T_{d_s a_y}$ going respectively from $c_1$ to $a_1$ and from $a_y$ to $b_1$. We distinguish two cases according to the index $y$ of $a_y$, the case $y = 2$ and the case $y > 2$.

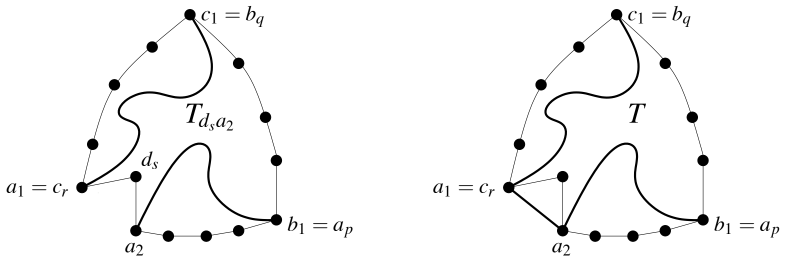


Fig. 8. Case 1.4.1.

**Case 1.4.1: $y = 2$ (see Figure 8).** The graph $T$ being a W-triangulation, the cycle $(a_1, a_2, d_s)$ bounds a face of $T$, so $V(T) = V(T_{d_s a_2})$. Let $P = P' \cup \{a_1a_2\} \cup Q'$. Since $a_1$ and $a_2$ are ends of respectively $P'$ and $Q'$ the graph $P$ is a path from $c_1$ to $b_1$. Finally since $V(P) = V(T)$ and $a_1a_2 \in E(P)$ the path $P$ fulfills Property 4.

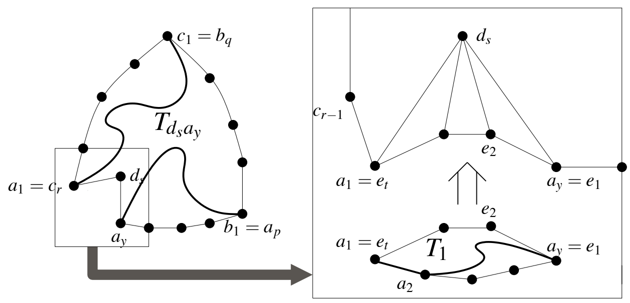


Fig. 9. Case 1.4.2.

**Case 1.4.2: $y > 2$ (see Figure 9).** Let $e_1, e_2, \ldots, e_t$ be the neighbors of $d_s$ in $T$ and inside the cycle $(d_s, a_1, a_2, \ldots, a_y)$, going from $a_y$ to $a_1$ included. This implies that $e_1 = a_y$ and $e_t = a_1$. Furthermore since $T$ has no chord $a_1a_y$, we have $t \geq 3$. The index $y$ being minimum we have $e_i \neq a_j$ for all $i$ and $j$ such that $1 < i < t$ and $1 < j < y$. Consider now the W-triangulation $T_1$ (*c.f.* Lemma 2), subgraph of $T$ inside the cycle $(a_2, \ldots, a_y, e_2, \ldots, e_t)$. It is clear that $V(T) = V(T_{d_s a_y}) \cup V(T_1)$ and $V(T_{d_s a_y}) \cap V(T_1) = \{a_1, a_y\}$. The W-triangulation $T$ having no separating 3-cycle $(d_s, e_i, e_j)$ there is no chord $e_ie_j$ in $T_1$. Furthermore since $y > 2$, $(a_2, a_1)$-$(e_t, \ldots, e_1)$-$(a_y, \ldots, a_2)$ is a 3-boundary of $T_1$. Since $a_1d_s \notin E(T_1)$, the W-triangulation $T_1$ has less edges than $T$, so Property 4 holds for $T_1$ with the mentioned 3-boundary. Let $P_1$ be a hamiltonian path of $T_1$ going from $a_y$ to $a_1$ and passing through the edge $a_2a_1$.

Let $P = P' \cup P_1 \cup Q'$. Since $a_1$ and $a_y$ are ends of respectively $P'$ and $Q'$, and since these two vertices are ends of $P_1$, the graph $P$ is a path from $c_1$ to $b_1$. Finally since $V(P) = V(P') \cup V(Q') \cup V(P_1) = V(T_{d_s a_y}) \cup V(T_1) = V(T)$, and since $a_1 a_2 \in E(P_1) \subset E(P)$, the path $P$ fulfills Property 4.

This concludes the proof of Case 1.

**Case 2: Proof of Property 5 for a W-triangulation $T_{d_x a_y}$ with $m$ edges.** The W-triangulation $T_{d_x a_y}$ is a subgraph of a W-triangulation $T$. This W-triangulation $T$ is 3-bounded by $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$. Furthermore, $T$ has no chord $a_i b_j$ or $a_i c_j$ and its adjacent path is $(d_1, \ldots, d_s, a_1)$, with $s \geq 1$. We distinguish the case where $d_x a_y = d_1 a_p$ and the case where $d_x a_y \neq d_1 a_p$.

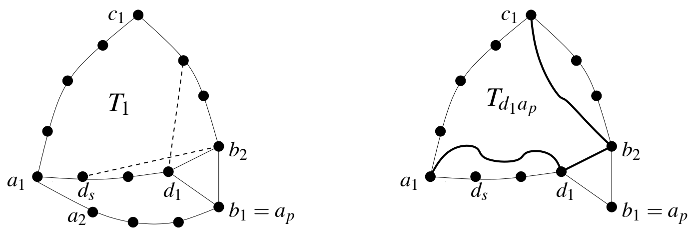


Fig. 10. Case 2.1.

**Case 2.1: $d_x a_y = d_1 a_p$ (see Figure 10).** Let $T_1$ be the W-triangulation (*c.f.* Lemma 2), subgraph of $T$ inside the cycle $(d_s, \ldots, d_1, b_2, \ldots, b_q, c_2, \ldots, c_r)$. The graph $T_{d_1 a_p}$ being a W-triangulation, the cycle $(d_1, a_p, b_2)$ bounds a face of $T_{d_x a_y}$ and so $V(T_{d_1 a_p}) = V(T_1) \cup \{a_p\}$. The W-triangulation $T_1$ has no chord $b_i b_j$, $c_i c_j$, $d_i d_j$ or $a_1 d_j$. We consider two cases according to the existence of an edge $d_1 b_i$ with $2 < i \leq q$.

- If $T_1$ has no chord $d_1 b_i$, with $2 < i \leq q$, then $(d_1, b_2, \ldots, b_q)$-$(c_1, \ldots, c_r)$-$(a_1, d_s, \ldots, d_1)$ is a 3-boundary of $T_1$.
- If $T_1$ has a chord $d_1 b_i$, with $2 < i \leq q$ (so $q > 2$), then $T_1$ has no chord $b_2 a_1$ or $b_2 d_j$, with $1 < j \leq s$. Indeed, this would contradict the planarity of $T$ (see Figure 10). In this case, $(b_2, d_1, \ldots, d_s, a_1)$-$(c_r, \ldots, c_1)$-$(b_q, \ldots, b_2)$ is a 3-boundary of $T_1$.

Since $a_p b_2 \notin E(T_1)$, the W-triangulation $T_1$ has less edges than $T_{d_1 a_p}$, so Property 4 holds for $T_1$ with one of the mentioned 3-boundaries. With both of these 3-boundaries, Property 4 gives a hamiltonian path $P_1$ of $T_1$, from $c_1$ to $a_1$ and passing through the edge $d_1 b_2$.

Let $Q$ be the trivial path of length 0 such that $V(Q) = \{a_p\}$. Since $V(P_1) \cup V(Q) = V(T_1) \cup \{a_p\} = V(T_{d_1 a_p})$ and since $V(P_1) \cap V(Q) = V(T_1) \cap \{a_p\} = \emptyset$, the paths $P_1$ and $Q$ fulfill Property 5.

**Case 2.2:** $d_x a_y \neq d_1 a_p$**.** In this case we consider an edge $d_z a_w \in E(T_{d_x a_y})$ such that $d_z a_w \neq d_x a_y$. Among all the possible edges $d_z a_w$ we choose the one that firstly maximizes $z$ and secondly minimizes $w$. Such an edge necessarily exists and actually one can see that $d_z = d_x$ or $d_z = d_{x+1}$. Indeed, if $d_x = d_1$ there is at least one edge $d_1 a_w$ with $w > y$, the edge $d_1 a_p$. If $x > 1$, it is clear by definition of the adjacent path that the vertex $d_{x-1}$ is adjacent to at least one vertex $a_w$ with $w \geq y$.

Since $d_x a_y \notin E(T_{d_z a_w})$, the W-triangulation $T_{d_z a_w}$ has less edges than $T_{d_x a_y}$, so Property 5 holds for $T_{d_z a_w}$. Let $P'$ and $Q'$ be the obtained paths, going respectively from $c_1$ to $a_1$ and from $a_w$ to $b_1$.

We distinguish 4 cases according to the edge $d_z a_w$. When $z = x$ we consider the case where $w = y + 1$ and the case where $w > y + 1$. When $z = x - 1$ we consider the case where $w = y$ and the case where $w > y$.

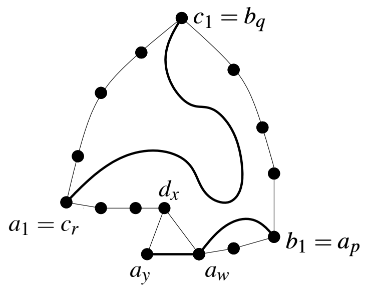


Fig. 11. Case 2.2.1.

**Case 2.2.1:** $d_z = d_x$**, and** $w = y + 1$ **(see Figure 11).** The graph $T_{d_x a_y}$ being a W-triangulation, the cycle $(d_x, a_y, a_w)$ bounds a face of $T_{d_x a_y}$ and so $V(T_{d_x a_y}) = V(T_{d_z a_w}) \cup \{a_y\}$. Since $a_w$ is an end of $Q'$ let $Q = Q' \cup \{a_y a_w\}$ be a path from $a_y$ to $b_1$. Since $V(P') \cup V(Q) = V(T_{d_z a_w}) \cup \{a_y\} = V(T_{d_x a_y})$ and since $V(P') \cap (V(Q) \setminus \{a_y a_w\}) \subseteq V(T_{d_z a_w}) \cap \{a_y\} = \emptyset$, the paths $P'$ and $Q$ fulfill Property 5.

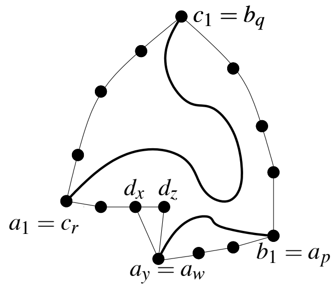


Fig. 12. Case 2.2.2.

**Case 2.2.2:** $z = x - 1$**, and** $a_w = a_y$ **(see Figure 12).** The graph $T_{d_x a_y}$ being a W-triangulation, the cycle $(d_x, a_y, d_z)$ bounds a face of $T_{d_x a_y}$ and so

$V(T_{d_x a_y}) = V(T_{d_z a_w})$. Thus the paths $P'$ and $Q'$ already fulfill Property 5.

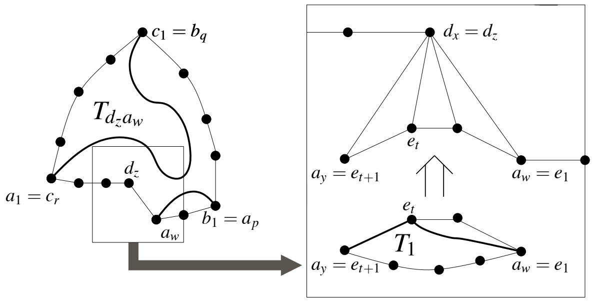


Fig. 13. Case 2.2.3.

**Case 2.2.3:** $d_z = d_x$**, and** $w > y + 1$ **(see Figure 13).** Let $e_1, e_2, \ldots, e_t, e_{t+1}$ be the neighbors of $d_x$ in $T$ and inside the cycle $(d_x, a_y, \ldots, a_w)$ going from $a_w$ to $a_y$ included. This implies that $e_1 = a_w$ and $e_{t+1} = a_y$. Furthermore $t \geq 2$, since there is no chord $a_y a_w$. By definition of $d_z a_w$ we have $e_i \neq a_j$ for all $i$ and $j$ such that $1 < i \leq t$ and $y < j < w$. Consider the W-triangulation $T_1$ (*c.f.* Lemma 2), subgraph of $T_{d_x a_y}$ inside the cycle $(a_y, \ldots, a_w, e_2, \ldots, e_t)$. It is clear that $V(T_{d_x a_y}) = V(T_{d_z a_w}) \cup V(T_1)$ and $V(T_{d_z a_w}) \cap V(T_1) = \{a_w\}$. The W-triangulation $T_{d_x a_y}$ having no separating 3-cycle $(d_x, e_i, e_j)$, there is no chord $e_i e_j$ in $T_1$. Furthermore since $t \geq 2$, $(e_t, e_{t+1})$-$(a_y, \ldots, a_w)$-$(e_1, \ldots, e_t)$ is a 3-boundary of $T_1$. Since $d_x a_y \notin E(T_1)$, the W-triangulation $T_1$ has less edges than $T_{d_x a_y}$, so Property 4 holds for $T_1$ with the mentioned 3-boundary and let $P_1$ be a hamiltonian path of $T_1$, going from $a_y$ to $a_w$.

Let $Q = Q' \cup P_1$. Since $a_w$ is an end in both $P_1$ and $Q'$ the graph $Q$ is a path from $a_y$ to $b_1$. Since $V(P') \cup (V(Q') \cup V(P_1)) = V(T_{d_z a_w}) \cup V(T_1) = V(T_{d_x a_y})$ and since $V(P') \cap (V(Q') \cup V(P_1)) = V(P') \cap (V(P_1) \setminus \{a_w\}) = \emptyset$, the paths $P'$ and $Q$ fulfill Property 5.

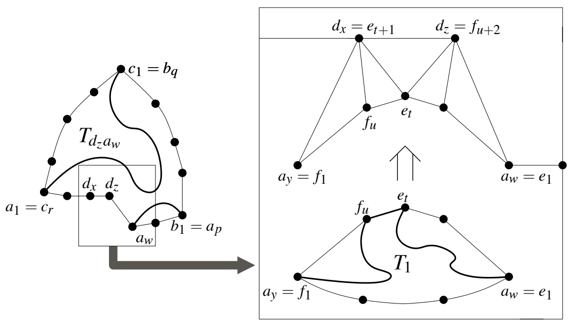


Fig. 14. Case 2.2.4.

**Case 2.2.4:** $z = x - 1$**, and** $1 < y < w$ **(see Figure 14).** Let $e_1, e_2, \ldots, e_t, e_{t+1}$ (resp. $f_1, f_2, \ldots, f_u, f_{u+1}, f_{u+2}$) be the neighbors of $d_z$ (resp. $d_x$) in $T$ and inside the cycle $(d_z, d_x, a_y, \ldots, a_w)$ going from $a_w$ to $d_x$ (resp. from $a_y$ to $d_z$) included.

This implies that $e_1 = a_w$, $e_t = f_{u+1}$, $e_{t+1} = d_x$, $f_1 = a_y$, and $f_{u+2} = d_z$. Furthermore, by definition of the edge $d_z a_w$, there is no edge $d_x a_w$ or $d_z a_y$, so $t \geq 2$ and $u \geq 1$. Also by definition of $d_z a_w$ we have $e_i \neq a_j$ (resp. $f_i \neq a_j$) for all $i$ and $j$ such that $1 < i \leq t$ (resp. $1 < i \leq u$) and $y < j < w$. Since there is no separating 3-cycle $(d_x, d_z, e_i)$ we have $e_i \neq f_j$ for all $i$ and $j$ such that $1 \leq i < t$ and $1 \leq j \leq u$. Consider the W-triangulation $T_1$ (*c.f.* Lemma 2), subgraph of $T_{d_x a_y}$ inside the cycle $(a_y, \ldots, a_w, e_2, \ldots, e_t, f_u, \ldots, f_2)$. It is clear that $V(T_{d_x a_y}) = V(T_{d_z a_w}) \cup V(T_1)$ and $V(T_{d_z a_w}) \cap V(T_1) = \{a_w\}$. The W-triangulation $T_{d_x a_y}$ having no separating 3-cycle $(d_z, e_i, e_j)$ or $(d_x, f_i, f_j)$, there is no chord $e_i e_j$ or $f_i f_j$ in $T_1$. Furthermore since there is no chord $a_i a_j$, since $t \geq 2$, and since $u \geq 1$, $(e_t, f_u, \ldots, f_1)$-$(a_y, \ldots, a_w)$-$(e_1, \ldots, e_t)$ is a 3-boundary of $T_1$. Since $d_x a_y \notin E(T_1)$, the W-triangulation $T_1$ has less edges than $T_{d_x a_y}$ and Property 4 holds for $T_1$ with the mentioned 3-boundary. Let $P_1$ be a hamiltonian path of $T_1$, going from $a_y$ to $a_w$.

Let $Q = Q' \cup P_1$. Since $a_w$ is an end in both $P_1$ and $Q'$ the graph $Q$ is a path from $a_y$ to $b_1$. Since $V(P') \cup (V(Q') \cup V(P_1)) = V(T_{d_z a_w}) \cup V(T_1) = V(T_{d_x a_y})$ and since $V(P') \cap (V(Q') \cup V(P_1)) \subseteq (V(T_{d_z a_w}) \setminus \{a_w\}) \cap V(T_1) = \emptyset$, the paths $P'$ and $Q$ fulfill Property 5.

This concludes the proof of Case 2 and so the joint proof of Property 4 and Property 5.

## 3 Outerplanar graphs

We consider a subclass of outerplanar graphs, the chordal outerplanar graphs (COGs).

**Lemma 6** *The set of chordal outerplanar graphs corresponds to the set of outerplanar graphs that have an outerplanar embedding in which every inner-face is a triangle.*

**PROOF.** Consider an outerplanarly embedded chordal outerplanar graph $G$. If $G$ had an inner-face $f$ bounded by a cycle $C$ of length at least 4, $C$ should have a chord. In this case, $C$ and its chord would form a graph containing a cycle with vertices inside, contradicting the definition of outerplanar embedding.

Conversely consider an outerplanarly embedded graph $G$ in which every inner-face is triangular. Any cycle $C \subseteq G$ of length $l \geq 4$ delimits a region of the plane which is the union of some inner-faces. Since there is no vertex inside $C$ and since these inner-faces are triangles, the cycle $C$ necessarily has a chord.

In an outerplanarly embedded graph $G$, a *side* is an edge $e \in E(G)$ incident to the outer-face. It is easy to see that in every outerplanar embedding of a graph $G$ the set of sides is exactly the same. So we extend the definition of side to every outerplanar graphs (not necessarily outerplanarly embedded). In a graph $G$, two vertices are *linked* if they belong to the same connected component. If they belong to distinct connected components these vertices are *unlinked*. We observe now that the class of chordal outerplanar graphs is closed under some operations.

**Lemma 7** *If $A$ is a COG with $c$ connected components and with a bridge $e$, then $A\backslash\{e\}$ is a COG with $c+1$ connected components. Furthermore:*

- *all the sides (resp. bridges) $f \neq e$ of $A$ are sides (resp. bridges) of $A\backslash\{e\}$, and*
- *any two vertices unlinked in $A$ are unlinked in $A\backslash\{e\}$.*

**PROOF.** Consider an outerplanar embedding of $A$. It is clear that deleting a bridge $e$ of $A$ does not modify the length of any inner-face. So the outerplanar embedding of $A\backslash\{e\}$ clearly implies the lemma.

**Lemma 8** *Let $A$ be a COG with $c$ connected components and with a vertex $u$ of degree 2 and such that its two neighbors, $v$ and $w$, are adjacent. The graph $A\backslash\{u\}$ is a COG with $c$ connected components and such that:*

- *the edge $vw$ is a side of $A\backslash\{u\}$,*
- *any side (resp. bridge) of $A$ that is not incident to $u$ is a side (resp. a bridge) of $A\backslash\{u\}$, and*
- *any two vertices unlinked in $A$ are unlinked in $A\backslash\{u\}$.*

**PROOF.** It is known that the set of chordal graphs is closed under vertex deletion. Furthermore given an outerplanar embedding of $A$, if we delete $u$ the embedding of $A\backslash\{u\}$ obtained clearly implies the lemma.

The *union* $A \cup B$ of two graphs $A$ and $B$ is a graph defined by $V(A \cup B) = V(A) \cup V(B)$ and $E(A \cup B) = E(A) \cup E(B)$. The *intersection* $A \cap B$ of two graphs $A$ and $B$ is a graph defined by $V(A \cap B) = V(A) \cap V(B)$ and $E(A \cap B) = E(A) \cap E(B)$. The following lemmas give us some conditions for the union of two COGs to be a COG.

**Lemma 9** *Let $A$ and $B$ be two COGs with respectively $c_A$ and $c_B$ connected components and such that their intersection is a single vertex $v$. Their union $A \cup B$ is a COG with $c_A + c_B - 1$ connected components such that:*

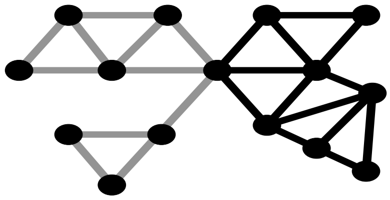

Fig. 15. Lemma 9.

- *any side (resp. bridge) of $A$ or $B$ is a side (resp. a bridge) of $A \cup B$, and*
- *any two vertices unlinked in $A$ (resp. $B$) are unlinked in $A \cup B$.*

**PROOF.** Divide the plane by a line $(\mathcal{D})$. Put the vertex $v$ on $(\mathcal{D})$ and then outerplanarly draw $A$ and $B$ in distinct half-planes. This gives us an outerplanar embedding of $A \cup B$. Furthermore, any inner-face of $A \cup B$ being an inner-face of $A$ or $B$, the inner-faces of $A \cup B$ are all triangular. So the embedding of $A \cup B$ clearly implies the lemma.

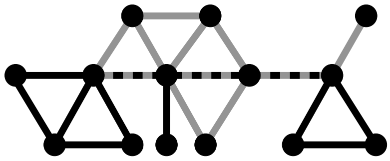

Fig. 16. Lemma 10.

**Lemma 10** *Let $A$ and $B$ be two COGs with respectively $c_A$ and $c_B$ connected components and such that their intersection is a path $P = (v_1, \ldots, v_k)$. If all the edges of $P$ are bridges of $B$, then $A \cup B$, is a COG with $c_A + c_B - 1$ connected components. Furthermore:*

- *any side (resp. bridge) $e \notin E(P)$ of $A$ or $B$ is a side (resp. a bridge) of $A \cup B$, and*
- *any two vertices unlinked in $A$ (resp. $B$) are unlinked in $A \cup B$.*

**PROOF.** The edges of $P$ being bridges of $B$, Lemma 7 implies that the graph $B' = B \backslash E(P)$ is a COG. Since $P \subseteq A$ we have $A \cup B = A \cup B'$ and so $A \cup B$ is the union of $A$ and each of the connected components of $B'$. The edges of $P$ being bridges of $B$, each connected component of $B'$ has at most one vertex in $A$. This implies, by Lemma 9 (applied for each union of a connected component), that $A \cup B$ is a COG with the desired properties.

**Lemma 11** *Let $A$ and $B$ be two COGs with respectively $c_A$ and $c_B$ connected components and such that their intersection is an edge $e$. If $e$ is a side of both $A$ and $B$ then $A \cup B$ is a COG with $c_A + c_B - 1$ connected components. Furthermore:*

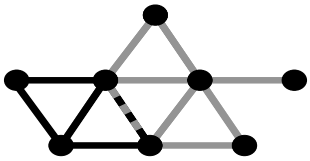

Fig. 17. Lemma 11.

- *any side (resp. bridge) $f \neq e$ of $A$ or $B$ is a side (resp. a bridge) of $A \cup B$, and*
- *any two vertices unlinked in $A$ (resp. $B$) are unlinked in $A \cup B$.*

**PROOF.** Divide the plane by a line $(\mathcal{D})$. Put the edge $e$ on $(\mathcal{D})$ and then outerplanarly draw $A$ and $B$ in distinct half-planes. This gives us an outerplanar embedding of $A \cup B$. Furthermore, any inner-face of $A \cup B$ being an inner-face of $A$ or $B$, the inner-faces of $A \cup B$ are all triangular. So the embedding of $A \cup B$ clearly implies the lemma.

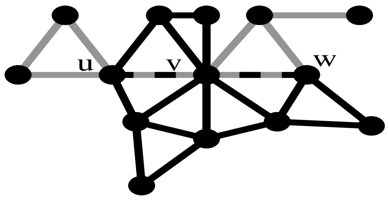


Fig. 18. Lemma 12.

**Lemma 12** *Let $A$ and $B$ be two COGs with respectively $c_A$ and $c_B$ connected components and such that their intersection is a path $(u, v, w)$. If $uv$ is a bridge of $A$ and if $vw$ is a side of both $A$ and $B$ then $A \cup B$ is a COG with $c_A + c_B - 1$ connected components. Furthermore:*

- *any side (resp. bridge) $e$ of $A$ or $B$, with $e \neq uv$ and $e \neq vw$, is a side (resp. a bridge) of $A \cup B$, and*
- *any two vertices unlinked in $A$ (resp. $B$) are unlinked in $A \cup B$.*

**PROOF.** The edge $uv$ being a bridge of $A$, by Lemma 7 the graph $A' = A \backslash \{uv\}$ is a COG with $c_A + 1$ connected components. Let $A'_u$ be the connected component of $A'$ containing the vertex $u$ and let $A'_v$ be the graph $A' \backslash A'_u$. The edge $vw$ being a side of both $A'_v$ and $B$, Lemma 11 applies to the union $A'_v \cup B$. Finally, this union having only the vertex $u$ in $A'_u$, Lemma 9 applies to the union $A'_u \cup (A'_v \cup B)$ and implies the lemma.

## 4 Partition of 4-connected triangulations

Since 4-connected triangulations have a hamiltonian cycle, they have a hamiltonian partition into two COGs. Let $T$ be a triangulation with $k$ 4-connected components $T_1, \ldots, T_k$. It is known that these 4-connected components are 4-connected triangulations and that we obtain them by cutting $T$ along its separating 3-cycles. So each $T_i$ has a hamiltonian cycle and let $A_i$ and $B_i$ be two COGs partitioning $T_i$, obtained by using the hamiltonian cycle method. It is not easy to combine the COGs $A_i$ (resp. $B_i$) to obtain a COG $A$ (resp. $B$), such that $A$ and $B$ form an edge-partition of $T$. For such process being successfull, each COG $A_i$ or $B_i$ should fulfill some special conditions. We prove in this section that some W-triangulations (including 4-connected triangulations) admit a partition into two COGs verifying these special conditions. In the next section we show how these conditions allow us to combine the COGs $A_i$ and $B_i$ of each 4-connected components of $T$ to obtain the partition of $T$ described in Theorem 1.

The *stellation* $T^*$ of a near-triangulation $T$, is the near-triangulation obtained from $T$ by adding inside each inner-face $abc$ of $T$ a new vertex $x$ and three new edges $xa$, $xb$, and $xc$. Such a vertex $x$ of $T^*$ is called an *$f$-vertex*. Given a partition of a stellation $T^*$ into two COGs $A$ and $B$, a $f$-vertex $v \in V(T^*)$ has its neighborhood partitioned in an *extendable* way (see Figure 19) if its three neighbors $a$, $b$, and $c$, are such that the edges $ab$, $va$, and $vb$ are in the same COG (*e.g.* $A$) and the edge $vc$ in the other one (*e.g.* $B$). The edges $ac$ and $bc$ are either in $A$ or $B$. In such partition of the edges in the neighborhood of an $f$-vertex $v$, the edge $ab$ is called the *support edge* of $v$. A partition of a stellation $T^*$ is *extendable* if every $f$-vertex has its neighborhood partitioned in an extendable way.

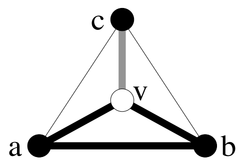


Fig. 19. Neighborhood of an $f$-vertex $v$ partitioned in an extendable way.

In this section there are many edge partitions depicted. Let us define a drawing convention for this figures.

**Drawing convention for the partitions into two COGs $A$ and $B$ (see Figure 20).** In these figures, the thin edges are edges that are either edges of $A$ or $B$. The bold edges are either grey or black, according to which COG they belong to. In each figure it is indicated which of the colors corresponds to $A$ or $B$. There are three types of edges in $A$ (resp. $B$): the "normal" ones, the "bulging" ones or the "dotted" ones. The "normal" edges are bridges of $A$

or $B$. The "bulging" ones are sides of $A$ or $B$. The "dotted" ones are edges of $A$ or $B$ which nature (bridge, side or other) are not indicated. Since a bridge $e$ of a COG $A$ is also a side of $A$, such an edge may be depicted as a "normal", a "bulging", or a "dotted" line.

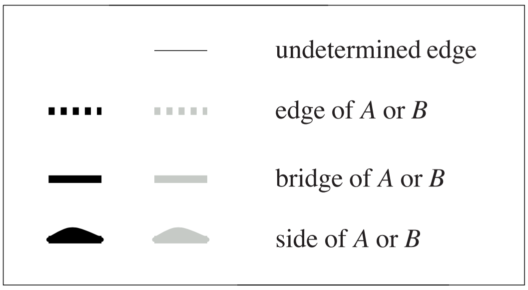


Fig. 20. Drawing convention for the figures depicting a partition into two COGs.

The following property concerns bipartitions of 3-bounded W-triangulations into COGs.

**Property 13** *For any* 3*-bounded W-triangulation* $T$ *and any* 3*-boundary* $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$ *of* $T$*, there is a partition of the stellation* $T^*$ *into two COGs* $A = (V(T^*), E(A))$ *and* $B = (V(T^*), E(B))$ *(see Figure 21). Furthermore,*

*(a) this partition is extendable,*
*(b)* $A$ *is connected,*
*(c)* $B$ *has exactly two connected components, one containing* $b_1$ *and the other one containing* $b_q$*,*
*(d) the edge* $a_1a_2$ *is a side of* $A$*,*
*(e) the edges* $a_ia_{i+1}$ *for* $2 \leq i < p$*, are bridges of* $B$*,*
*(f) the edges* $b_ib_{i+1}$ *for* $1 \leq i < q$*, are bridges of* $A$*, and*
*(g) the edges* $c_ic_{i+1}$ *for* $1 \leq i < r$*, are bridges of* $B$*.*

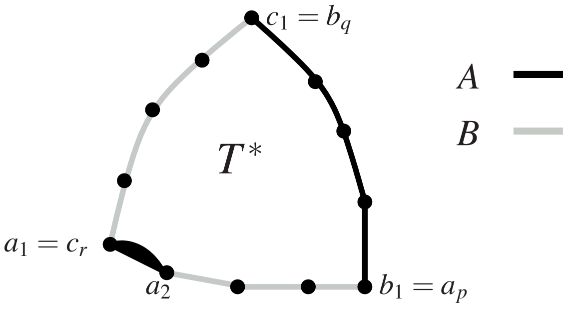


Fig. 21. Property 13.

Note that Property 13 holds for 4-connected triangulations. Indeed, a 4-connected triangulation $T$ with outer-boundary $abc$ is a W-triangulation 3-bounded by $(a, b)$-$(b, c)$-$(c, a)$. The following property is related to Property 13.

**Property 14** *Let $T$ be a W-triangulation 3-bounded by $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$, without chord $a_ib_j$ or $a_ic_j$ and with adjacent path $(d_1, d_2, \ldots, d_s, a_1)$. For any edge $d_xa_y \in E(T)$, with $1 \le x \le s$ and $1 < y \le p$, there is a partition of the stellation $T^*_{d_xa_y}$ into two COGs $A = (V(T^*_{d_xa_y}), E(A))$ and $B = (V(T^*_{d_xa_y}), E(B))$ (see Figure 22). Furthermore,*

*(a) the partition is extendable,*
*(b) $A$ is connected,*
*(c) $B$ has exactly two connected components, one containing $b_1$ and the other one containing $b_q$,*
*(d) the edge $a_1d_s$ and the edges $d_id_{i+1}$ for $x \le i < s$, are bridges of $A$,*
*(e) the edge $d_xa_y$ is a side of $A$,*
*(f) the edges $a_ia_{i+1}$ for $y \le i < p$, are bridges of $B$,*
*(g) the edges $b_ib_{i+1}$ for $1 \le i < q$, are bridges of $A$, and*
*(h) the edges $c_ic_{i+1}$ for $1 \le i < r$, are bridges of $B$.*

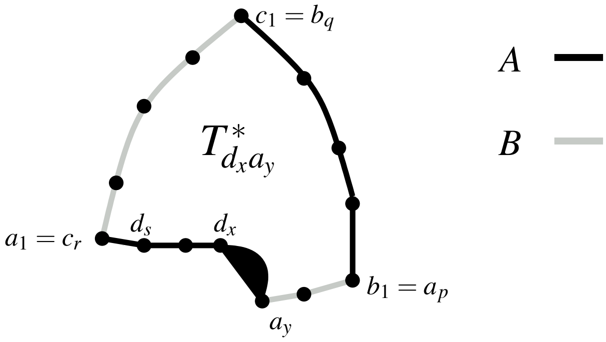


Fig. 22. Property 14.

We need Property 13 for proving Theorem 1 in the next section. Even if Property 14 is not used there, this property is needed to prove Property 13. Indeed, as in Section 2, we prove these two properties by doing a crossed induction.

**PROOF of Property 13 and Property 14.** We prove, by induction on $m \ge 3$, that the following two statements hold:

- Property 13 holds if $T$ has at most $m$ edges.
- Property 14 holds if $T_{d_xa_y}$ has at most $m$ edges.

The initial case, $m = 3$, is easy to prove since there is only one W-triangulation having at most 3 edges, $K_3$. For Property 13 we have to consider all the possible 3-boundaries of $K_3$. Since they are all equivalent, we denote $a_1$, $b_1$, and $c_1$ the vertices of $K_3$ and we consider the 3-boundary $(a_1, b_1)$-$(b_1, c_1)$-$(c_1, a_1)$. In Figure 23 there is a partitions of $K_3^*$ verifying Property 13 for the considered 3-boundary. Note in particular that, since $V(B) = V(K_3^*) = \{a_1, b_1, c_1, v\}$, the graph $B$ has two connected components, the path $(a_1, c_1, v)$ and the vertex $b_1$.

For Property 14, recall that there is no W-triangulation $T_{d_x a_y}$ with at most 3 edges. So by vacuity, Property 14 holds for $m = 3$.

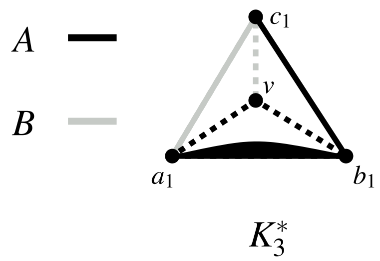


Fig. 23. Initial case of Property 13.

The induction step applies to both Property 13 and Property 14. This means that we prove Property 13 (resp. Property 14) for the W-triangulations $T$ (resp. $T_{d_x a_y}$) with $m$ edges using both Property 13 and Property 14 on W-triangulations with less than $m$ edges. We first prove the induction for Property 13.

**Case 1: Proof of Property 13 for a W-triangulation $T$ with $m$ edges.** Let $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$ be the 3-boundary of $T$. As in Section 2 we consider various cases according to the existence of a chord $a_i b_j$ or $a_i c_j$ in $T$.

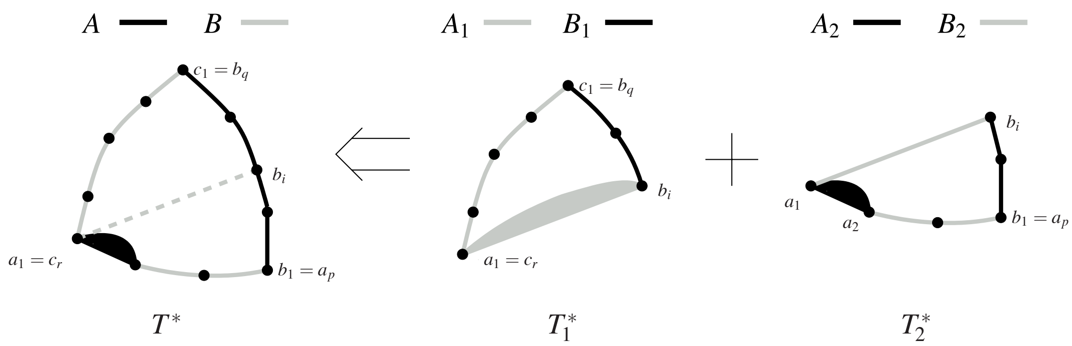


Fig. 24. Case 1.1: chord $a_1 b_i$.

**Case 1.1: There is a chord $a_1 b_i$, with $1 < i < q$ (see Figure 24).** Let $T_1$ and $T_2$ be the W-triangulations respectively delimited by $(b_i, \ldots, b_q, c_2, \ldots, c_r)$ and $(a_1, a_2, \ldots, a_p, b_2, \ldots, b_i)$. We have already seen that these graphs have less edges than $T$ and are respectively 3-bounded by $(b_i a_1)$-$(c_r, \ldots, c_1)$-$(b_q, \ldots, b_i)$ and $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_i)$-$(b_i a_1)$. Thus Property 13 holds for $T_1$ and $T_2$ with the mentioned 3-boundaries. This implies that there exists a partition of $T_1^*$ into $A_1 = (V(T_1^*), E(A_1))$ and $B_1 = (V(T_1^*), E(B_1))$ such that:

(a1) the partition of $T_1^*$ is extendable,
(b1) $A_1$ is connected,

(c1) $B_1$ has exactly two connected components, one containing $c_1$ and one containing $c_r$,
(d1) the edge $a_1b_i$ is a side of $A_1$,
(f1) the edges $c_jc_{j+1}$ are bridges of $A_1$, and
(g1) the edges $b_jb_{j+1}$, for $j \geq i$, are bridges of $B_1$.

Property 13 implies that there exists a partition of $T_2^*$ into $A_2 = (V(T_2^*), E(A_2))$ and $B_2 = (V(T_2^*), E(B_2))$ such that:

(a2) the partition of $T_2^*$ is extendable,
(b2) $A_2$ is connected,
(c2) $B_2$ has exactly two connected components, one containing $b_1$ and one containing $b_i$,
(d2) the edge $a_1a_2$ is a side of $A_2$,
(e2) the edges $a_ja_{j+1}$, for $j \geq 2$, are bridges of $B_2$,
(f2) the edges $b_jb_{j+1}$, for $j < i$, are bridges of $A_2$, and
(g2) the edge $a_1b_i$ is a bridge of $B_2$.

Let $A = B_1 \cup A_2$ and $B = A_1 \cup B_2$. All the edges of $T^*$ being in $A_1$, $B_1$, $A_2$ or $B_2$, the graphs $A$ and $B$ cover $T^*$. Furthermore, the only edge belonging to both $T_1^*$ and $T_2^*$, $a_1b_i$, is in $A_1$ and $B_2$ (*c.f.* (f1) and (d2)). So the sets $E(A)$ and $E(B)$ do not intersect and they form a partition of $T^*$. We now prove that $A$ and $B$ are COGs and that they verify Property 13.

(a) Each inner-face of $T$ being an inner-face of $T_1$ or $T_2$, any $f$-vertex of $T^*$ is an $f$-vertex of $T_1^*$ or $T_2^*$. For each $f$-vertex of $T^*$, the partition of its neighborhood is as in $T_1^*$ or $T_2^*$. So the partitions of $T_1^*$ and $T_2^*$ being both extendable (*c.f.* (a1) and (a2)), the partition of $T^*$ into $A$ and $B$ is extendable too. Thus point (a) of Property 13 holds.

The COGs $B_1$ and $A_2$ intersect on two vertices, $a_1$ and $b_i$. $B_1$ has two connected components, one containing $a_1$ and one containing $b_i$. Indeed, the connected component containing the vertex $c_1$ also contains the path $(b_i, \ldots, b_q)$ (*c.f.* (c1) and (g1)). Let $B_1'$ (resp. $B_1''$) be the connected component of $B_1$ containing the vertex $a_1$ (resp. $b_i$). We consider the union of $B_1$ and $A_2$ as a succession of two unions in which the graphs intersect on a single vertex: $A = A_2 \cup B_1 = (A_2 \cup B_1') \cup B_1''$. Lemma 9 holds for each of these unions and it implies that $A = A_2 \cup B_1$ is a COG that fulfills points (b), (d), and (f) of Property 13. Indeed:

(b) Since $A_2$, $B_1'$, and $B_1''$ are connected (*c.f.* (b2) and (c1)), $A$ is connected.
(d) The edge $a_1a_2$ being a side of $A_2$ (*c.f.* (d2)), it is a side of $A$.
(f) The edges $b_jb_{j+1}$ being bridges of $A_2$ or $B_1$ (*c.f.* (f2) and (g1)), these edges are bridges of $A$.

The intersection of the COGs $A_1$ and $B_2$ is the edge $a_1b_i$. This edge being a bridge of $B_2$ (*c.f.* (g2)), Lemma 10 implies that $B = A_1 \cup B_2$ is a COG that fulfills points (c), (e), and (g) of Property 13. Indeed:

(c) Since $A_1$ is connected and contains the vertices $b_i$ and $b_q$ (*c.f.* (b1) and (g1)) and since $B_2$ has two connected components, one containing $b_1$ and one containing $b_i$ (*c.f.* (c2)), $B$ has two connected components, one containing $b_1$ and one containing $b_i$. Furthermore since $b_i$ and $b_q$ are in the same connected component of $B$ ($A_1$ being connected), the vertices $b_1$ and $b_q$ are in distinct connected components of $B$.
(e) The edges $a_j a_{j+1}$, for $j \geq 2$, being bridges of $B_2$ (*c.f.* (e2)), these edges are bridges of $B$.
(g) The edges $c_j c_{j+1}$ being bridges of $A_1$ (*c.f.* (f1)), these edges are bridges of $B$.

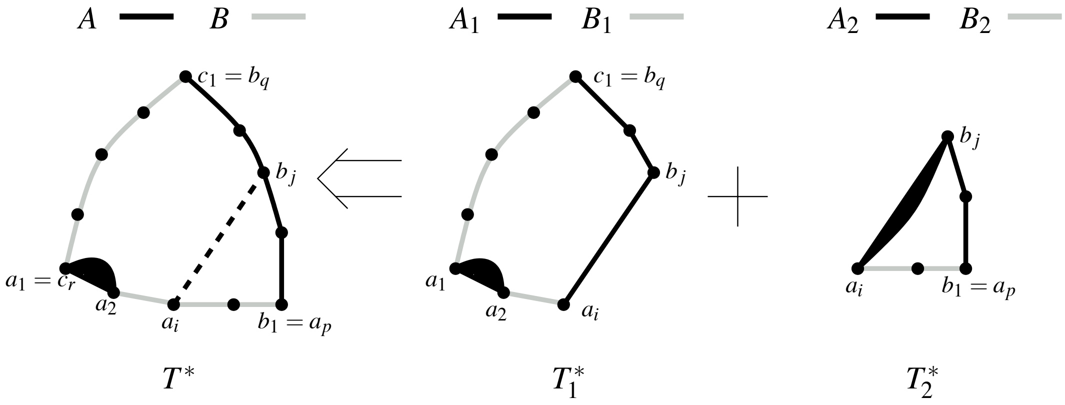


Fig. 25. Case 1.2: chord $a_i b_j$.

**Case 1.2: There is a chord $a_i b_j$, with $1 < i < p$ and $1 < j \leq q$ (see Figure 25).** If there are several chords $a_i b_j$ consider one that maximizes $j$. Let $T_1$ and $T_2$ be the W-triangulations respectively delimited by $(a_2, \ldots, a_i, b_j, \ldots, b_q, c_2, \ldots, c_r)$ and $(a_i, \ldots, a_p, b_2, \ldots, b_j)$. We have already seen that these graphs have less edges than $T$ and are respectively 3-bounded by $(a_1, \ldots, a_i)$-$(a_i, b_j, \ldots, b_q)$-$(c_1, \ldots, c_r)$ and $(a_i, b_j)$-$(b_j, \ldots, b_1)$-$(a_p, \ldots, a_i)$. Thus Property 13 holds for $T_1$ and $T_2$ with the mentioned 3-boundaries. This implies that there exists a partition of $T_1^*$ into $A_1 = (V(T_1^*), E(A_1))$ and $B_1 = (V(T_1^*), E(B_1))$ such that:

(a1) the partition of $T_1^*$ is extendable,
(b1) $A_1$ is connected,
(c1) $B_1$ has exactly two connected components, one containing $a_i$ and one containing $b_q$,
(d1) the edge $a_1 a_2$ is a side of $A_1$,
(e1) the edges $a_k a_{k+1}$, for $2 \leq k < i$, are bridges of $B_1$,
(f1) the edge $a_i b_j$ and the edges $b_k b_{k+1}$, for $k \geq j$, are bridges of $A_1$, and
(g1) the edges $c_k c_{k+1}$ are bridges of $B_1$.

Property 13 implies that there exists a partition of $T_2^*$ into $A_2 = (V(T_2^*), E(A_2))$ and $B_2 = (V(T_2^*), E(B_2))$ such that:

(a2) the partition of $T_2^*$ is extendable,
(b2) $A_2$ is connected,
(c2) $B_2$ has exactly two connected components, one containing $b_1$ and one containing $b_j$,
(d2) the edge $a_i b_j$ is a side of $A_2$,
(f2) the edges $b_k b_{k+1}$, for $k < j$, are bridges of $A_2$, and
(g2) the edges $a_k a_{k+1}$, for $k \geq i$, are bridges of $B_2$.

Let $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$. The graphs $A$ and $B$ covering all the edges of $T^*$ and having no common edge ($a_i b_j \in E(A) \setminus E(B)$), they form a partition of $T^*$. We now prove that $A$ and $B$ are COGs and that they verify Property 13.

(a) The neighborhood of every $f$-vertex of $T^*$ is partitioned as in $T_1^*$ or as in $T_2^*$. Thus (*c.f.* (a1) and (a2)) the partition of $T^*$ into $A$ and $B$ is extendable.

The intersection of the COGs $A_1$ and $A_2$ is the edge $a_i b_j$. This edge being a bridge of $A_1$ (*c.f.* (e1)), Lemma 10 implies that $A = A_1 \cup A_2$ is a COG that fulfills points (b), (d), and (f) of Property 13. Indeed:

(b) Since $A_1$ and $A_2$ are connected (*c.f.* (b1) and (b2)), $A$ is connected.
(d) The edge $a_1 a_2$ being a side of $A_1$ (*c.f.* (d1)), it is a side of $A$.
(f) The edges $b_k b_{k+1}$ being bridges of $A_1$ or $A_2$ (*c.f.* (f1) and (f2)), these edges are bridges of $A$.

The COGs $B_1$ and $B_2$ intersect on two vertices, $a_i$ and $b_j$. The COG $B_2$ has two connected components, one containing $b_1$ and $a_i$ and one containing $b_j$ (*c.f.* (c2) and (g2)). We consider the union of $B_1$ and $B_2$ as a succession of two unions in which the graphs intersect on a single vertex. Lemma 9 implies that $B = B_1 \cup B_2$ is a COG that fulfills points (c), (e), and (g) of Property 13. Indeed:

(c) Since $B_1$ has two connected components, one containing $a_i$ and one containing $b_q$ (*c.f.* (c1)), and since $B_2$ has two connected components, one containing $b_1$ and $a_i$ and one containing $b_j$ (*c.f.* (c2) and (g2)), $B$ has two connected components, one containing $b_1$ and one containing $b_q$.
(e)(g) The edges $a_k a_{k+1}$, for $k \geq 2$, being bridges of $B_1$ or $B_2$ (*c.f.* (e1) and (g2)), and the edges $c_k c_{k+1}$ being bridges of $B_1$ (*c.f.* (g1)), these edges are bridges of $B$.

**Case 1.3: There is a chord $a_i c_j$, with $1 < i \leq p$ and $1 < j < r$ (see Fig-**

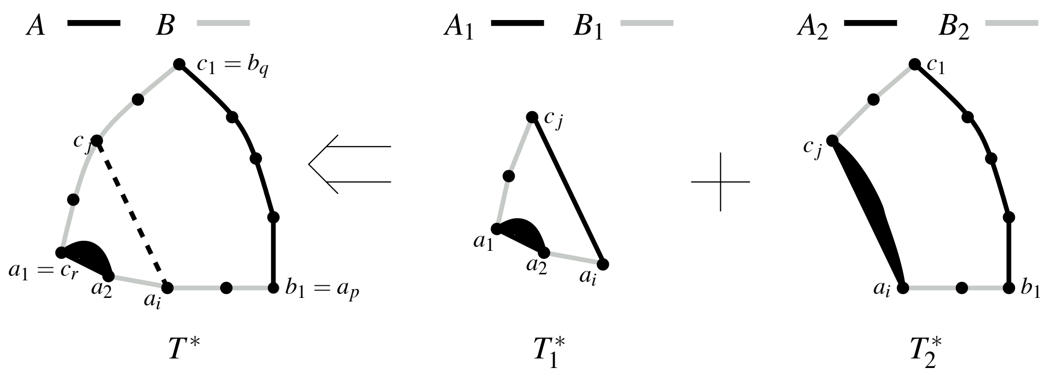


Fig. 26. Case 1.3: chord $a_ic_j$.

**ure 26).** If there are several chords $a_ic_j$ consider one that maximizes $i$. Let $T_1$ and $T_2$ be the W-triangulations respectively delimited by $(a_2, \ldots, a_i, c_j, \ldots, c_r)$ and $(a_i, \ldots, a_p, b_2, \ldots, b_q, c_2, \ldots, c_j)$. We have already seen that these graphs have less edges than $T$ and are respectively 3-bounded by $(a_1, \ldots, a_i)$-$(a_i, c_j)$-$(c_j, \ldots, c_r)$ and $(c_j, a_i, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_j)$. Thus Property 13 holds for $T_1$ and $T_2$ with the mentioned 3-boundaries. This implies that there exists a partition of $T_1^*$ into $A_1 = (V(T_1^*), E(A_1))$ and $B_1 = (V(T_1^*), E(B_1))$ such that:

(a1) the partition of $T_1^*$ is extendable,
(b1) $A_1$ is connected,
(c1) $B_1$ has exactly two connected components, one containing $a_i$ and one containing $c_j$,
(d1) the edge $a_1a_2$ is a side of $A_1$,
(e1) the edges $a_ka_{k+1}$, for $2 \leq k < i$, are bridges of $B_1$,
(f1) the edge $a_ic_j$ is a bridge of $A_1$, and
(g1) the edges $c_kc_{k+1}$, for $k \geq j$, are bridges of $B_1$.

Property 13 implies that there exists a partition of $T_2^*$ into $A_2 = (V(T_2^*), E(A_2))$ and $B_2 = (V(T_2^*), E(B_2))$ such that:

(a2) the partition of $T_2^*$ is extendable,
(b2) $A_2$ is connected,
(c2) $B_2$ has exactly two connected components, one containing $b_1$ and one containing $b_q$,
(d2) the edge $a_ic_j$ is a side of $A_2$,
(e2) the edges $a_ka_{k+1}$, for $k \geq i$, are bridges of $B_2$,
(f2) the edges $b_kb_{k+1}$ are bridges of $A_2$, and
(g2) the edges $c_kc_{k+1}$, for $k < j$, are bridges of $B_2$.

Let $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$. The graphs $A$ and $B$ covering all the edges of $T^*$ and having no common edge ($a_ic_j \in E(A) \setminus E(B)$), they form a partition of $T^*$. We now prove that $A$ and $B$ are COGs and that they verify Property 13.

(a) The neighborhood of every $f$-vertex of $T^*$ is partitioned as in $T_1^*$ or as in $T_2^*$. Thus (*c.f.* (a1) and (a2)) the partition of $T^*$ into $A$ and $B$ is extendable.

The intersection of the COGs $A_1$ and $A_2$ is the edge $a_ic_j$. This edge being a bridge of $A_1$ (*c.f.* (f1)), Lemma 10 implies that $A = A_1 \cup A_2$ is a COG that fulfills points (b), (d), and (f) of Property 13. Indeed:

(b) Since $A_1$ and $A_2$ are connected (*c.f.* (b1) and (b2)), $A$ is connected.
(d) The edge $a_1a_2$ being a side of $A_1$ (*c.f.* (d1)), it is a side of $A$.
(f) The edges $b_kb_{k+1}$ being bridges of $A_2$ (*c.f.* (f2)), these edges are bridges of $A$.

The COGs $B_1$ and $B_2$ intersect on two vertices, $a_i$ and $c_j$. The COG $B_1$ has two connected components, one containing the vertex $a_i$ and one containing the vertex $c_j$ (*c.f.* (c1)). We consider the union of $B_1$ and $B_2$ as a succession of two unions in which the graphs intersect on a single vertex. Lemma 9 implies that $B = B_1 \cup B_2$ is a COG that fulfills points (c), (e), and (g) of Property 13. Indeed:

(c) Since $B_1$ has two connected components, one containing $a_i$ and one containing $c_j$ (*c.f.* (c1)), and since $B_2$ has two connected components, one containing $b_1$ and $a_i$ and one containing $b_q$ and $c_j$ (*c.f.* (c2), (e2), and (g2)), $B$ has two connected components, one containing $b_1$ and one containing $b_q$.
(e)(g) The edges $a_ka_{k+1}$, for $k \geq 2$, being bridges of $B_1$ or $B_2$ (*c.f.* (e1) and (e2)), and the edges $c_kc_{k+1}$ being bridges of $B_1$ or $B_2$ (*c.f.* (g1) and (g2)), these edges are bridges of $B$.

**Case 1.4: There is no chord $a_ib_j$ or $a_ic_j$.** As in Section 2 we consider the adjacent path $(d_1, \ldots, d_s, a_1)$ (see Figure 3) of $T$ for the 3-boundary $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$. Let $d_sa_y \in E(T)$ be the edge with $1 < y \leq p$ such that $y$ is minimum. The W-triangulation $T_{d_sa_y}$ having less edges than $T$, Property 14 holds for $T_{d_sa_y}$. This implies that there exists a partition of $T^*_{d_sa_y}$ into $A' = (V(T^*_{d_sa_y}), E(A'))$ and $B' = (V(T^*_{d_sa_y}), E(B'))$ such that:

(a') the partition of $T^*_{d_sa_y}$ is extendable,
(b') $A'$ is connected,
(c') $B'$ has exactly two connected components, one containing $b_1$ and one containing $b_q$,
(d') the edge $a_1d_s$ is a bridge of $A'$,
(e') the edge $d_sa_y$ is a side of $A'$,
(f') the edges $a_ia_{i+1}$, for $i \geq y$, are bridges of $B'$,
(g') the edges $b_ib_{i+1}$ are bridges of $A'$, and
(h') the edges $c_ic_{i+1}$ are bridges of $B'$.

We extend these two COGs in order to obtain a partition of $T^*$. We distinguish two cases according to the index $y$ of $a_y$, the case $y = 2$ and the case $y > 2$.

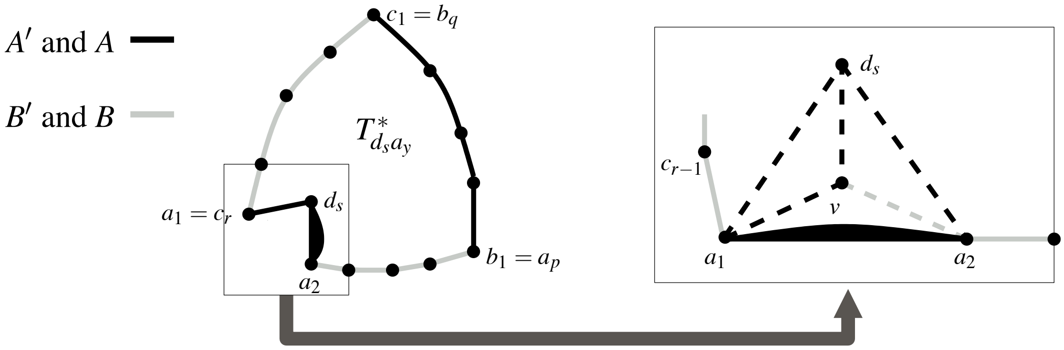


Fig. 27. Case 1.4.1.

**Case 1.4.1: $y = 2$ (see Figure 27).** Let $v$ be the $f$-vertex of $T^*$ adjacent to $a_1$, $a_2$, and $d_s$. Let $G_A$ be the connected COG which is the union of the cycle $(v, a_1, a_2, d_s)$ and the edge $a_1d_s$, and let $G_B$ be the connected COG with only one edge, $a_2v$. Let $A = A' \cup G_A$ and $B = B' \cup G_B$. The graphs $A$ and $B$ covering all the edges of $T^*$ and having no common edge, they form a partition of $T^*$. These graphs are COGs and they verify Property 13.

(a) The partition of $v$'s neighborhood being extendable, and the neighborhood of the other $f$-vertices of $T^*$ being partitioned as in $T^*_{d_sa_2}$, the partition of $T^*$ into $A$ and $B$ is extendable.

The intersection of $A'$ and $G_A$ is the path $(a_1, d_s, a_2)$. The edge $a_1d_s$ being a bridge of $A'$ and the edge $d_sa_2$ being a side of both $A'$ and $G_A$ (*c.f.* (d') and (e')), Lemma 12 implies that $A = A' \cup G_A$ is a COG that fulfills points (b), (d), and (f) of Property 13. Indeed:

(b) Since $A'$ and $G_A$ are connected (*c.f.* (b')), $A$ is connected.
(d) The edge $a_1a_2$ being a side of $G_A$, it is a side of $A$.
(f) The edges $b_ib_{i+1}$ being bridges of $A'$ (*c.f.* (g')), these edges are bridges of $A$.

The intersection of $B'$ and $G_B$ being the vertex $a_2$, Lemma 9 implies that the graph $B = B' \cup G_B$ is a COG that fulfills points (c), (e), and (g) of Property 13. Indeed:

(c) Since $G_B$ is connected and since $B'$ has two connected components, one containing $b_1$ and one containing $b_q$ (*c.f.* (c')), $B$ has two connected components, one containing $b_1$ and one containing $b_q$.
(e)(g) The edges $a_ia_{i+1}$, for $k \geq 2$, and the edges $c_ic_{i+1}$ being bridges of $B'$ (*c.f.* (f') and (h')), these edges are bridges of $B$.

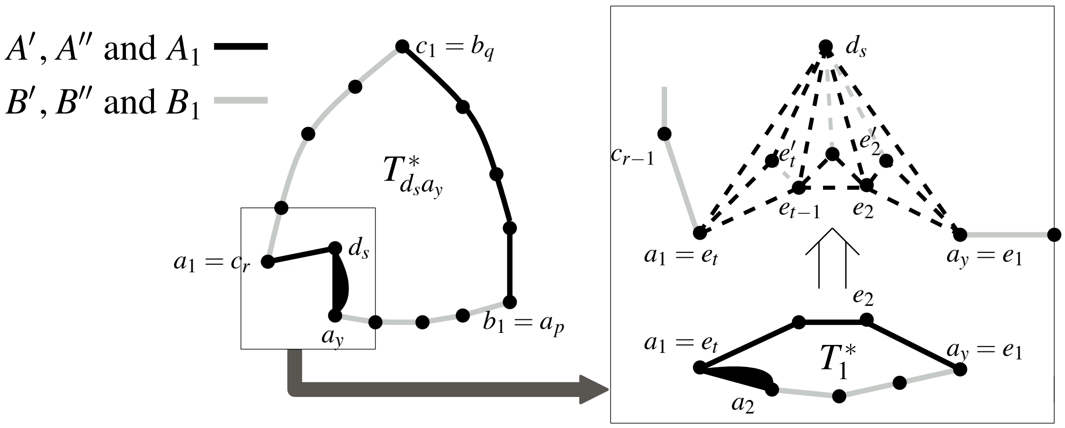


Fig. 28. Case 1.4.2.

**Case 1.4.2: $y > 2$ (see Figure 28).** Let $e_1, e_2, \ldots, e_t$ be the neighbors of $d_s$ in $T$ and inside the cycle $(d_s, a_1, a_2, \ldots, a_y)$, going from $a_y$ to $a_1$ included. This implies that $e_1 = a_y$, $e_t = a_1$, and $t \geq 3$. For each $i \in \{2, \ldots, t\}$, let $e'_i$ be the $f$-vertex of $T^*$ adjacent to $d_s$, $e_i$, and $e_{i-1}$.

Let $G_A$ be the connected COG with the edges $e_i e_{i+1}$, for $1 \leq i < t$, the edges $d_s e_i$, for $1 \leq i \leq t$, the edges $e_i e'_{i+1}$, for $1 \leq i < t-1$, the edges $e_i e'_i$, for $2 \leq i < t$, and the edges $d_s e'_t$ and $a_1 e'_t$. The intersection of $A'$ and $G_A$, the path $(a_1, d_s, a_y)$, is such that the edge $a_1 d_s$ is a bridge of $A'$ and such that the edge $d_s a_y$ is a side in both $A'$ and $G_A$ (*c.f.* (d') and (e')). So Lemma 12 implies that $A'' = A' \cup G_A$ is a COG:

(a”) that is connected (*c.f.* (b')), and
(b”) which edges $b_i b_{i+1}$ are bridges (*c.f.* (g')).

Let $G_B$ be the COG which is the union of the star with edges $d_s e'_i$, for $2 \leq i < t$, and the edge $e_{t-1} e'_t$. Since $B'$ and $G_B$ intersect on $d_s$, Lemma 9 implies that $B'' = B' \cup G_B$ is a COG:

(c”) having three connected components, one containing $e_1$, one containing $e_{t-1}$ and one containing $e_t$ (*c.f.* (c'), (f'), (h')),
(d”) which edges $a_i a_{i+1}$, for $i \geq y$, are bridges (*c.f.* (f')), and
(e”) which edges $c_i c_{i+1}$ are bridges (*c.f.* (h')).

Consider now the W-triangulation $T_1$ delimited by $(a_2, \ldots, a_y, e_2, \ldots, e_t)$. We have already seen that this graph has less edges than $T$ and is 3-bounded by $(a_2, a_1)$-$(e_t, \ldots, e_1)$-$(a_y, \ldots, a_2)$. Thus Property 13 holds for $T_1$ with the mentioned 3-boundary and there exists a partition of $T^*_1$ into $A_1 = (V(T^*), E(A_1))$ and $B_1 = (V(T^*), E(B_1))$ such that:

(a1) the partition of $T^*_1$ is extendable,
(b1) $A_1$ is connected,

(c1) $B_1$ has exactly two connected components, one containing $a_1$ and one containing $a_y$,
(d1) the edge $a_1a_2$ is a side of $A_1$,
(f1) the edges $e_ie_{i+1}$, for $1 \leq i < t$, are bridges of $A_1$, and
(g1) the edges $a_ia_{i+1}$, for $2 \leq i < y$, are bridges of $B_1$.

Let $A = A'' \cup A_1$ and $B = B'' \cup B_1$. The graphs $A$ and $B$ covering all the edges of $T^*$ and having no common edge, they form a partition of $T^*$. We now prove that these graphs are COGs and that they verify Property 13.

(a) The partition of $e_i'$'s neighborhoods being extendable, and the neighborhood of the other $f$-vertices of $T^*$ being partitioned as in $T^*_{d_sa_y}$ or as in $T^*_1$, the partition of $T^*$ into $A$ and $B$ is extendable.

The intersection of the COGs $A''$ and $A_1$ is the path $(e_1, e_2, \ldots, e_t)$ which edges are all bridges of $A_1$ (*c.f.* (f1)). So Lemma 10 implies that $A = A'' \cup A_1$ is a COG that fulfills points (b), (d), and (f) of Property 13. Indeed:

(b) Since $A''$ and $A_1$ are connected (*c.f.* (a") and (b1)), $A$ is connected.
(d) The edge $a_1a_2$ being a side of $A_1$ (*c.f.* (d1)), it is a side of $A$.
(f) The edges $b_ib_{i+1}$ being bridges of $A''$ (*c.f.* (b")), these edges are bridges of $A$.

The COGs $B''$ and $B_1$ intersect on the vertices $e_1$, $e_{t-1}$, and $e_t$. $B''$ has three connected components, one containing $e_1$, one containing $e_{t-1}$ and one containing $e_t$ (*c.f.* (c")). We consider the union of $B''$ and $B_1$ as a succession of three unions in which the graphs intersect on a single vertex. So Lemma 9 implies that $B = B'' \cup B_1$ is a COG that fulfills points (c), (e), and (g) of Property 13. Indeed:

(c) Since $B''$ has three connected components, one containing $e_1$ and $b_1$, one containing $e_{t-1}$, and one containing $e_t$ and $b_q$ (*c.f.* (c"), (d"), and (e")), and since $B_1$ has two connected components, one containing $e_1$ and one containing $e_t$ (*c.f.* (c1)), $B$ has two connected components, one containing $b_1$ and one containing $b_q$.
(e)(g) The edges $a_ia_{i+1}$, for $k \geq 2$, and the edges $c_ic_{i+1}$ being bridges of $B''$ or $B_1$ (*c.f.* (d"), (e"), and (g1)), these edges are bridges of $B$.

This concludes the proof of Case 1.

**Case 2: Proof of Property 14 for a W-triangulation $T_{d_xa_y}$ with $m$ edges.** As in Section 2, we consider one case where $d_xa_y = d_1a_p$ and four cases where $d_xa_y \neq d_1a_p$.

**Case 2.1: $d_xa_y = d_1a_p$ (see Figure 29).** Let $T_1$ be the W-triangulation delimited by $(d_s, \ldots, d_1, b_2, \ldots, b_q, c_2, \ldots, c_r)$. We have seen that $T_1$ has less

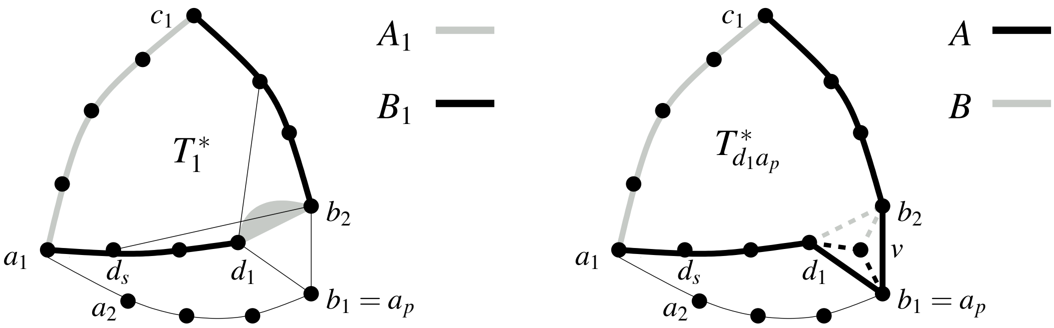


Fig. 29. Case 2.1.

edges than $T_{d_1 a_p}$ and is 3-bounded by $(d_1, b_2, \ldots, b_q)$-$(c_1, \ldots, c_r)$-$(a_1, d_s, \ldots, d_1)$ or by $(b_2, d_1, \ldots, d_s, a_1)$-$(c_r, \ldots, c_1)$-$(b_q, \ldots, b_2)$. Applying Property 13 to $T_1$ for any of these 3-boundaries we obtain a partition of $T_1^*$, into two COGs $A_1 = (V(T_1^*), E(A_1))$ and $B_1 = (V(T_1^*), E(B_1))$, such that:

- (a1) the partition of $T_1^*$ is extendable,
- (b1) $A_1$ is connected,
- (c1) $B_1$ has exactly two connected components, one containing $c_1$ and one containing $c_r$,
- (d1) the edge $d_1 b_2$ is a side of $A_1$,
- (e1-g1) the edges $b_i b_{i+1}$, for $i \geq 2$, the edges $d_i d_{i+1}$, and the edge $a_1 d_s$ are bridges of $B_1$, and
- (f1) the edges $c_i c_{i+1}$ are bridges of $A_1$.

We extend $A_1$ and $B_1$ to obtain the desired partition of $T^*_{d_1 a_p}$. Let $v$ be the $f$-vertex adjacent to $a_p$, $b_2$, and $d_1$ (see Figure 29). Let $G_A$ be the union of the cycle $(d_1, b_1, v)$ and the edge $b_1 b_2$, and let $G_B$ be the union of the path $(d_1, b_2, v)$ and the vertex $b_1$. Note that $G_A$ and $G_B$ are COGs and let $A = B_1 \cup G_A$ and $B = A_1 \cup G_B$. The graphs $A$ and $B$ covering all the edges of $T^*_{d_1 a_p}$ and having no common edge, they form a partition of $T^*_{d_x a_y}$. We now prove that these graphs are COGs and that they verify Property 14.

- (a) The partition of $v$'s neighborhood being extendable, and the neighborhood of the other $f$-vertices of $T^*_{d_1 a_p}$ being partitioned as in $T_1^*$, the partition of $T^*_{d_1 a_p}$ into $A$ and $B$ is extendable.

The COGs $B_1$ and $G_A$ intersect on $d_1$ and $b_2$. The COG $B_1$ has two connected components, one containing $c_r$ and $d_1$ and one containing $c_1$ and $b_2$ (*c.f.* (c1) and (e1-g1)). Thus we consider the union of $B_1$ and $G_A$ as a succession of two unions in which the graphs intersect on a single vertex. So Lemma 9 implies that $A = B_1 \cup G_A$ is a COG that fulfills points (b), (d), (e), and (g) of Property 14. Indeed:

- (b) Since $G_A$ is connected and since $B_1$ has two connected components (*c.f.* (c1)), one containing $d_1$ and one containing $b_2$, $A$ is connected.

(e) The edge $d_1a_p$ being a side of $G_A$, it is a side of $A$.

(d)(g) The edge $b_1b_2$ being a bridge of $G_A$; the edge $a_1d_s$, the edges $d_id_{i+1}$ and the edges $b_ib_{i+1}$, for $i \geq 2$, being bridges of $B_1$ (*c.f.* (e1-g1)), these edges are bridges of $A$.

The intersection of the COGs $A_1$ and $G_B$ is the edge $d_1b_2$. This edge being a bridge of $G_B$, Lemma 10 implies that $B = A_1 \cup G_B$ is a COG that fulfills points (c), (f), and (h) of Property 14. Indeed:

(c) Since $A_1$ is connected and contains $b_2$ and $b_q$ (*c.f.* (b1)), and since $G_B$ has two connected components, one containing $b_1$ and one containing $b_2$, $B$ has two connected components, one containing $b_1$ and one containing $b_q$.

(f) Since there is no edge $a_ia_{i+1}$ in $T_{d_1a_p}$, $B$ fulfills point (f) by vacuity.

(h) The edges $c_ic_{i+1}$ being bridges of $A_1$ (*c.f.* (f1)), these edges are bridges of $B$.

**Case 2.2:** $d_xa_y \neq d_1a_p$**.** In this case we consider an edge $d_za_w \in E(T_{d_xa_y})$ such that $d_za_w \neq d_xa_y$. Among all the possible edges $d_za_w$ we choose the one that firstly maximizes $z$ and secondly minimizes $w$. As we have already seen, such an edge necessarily exists and actually $d_z = d_x$ or $d_z = d_{x+1}$.

We have seen that $T_{d_za_w}$ is a W-triangulation with less edges than $T_{d_xa_y}$. Thus Property 14 applies and there exists a partition of $T^*_{d_za_w}$ into $A' = (V(T^*_{d_za_w}), E(A'))$ and $B' = (V(T^*_{d_za_w}), E(B'))$ such that:

(a') the partition of $T^*_{d_za_w}$ is extendable,

(b') $A'$ is connected,

(c') $B'$ has exactly two connected components, one containing $b_1$ and one containing $b_q$,

(d') the edge $a_1d_s$ and the edges $d_id_{i+1}$, for $i \geq z$, are bridges of $A'$,

(e') the edge $d_za_w$ is a side of $A'$,

(f') the edges $a_ia_{i+1}$, for $i \geq w$, are bridges of $B'$,

(g') the edges $b_ib_{i+1}$ are bridges of $A'$, and

(h') the edges $c_ic_{i+1}$ are bridges of $B'$.

We now extend this partition of $T^*_{d_za_w}$ to $T^*_{d_xa_y}$. We proceed by distinguishing 4 cases according to the edge $d_za_w$.

**Case 2.2.1:** $d_z = d_x$**, and** $w = y+1$ **(see Figure 30).** Let $v$ be the $f$-vertex adjacent to $d_x$, $a_y$, and $a_w$. Let $G_A$ be the cycle $(v, d_x, a_y)$ and $G_B$ be the path $(v, a_w, a_y)$. Note that $G_A$ and $G_B$ are COGs and let $A = A' \cup G_A$ and $B = B' \cup G_B$. The graphs $A$ and $B$ covering all the edges of $T^*_{d_xa_y}$ and having no common edge, they form a partition of $T^*_{d_xa_y}$. We now prove that these graphs are COGs and that they verify Property 14.

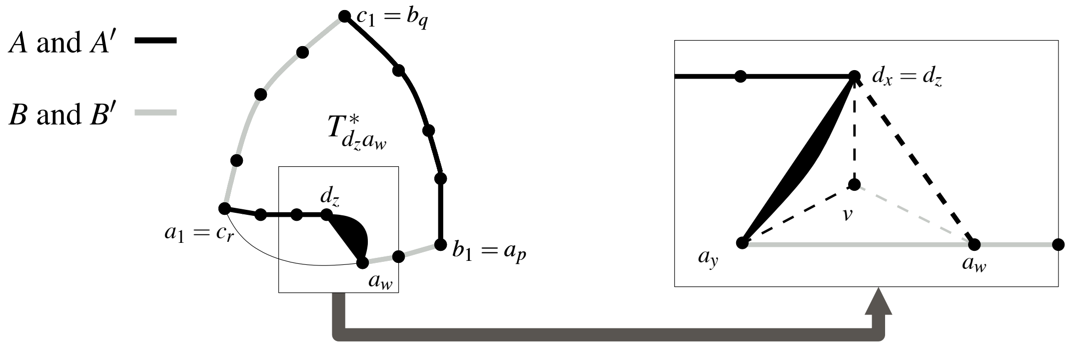


Fig. 30. Case 2.2.1.

(a) The partition of $v$'s neighborhood being extendable, and the neighborhood of the other $f$-vertices of $T^*_{d_x a_y}$ being partitioned as in $T^*_{d_z a_w}$, the partition of $T^*_{d_x a_y}$ into $A$ and $B$ is extendable.

The intersection of $A'$ and $G_A$ is the vertex $d_x$, so Lemma 9 implies that $A = A' \cup G_A$ is a COG that fulfills points (b), (d), (e), and (g) of Property 14. Indeed:

(b) Since $A'$ and $G_A$ are connected (*c.f.* (b')), $A$ is connected.
(e) The edge $d_x a_y$ being a side of $G_A$, it is a side of $A$.
(d)(g) The edge $a_1 d_s$, the edges $d_i d_{i+1}$, for $i \geq x$, and the edges $b_i b_{i+1}$ being bridges of $A'$ (*c.f.* (d') and (g')), these edges are bridges of $A$.

The intersection of $B'$ and $G_B$ is the vertex $a_w$, so Lemma 9 implies that $B = B' \cup G_B$ is a COG that fulfills points (c), (f), and (h) of Property 14. Indeed:

(c) Since $G_B$ is connected and since $B'$ has two connected components, one containing $b_1$ and one containing $b_q$ (*c.f.* (c')), $B$ has two connected components, one containing $b_1$ and one containing $b_q$.
(f)(h) The edge $a_y a_w$ being a bridge of $G_B$; the edges $a_i a_{i+1}$, for $i \geq w$, and the edges $c_i c_{i+1}$ being bridges of $B'$ (*c.f.* (f') and (h')), these edges are bridges of $B$.

**Case 2.2.2: $z = x - 1$, and $a_w = a_y$ (see Figure 31).** Let $v$ be the $f$-vertex adjacent to $d_x$, $a_y$, and $d_z$. Let $G_A$ be the cycle $(a_y, d_z, v, d_x)$ and the edge $d_x d_z$ and let $G_B$ be the path $(a_y, v)$. Note that $G_A$ and $G_B$ are COGs and let $A = A' \cup G_A$ and $B = B' \cup G_B$. The graphs $A$ and $B$ covering all the edges of $T^*_{d_x a_y}$ and having no common edge, they form a partition of $T^*_{d_x a_y}$. We now prove that these graphs are COGs and that they verify Property 14.

(a) The partition of $v$'s neighborhood being extendable, and the neighborhood of the other $f$-vertices of $T^*_{d_x a_y}$ being partitioned as in $T^*_{d_z a_w}$, the

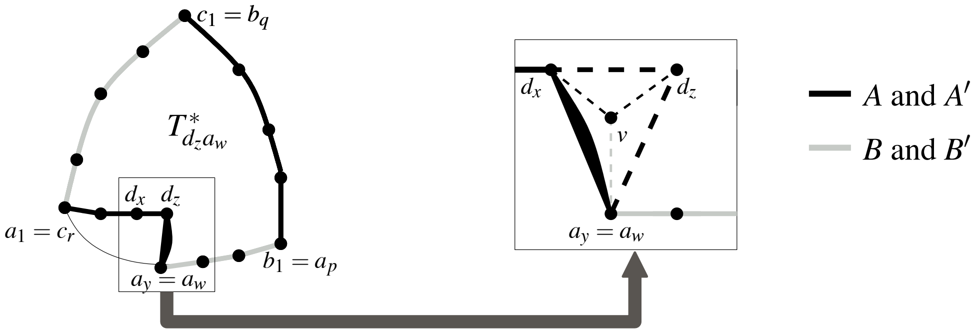


Fig. 31. Case 2.2.2.

partition of $T^*_{d_x a_y}$ into $A$ and $B$ is extendable.

The intersection of $A'$ and $G_A$ is the path $(d_x, d_z, a_y)$. The edge $d_x d_z$ is a bridge of $A'$ and the edge $d_z a_y$ is a side of both $A'$ and $G_A$. So Lemma 12 implies that $A = A' \cup G_A$ is a COG that fulfills points (b), (d), (e), and (g) of Property 14. Indeed:

(b) Since $A'$ and $G_A$ are connected (*c.f.* (b')), $A$ is connected.
(e) The edge $d_x a_y$ being a side of $G_A$, it is a side of $A$.
(d)(g) The edge $a_1 d_s$, the edges $d_i d_{i+1}$, for $i \geq x$ and the edges $b_i b_{i+1}$ being bridges of $A'$ (*c.f.* (d') and (g')), these edges are bridges of $A$.

The COGs $B'$ and $G_B$ intersect on $a_y$, so Lemma 9 implies that $B = B' \cup G_B$ is a COG that fulfills points (c), (f), and (h) of Property 14. Indeed:

(c) Since $G_B$ is connected and since $B'$ has two connected components, one containing $b_1$ and one containing $b_q$ (*c.f.* (c')), $B$ has two connected components, one containing $b_1$ and one containing $b_q$.
(f)(h) The edges $a_i a_{i+1}$, for $i \geq y$, and the edges $c_i c_{i+1}$ being bridges of $B'$ (*c.f.* (f') and (h')), these edges are bridges of $B$.

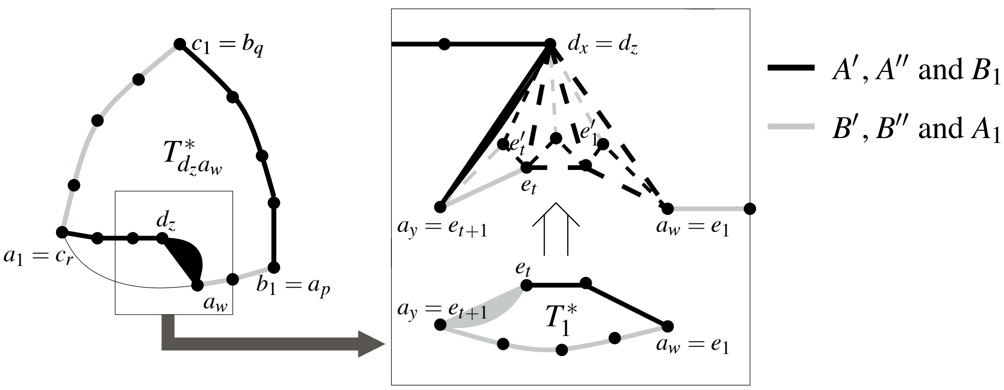


Fig. 32. Case 2.2.3.

**Case 2.2.3: $d_z = d_x$, and $w > y + 1$ (see Figure 32).** Let $e_1, e_2, \ldots, e_t, e_{t+1}$

be the neighbors of $d_x$ in $T$ and inside the cycle $(d_x, a_y, \dots, a_w)$ going from $a_w$ to $a_y$ included. This implies that $e_1 = a_w$, $e_{t+1} = a_y$, and $t \geq 2$. For each $i \in \{1, \dots, t\}$, let $e'_i$ be the $f$-vertex of $T^*$ adjacent to $d_x$, $e_i$, and $e_{i+1}$.

Let $G_A$ be the connected COG which edges are the edges $e_ie_{i+1}$, for $1 \leq i < t$, the edges $d_xe_i$, for $1 \leq i \leq t+1$, the edges $e_ie'_i$, for $1 \leq i \leq t$, the edges $e'_ie_{i+1}$, for $1 \leq i < t$, and the edge $d_xe'_t$. Since the intersection of $A'$ and $G_A$, the edge $d_xa_w$, is a side in both of these COGs (*c.f.* (e')), Lemma 11 implies that $A'' = A' \cup G_A$ is a COG:

(a") that is connected (*c.f.* (b')),
(b") which edge $a_1d_s$ and edges $d_id_{i+1}$, for $i \geq z$, are bridges (*c.f.* (d')),
(c") which edge $d_xa_y$ is a side, and
(d") which edges $b_ib_{i+1}$ are bridges (*c.f.* (g')).

Let $G_B$ be the COG which is the union of the path $(e'_t, e_t, e_{t+1})$ and the star with edges $d_xe'_i$, for $1 \leq i < t$. Since $B'$ and $G_B$ intersect on $d_x$ Lemma 9 implies that $B'' = B' \cup G_B$ is a COG:

(e") having three connected components, one containing $a_w$ and $a_p$, one containing $b_q$, and one containing the edge $a_ye_t$ (*c.f.* (c') and (f')),
(f") which edge $a_ye_t$ is a bridge,
(g") which edges $a_ia_{i+1}$, for $i \geq w$, are bridges (*c.f.* (f')), and
(h") which edges $c_ic_{i+1}$ are bridges (*c.f.* (h')).

Consider now the W-triangulation $T_1$ delimited by $(a_y, \dots, a_w, e_2, \dots, e_t)$. We have already seen that this graph has less edges than $T_{d_xa_y}$ and is 3-bounded by $(e_t, e_{t+1})$-$(a_y, \dots, a_w)$-$(e_1, \dots, e_t)$. Thus Property 13 holds for $T_1$ with the mentioned 3-boundary. This implies that there exists a partition of $T_1^*$ into $A_1 = (V(T_1^*), E(A_1))$ and $B_1 = (V(T_1^*), E(B_1))$ such that:

(a1) the partition of $T_1^*$ is extendable,
(b1) $A_1$ is connected,
(c1) $B_1$ has exactly two connected components, one containing $a_y$ and one containing $a_w$,
(d1) the edge $a_ye_t$ is a side of $A_1$,
(f1) the edges $a_ia_{i+1}$, for $y \leq i < w$, are bridges of $A_1$, and
(g1) the edges $e_ie_{i+1}$, for $1 \leq i < t$, are bridges of $B_1$.

Let $A = A'' \cup B_1$ and $B = B'' \cup A_1$. The graphs $A$ and $B$ covering all the edges of $T^*_{d_xa_y}$ and having no common edge, they form a partition of $T^*_{d_xa_y}$. We now prove that these graphs are COGs and that they verify Property 14.

(a) The partition of $e'_i$'s neighborhoods being extendable, and the neighborhood of the other $f$-vertices of $T^*_{d_xa_y}$ being partitioned as in $T^*_{d_za_w}$ or as in $T_1^*$, the partition of $T^*_{d_xa_y}$ into $A$ and $B$ is extendable.

The COGs $A''$ and $B_1$ intersect on the path $(e_1, e_2, \ldots, e_t)$ and on the vertex $a_y$. $B_1$ has two connected components, one containing the path $(e_1, e_2, \ldots, e_t)$ and one containing the vertex $a_y$ (*c.f.* (c1) and (g1)). We consider the union of $A''$ and $B_1$ as two successive unions, one for each connected component of $B_1$. For the union concerning the connected component of $B_1$ containing the path $(e_1, e_2, \ldots, e_t)$, the edges of this path being bridges of $B_1$, we apply Lemma 10. For the union concerning the other connected component of $B_1$ we apply Lemma 9. Lemma 10 and Lemma 9 imply that $A = A'' \cup B_1$ is a COG that fulfills points (b), (d), (e), and (g) of Property 14. Indeed:

- (b) Since $A''$ is connected (*c.f.* (a")) and since $B_1$ has two connected components, one containing the vertex $a_y$ and one containing the path $(e_1, \ldots, e_t)$ (*c.f.* (c1) and (g1)), $A$ is connected.
- (e) The edge $d_x a_y$ being a side of $A''$ (*c.f.* (c")), it is a side of $A$.
- (d)(g) The edge $a_1 d_s$, the edges $d_i d_{i+1}$, for $i \geq x$, and the edges $b_i b_{i+1}$ being bridges of $A''$ (*c.f.* (b") and (d")), these edges are bridges of $A$.

The COGs $B''$ and $A_1$ intersect on the edge $a_y e_t$ and on the vertex $a_w$. $B''$ has three connected components, one containing the edge $a_y e_t$, one containing $a_w$ and one other (*c.f.* (e")). We consider the union of $B''$ and $A_1$ as two successive unions, one with the connected component of $B''$ containing the edge $e_t a_y$, and one with the rest of the graph $B''$. For the first union, the edge $e_t a_y$ being a bridge of $B''$ (*c.f.* (f")), we apply Lemma 10. For the second union, the intersection being the vertex $a_w$ we apply Lemma 9. Lemma 10 and Lemma 9 imply that $B = B'' \cup A_1$ is a COG that fulfills points (c), (f), and (h) of Property 14. Indeed:

- (c) Since $A_1$ is connected (*c.f.* (b1)) and since $B''$ has three connected components, one containing the edge $a_y e_t$, one containing $a_w$ and $b_1$, and one containing $b_q$ (*c.f.* (e") and (g")), $B$ has two connected components, one containing $b_1$ and one containing $b_q$.
- (f)(h) The edges $a_i a_{i+1}$, for $i \geq y$, and the edges $c_i c_{i+1}$ being bridges of $A_1$ or $B'$ (*c.f.* (f1), (g") and (h")), these edges are bridges of $B$.

**Case 2.2.4:** $z = x-1$**, and** $1 < y < w$ **(see Figure 33).** Let $e_1, e_2, \ldots, e_t, e_{t+1}$ (resp. $f_1, f_2, \ldots, f_u, f_{u+1}, f_{u+2}$) be the neighbors of $d_z$ (resp. $d_x$) in $T$ and inside the cycle $(d_z, d_x, a_y, \ldots, a_w)$ going from $a_w$ to $d_x$ (resp. from $a_y$ to $d_z$) included. This implies that $e_1 = a_w$, $e_t = f_{u+1}$, $e_{t+1} = d_x$, $f_1 = a_y$, $f_{u+2} = d_z$, $t \geq 2$, and $u \geq 1$. For each $i \in \{1, \ldots, t\}$ (resp. $i \in \{1, \ldots, u\}$), let $e'_i$ (resp. $f'_i$) be the $f$-vertex of $T^*$ adjacent to $d_z$, $e_i$, and $e_{i+1}$ (resp. $d_x$, $f_i$, and $f_{i+1}$).

Let $G_A$ be the connected COG which edges are the edges $e_i e_{i+1}$, for $1 \leq i \leq t$, the edges $d_z e_i$, for $1 \leq i \leq t+1$, the edges $e_i e'_i$, for $1 \leq i < t$, the edges $e'_i e_{i+1}$, for $1 \leq i < t$, the edges $d_x e'_t$ and $d_z e'_t$, the edges $f_i f_{i+1}$, for $1 \leq i < u$, the

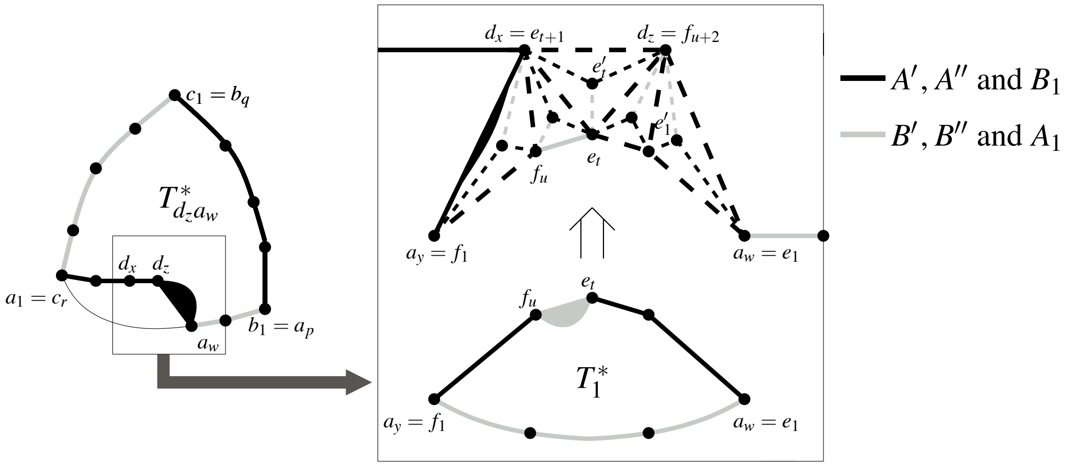


Fig. 33. Case 2.2.4.

edges $d_x f_i$, for $1 \leq i \leq u$, the edges $f_i f'_i$, for $1 \leq i < u$, the edges $f'_i f_{i+1}$, for $1 \leq i \leq u$, and the edge $d_x f'_u$. The intersection of $A'$ and $G_A$ is the path $(d_x, d_z, a_w)$ which edge $d_x d_z$ is a bridge of $A'$ and which edge $d_z a_w$ is a side of both $A'$ and $G_A$ (*c.f.* (d') and (e')). So Lemma 12 implies that $A'' = A' \cup G_A$ is a COG:

(a") that is connected (*c.f.* (a')),
(b") which edge $a_1 d_s$ and edges $d_i d_{i+1}$, for $i \geq x$, are bridges (*c.f.* (d')),
(c") which edge $d_x a_y$ is a side, and
(d") which edges $b_i b_{i+1}$ are bridges (*c.f.* (g')).

Let $G_B$ be the COG which edges are the edges $d_z e'_i$, for $1 \leq i < t$, the edges $d_x f'_i$, for $1 \leq i < u$, and the edges $f_u e_t$, $f_u f'_u$, and $e_t e'_t$. The intersection of $B'$ and $G_B$, the vertices $d_x$ and $d_z$, are in two distinct connected components of $G_B$, so Lemma 9 implies that $B'' = B' \cup G_B$ is a COG:

(e") having three connected components, one containing $a_w$ and $b_1$, one containing $b_q$ and one containing the edge $f_u e_t$ (*c.f.* (c') and (f')),
(f") which edge $f_u e_t$ is a bridge,
(g") which edges $a_i a_{i+1}$, for $i \geq w$, are bridges (*c.f.* (f')), and
(h") which edges $c_i c_{i+1}$ are bridges (*c.f.* (h')).

Consider now the W-triangulation $T_1$ delimited by $(a_y, \ldots, a_w, e_2, \ldots, e_t, f_u, \ldots, f_2)$. We have already seen that this graph has less edges than $T_{d_x a_y}$ and is 3-bounded by $(e_t, f_u, \ldots, f_1)$-$(a_y, \ldots, a_w)$-$(e_1, \ldots, e_t)$. Thus Property 13 holds for $T_1$ with the mentioned 3-boundary. This implies that there exists a partition of $T^*_1$ into $A_1 = (V(T^*_1), E(A_1))$ and $B_1 = (V(T^*_1), E(B_1))$ such that:

(a1) the partition of $T^*_1$ is extendable,
(b1) $A_1$ is connected,
(c1) $B_1$ has exactly two connected components, one containing $a_y$ and one

containing $a_w$,

(d1) the edge $f_u e_t$ is a side of $A_1$,

(e1) the edges $f_i f_{i+1}$, for $1 \leq i < u$, are bridges of $B_1$,

(f1) the edges $a_i a_{i+1}$, for $y \leq i < w$, are bridges of $A_1$, and

(g1) the edges $e_i e_{i+1}$, for $1 \leq i < t$, are bridges of $B_1$.

Let $A = A'' \cup B_1$ and $B = B'' \cup A_1$. The graphs $A$ and $B$ covering all the edges of $T^*_{d_x a_y}$ and having no common edge, they form a partition of $T^*_{d_x a_y}$. We now prove that these graphs are COGs and that they verify Property 14.

(a) The partition of $e'_i$'s and $f'_j$'s neighborhoods being extendable, and the neighborhood of the other $f$-vertices of $T^*_{d_x a_y}$ being partitioned as in $T^*_{d_z a_w}$ or as in $T^*_1$, the partition of $T^*_{d_x a_y}$ into $A$ and $B$ is extendable.

The COGs $A''$ and $B_1$ intersect on the paths $(e_1, e_2, \ldots, e_t)$ and $(f_1, f_2, \ldots, f_u)$. $B_1$ has two connected components, one containing the path $(e_1, e_2, \ldots, e_t)$ and one containing the path $(f_1, f_2, \ldots, f_u)$ (*c.f.* (c1), (e1), and (g1)). We consider the union of $A''$ and $B_1$ as a succession of two unions in which the graphs intersect on one path. All the edges of these paths being bridges of $B_1$ (*c.f.* (e1) and (g1)), we apply Lemma 10 to each of these unions and this implies that $A = A'' \cup B_1$ is a COG that fulfills points (b), (d), (e), and (g) of Property 14. Indeed:

(b) Since $A''$ is connected (*c.f.* (a")) and since $B_1$ has two connected components, one containing the path $(e_1, e_2, \ldots, e_t)$ and one containing the path $(f_1, f_2, \ldots, f_u)$ (*c.f.* (c1), (e1), and (g1)), $A$ is connected.

(e) The edge $d_x a_y$ being a side of $A''$ (*c.f.* (c")), it is a side of $A$.

(d)(g) The edge $a_1 d_s$, the edges $d_i d_{i+1}$, for $i \geq x$, and the edges $b_i b_{i+1}$ being bridges of $A''$ (*c.f.* (b") and (d")), these edges are bridges of $A$.

The COGs $B''$ and $A_1$ intersect on the edge $e_t f_u$ and on the vertex $a_w$. $B''$ has three connected components, one containing the edge $e_t f_u$, one containing the vertex $a_w$ and another one (*c.f.* (e")). We consider the union of $B''$ and $A_1$ as a succession of two unions, one with the connected component of $B''$ containing the edge $e_t f_u$, and one with the rest of $B''$. In the first union, the edge $e_t f_u$ being a bridge of $B''$ (*c.f.* (f")), we apply Lemma 10. In the second union, the intersection of the graphs being the vertex $a_w$, we apply Lemma 9. These two lemmas imply that $B = B'' \cup A_1$ is a COG that fulfills points (c), (f), and (h) of Property 14. Indeed:

(c) Since $A_1$ is connected (*c.f.* (b1)) and since $B''$ has three connected components, one containing the edge $e_t f_u$, one containing $a_w$ and $b_1$, and one containing $b_q$ (*c.f.* (e")and (g")), $B$ has two connected components, one containing $b_1$ and one containing $b_q$,

(f)(h) The edges $a_i a_{i+1}$, for $i \geq y$, and the edges $c_i c_{i+1}$ being bridges of $A_1$ or $B''$ (*c.f.* (f1), (g") and (h")), these edges are bridges of $B$.

This concludes the Case 2 of the induction and so the joint proof of Property 13 and Property 14.

## 5 Partition of triangulations: Proof of Theorem 1

### *5.1 The case of 4-connected triangulations*

Let $T$ be a 4-connected triangulation with outer-vertices $a$, $b$, and $c$. Since Property 13 applies to $T$ according to $(a,b)$-$(b,c)$-$(c,a)$, let $A$ and $B$ be the two COGs that form an extendable partition of $T^*$. This partition induces a partition of $T$ into $A'$ and $B'$ which respectively correspond to the graphs $A$ and $B$ where the $f$-vertices are deleted. Since the partition of $T^*$ is extendable, the $f$-vertices are either vertices of degree one in $A$ (resp. $B$) or vertices of degree two in a 3-cycle of $A$ (resp. $B$). So Lemma 7 and Lemma 8 imply that $A'$ and $B'$ are two COGs.

**The bipartition is hamiltonian.** Property 13 and Property 14 are closely related to Property 4 and Property 5, their proofs clearly use the same induction scheme. The reader can observe that by merging these proofs we obtain a proof of the following two properties.

**Property 15** *Given any 3-bounded W-triangulation $T$ and any of its 3-boundaries, Property 13 and Property 4 hold. Moreover, the path $P$ (going from $b_1$ to $b_q$, two vertices on $T$'s outer-boundary) divides $T$ into two parts (say the right and the left according to our figures) in such a way that the edges of $A' = A \cap T$ (resp. $B' = B \cap T$) are on $P$ or on its right (resp. on $P$ or on its left).*

**Property 16** *Given any $T_{d_x a_y}$, Property 14 and Property 5 hold. Moreover, the paths $P$ and $Q$ (being disjoint and both having their ends on $T_{d_x a_y}$'s outer-boundary) divide $T_{d_x a_y}$ into three parts (say the middle and the sides) in such a way that the edges of $A \cap T_{d_x a_y}$ (resp. $B \cap T_{d_x a_y}$) are either on $P$, on $Q$ or in the middle (resp. on $P$, on $Q$ or in one of the sides).*

Property 15 implies that in a 4-connected triangulation $T$ 3-bounded by $(a,b)$-$(b,c)$-$(c,a)$, there is a partition of $T$ into the COGs $A'$ and $B'$ such that the edges of $A'$ (resp. $B'$) are on or inside (resp. on or outside) the hamiltonian cycle formed by $P$ and the edge $bc$.

**$A'$ and $B'$ are $S$-free.** Recall that $S$ is the cycle $(x_1, y_1, x_2, y_2, x_3, y_3)$ with chords $y_1y_2$, $y_1y_3$, and $y_2y_3$ (see Figure 1). If $S$ was a subgraph of $A'$, $T$ having no separating 3-cycle, the cycle $(y_1, y_2, y_3)$ of $S$ would bound a face of $T$. This face could not be the outer-face since $ab \in A$ and $ac \in B$. So let $v$ be the

$f$-vertex of $T^*$ inside the cycle $(y_1, y_2, y_3)$. The partition of $T^*$ into $A$ and $B$ being extendable and the three edges $y_1y_2$, $y_1y_3$, and $y_2y_3$ being in $A$, the support edge of $v$, say $y_1y_2$, belongs to $A$. This implies that the edges $vy_1$ and $vy_2$ belongs to $A$ and so that the edges $y_1x_2$, $y_2x_2$, $y_1v$, $y_2v$, $y_1y_3$, and $y_2y_3$, which form a $K_{2,3}$, all belong to $A$. This is impossible since outerplanar graphs are $K_{2,3}$-minor free. Similarly, $B'$ is $S$-free.

Thus Theorem 1 holds for 4-connected triangulations.

### 5.2 *The case of general triangulations*

Now let $T$ be a triangulation having a separating 3-cycle $(a, b, c)$. Let $T_{int}$ (resp. $T_{ext}$) be the triangulation induced by the vertices on and inside (resp. on and outside) the cycle $(a, b, c)$. Assume that $T_{int}$ (resp. $T_{ext}$) has an edge-partition into two outerplanar graphs, $A_{int}$ and $B_{int}$ (resp. $A_{ext}$ and $B_{ext}$). For $A_{ext}$, $B_{ext}$, $A_{int}$ and $B_{int}$ being such that the graphs $A_{ext} \cup A_{int}$ and $B_{ext} \cup B_{int}$ are two outerplanar graphs that cover $T$, they have to verify some properties allowing a gluing along the cycle $(a, b, c)$. Since the cycle $(a, b, c)$ bounds an inner-face of $T_{ext}$, the partition of $T_{ext}$ into $A_{ext}$ and $B_{ext}$ has to verify some properties for each inner-face of $T_{ext}$. Similarly since $(a, b, c)$ bounds the outer-boundary of $T_{int}$, the partition of $T_{int}$ into $A_{int}$ and $B_{int}$ has to verify some properties around the outer-face of $T_{int}$.

**Property 17** *Given a triangulation $T$ with outer-face abc, there is an edge partition of $T^*$ into two COGs $A = (V(T^*), E(A))$ and $B = (V(T^*), E(B))$ (see Figure 34), such that:*

*(a) the partition is extendable,*
*(b) $A$ is connected,*
*(c) $B$ has exactly two connected components, one containing $b$ and one containing $c$,*
*(d) the edge $ab$ is a side of $A$,*
*(e) the edge $bc$ is a bridge of $A$, and*
*(f) the edge $ac$ is a bridge of $B$.*

This property clearly implies Theorem 1 for general triangulations.

**PROOF of Property 17.** Let $T$ be any triangulation with outer-face *abc*. We proceed by induction on the number of separating 3-cycles in $T$. If $T$ has no separating 3-cycle (i.e. $T$ is 4-connected) we apply Property 13 to $T$ for the 3-boundary $(a, b)$-$(b, c)$-$(c, a)$. It is easy to see that the obtained partition of $T^*$ fulfills Property 17.

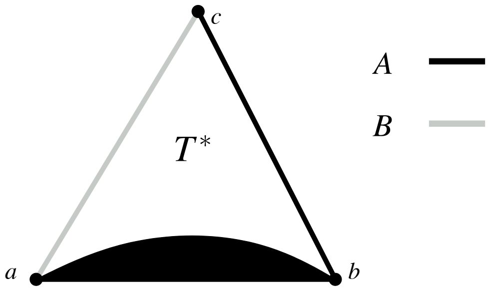


Fig. 34. Property 17.

If $T$ has a separating 3-cycle $C$, let $T_{ext}$ and $T_{int}$ be the triangulations respectively induced by the vertices on and outside $C$ and by the vertices on and inside $C$. The cycle $C$ is no more a separating 3-cycle in $T_{ext}$ or $T_{int}$. So both $T_{ext}$ and $T_{int}$ have less separating 3-cycles than $T$. Then by induction hypothesis Property 17 applies to both $T^*_{ext}$ and $T^*_{int}$.

We apply the induction hypothesis to $T_{ext}$ and obtain a partition of $T^*_{ext}$ into two COGs $A_e = (V(T^*_{ext}), E(A_e))$ and $B_e = (V(T^*_{ext}), E(B_e))$ such that:

($a_e$) the partition is extendable,
($b_e$) $A_e$ is connected,
($c_e$) $B_e$ has exactly two connected components, one containing the vertex $b$ and one containing $c$,
($d_e$) the edge $ab$ is a side of $A_e$,
($e_e$) the edge $bc$ is a bridge of $A_e$, and
($f_e$) the edge $ac$ is a bridge of $B_e$.

Let $v$ be the $f$-vertex inside the face delimited by $C$ in $T^*_{ext}$. The partition of $T^*_{ext}$ being extendable, it is possible to denote the vertices of $C$ by $a'$, $b'$, and $c'$, so that the support edge of $v$ is $a'b'$. Without loss of generality let $a'b' \in E(A_e)$. This implies that $va'$ and $vb' \in E(A_e)$ and that $vc' \in E(B_e)$. We now apply the induction hypothesis to the triangulation $T_{int}$ with outer-face $a'b'c'$ and we obtain a partition of $T^*_{int}$ into two COGs $A_i = (V(T^*_{int}), E(A_i))$ and $B_i = (V(T^*_{int}), E(B_i))$ such that:

($a_i$) the partition is extendable,
($b_i$) $A_i$ is connected,
($c_i$) $B_i$ has exactly two connected components, one containing the vertex $b'$ and one containing $c'$,
($d_i$) the edge $a'b'$ is a side of $A_i$,
($e_i$) the edge $b'c'$ is a bridge of $A_i$, and
($f_i$) the edge $a'c'$ is a bridge of $B_i$.

We now define the partition of $T^*$ into $A$ and $B$ by $A = (A_e \setminus \{v\}) \cup (A_i \setminus \{a'c', b'c'\})$ and $B = (B_e \setminus \{v\}) \cup (B_i \setminus \{a'c', b'c'\})$ (see Figure 35). In the case $a'b' \in E(B_e)$, we would have $A = (A_e \setminus \{v\}) \cup (B_i \setminus \{a'c', b'c'\})$ and

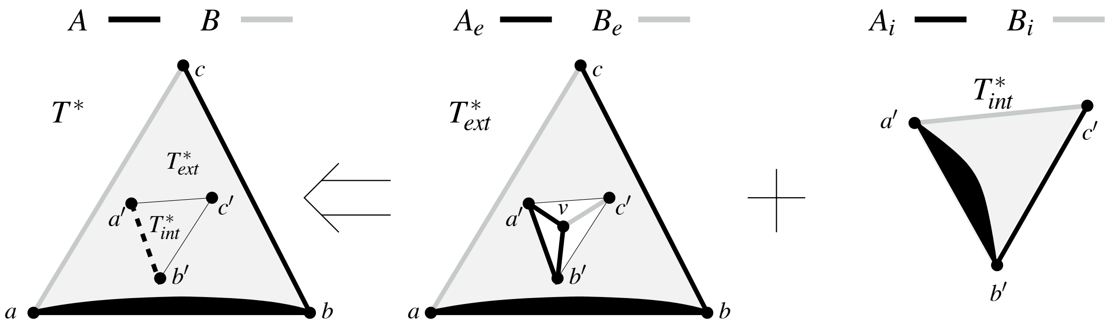


Fig. 35. The COGs $A$ and $B$.

$B = (B_e \setminus \{v\}) \cup (A_i \setminus \{a'c', b'c'\})$. The graphs $A$ and $B$ form a partition of $T^*$. Indeed:

- the cycle $C = (a', b', c')$ does not bound any face of $T$, so there is no $f$-vertex $v$ and no edges $va'$, $vb'$, and $vc'$ in $T^*$; and
- the edges $a'c'$ and $b'c'$ are covered by $A_e$ or $B_e$.

Let $A'_e = A_e \backslash v$. By Lemma 8, the graph $A'_e$ is a COG:

(e1) that is connected,
(e2) which edge $ab$ is a side,
(e3) which edge $bc$ is a bridge, and
(e4) which edge $a'b'$ is a side.

Let $A'_i = A_i \backslash \{a'b', b'c'\}$ which equals to $A_i \backslash \{b'c'\}$ since $a'c' \notin E(A_i)$. By Lemma 7, the graph $A'_i$ is a COG:

(i1) having two connected components, one containing $c'$ and one containing the edge $a'b'$, and
(i2) which edge $a'b'$ is a side.

Let $B'_e = B_e \backslash v$. By Lemma 7, the graph $B'_e$ is a COG:

(e5) having two connected components, one containing $b$ and one containing $c$, and
(e6) which edge $ac$ is a bridge.

Let $B'_i = B_i \backslash \{a'c', b'c'\}$ which equals to $B_i \backslash \{a'c'\}$ since $b'c' \notin E(B_i)$. By Lemma 7, the graph $B'_i$ is a COG:

(i3) having three connected components, one containing $a'$, one containing $b'$, and one containing $c'$.

We prove now that $A = A'_e \cup A'_i$ and $B = B'_e \cup B'_i$ are COGs that fulfill the property.

**The partition of $T^*$ is extendable.** Most of the $f$-vertices of $T^*$ have their neighborhood partitioned as in $T^*_{int}$ or $T^*_{ext}$. The only $f$-vertices for which this may not be the case are the $f$-vertex $v_1$ of $T^*_{int}$ adjacent to $b'$ and $c'$, and the $f$-vertex $v_2$ of $T^*_{int}$ adjacent to $a'$ and $c'$. According to ($e_i$) (resp. ($f_i$)), the edge $b'c'$ (resp. $c'a'$) is a bridge of $A_i$ (resp. $B_i$), so the support edge of $v_1$ (resp. $v_2$) is not $b'c'$ (resp. $a'c'$). In such case there would be a cycle $(v_1, b', c') \in A_i$ (resp. $(v_2, c', a') \in B_i$) and the edge $b'c'$ (resp. $a'c'$) would not be a bridge. So the edges incident to $v_1$ (resp. $v_2$) and its support edge are partitonned as in $T^*_{int}$ and the partition of $v_1$'s (resp. $v_2$'s) neighborhood is extendable. Thus point (a) of the property holds.

**The graph $A$ is a COG.** We consider the union of $A'_e$ and $A'_i$ as two successive unions. At each step we consider one of the connected components of $A'_i$. We begin with the union of $A'_e$ and the connected component of $A'_i$ containing $a'b'$. These two graphs intersect on $a'b'$. The edge $a'b'$ being a side in both of these COGs (*c.f.* (e4) and (i2)), Lemma 11 applies. Since this graph and the connected component of $A'_i$ containing the vertex $c'$ intersect on $c'$, Lemma 9 applies. Lemma 11 and Lemma 9 imply that the graph $A$ is a COG that fulfills points (b), (d), and (e) of the property:

(b) Since $A'_e$ is connected (*c.f.* (e1)) and since $A'_i$ has two connected components, one containing $c'$ and one containing $a'b'$ (*c.f.* (i1)), $A$ is connected.
(d) If $ab \neq a'b'$, the edge $ab$ being a side of $A'_e$ (*c.f.* (e2)), it is a side of $A$. If $ab = a'b'$, the edge $ab$ being a side of $A_e$ (*c.f.* ($d_e$)), it is a bridge of $A'_e$. In this case, by applying Lemma 10 instead of Lemma 11, since $a'b'$ is a side of $A'_i$ we obtain that $ab$ is a side of $A$.
(e) Since $bc \neq a'b'$ (the support edge of $v$ cannot be a bridge), the edge $bc$ being a bridge of $A'_e$ (*c.f.* (e3)), it is a bridge of $A$.

**The graph $B$ is a COG.** We consider the union of $B'_e$ and $B'_i$ as three successive unions. At each step we consider one of the connected components of $B'_i$. For each of these unions the two graphs intersect on a single vertex, $a'$, $b'$, or $c'$, so Lemma 9 applies at each step. Lemma 9 implies that the graph $B$ is a COG that fulfills points (c) and (f) of the property:

(c) Since $B'_e$ has two connected components, one containing $b$ and one containing $c$ (*c.f.* (e5)), and since $B'_i$ has three connected components, one containing $a'$, one containing $b'$, and one containing $c'$ (*c.f.* (i3)), $B$ has two connected components, one containing $b$ and one containing $c$.
(f) The edge $ac$ being a bridge of $B'_e$, it is a bridge of $B$ (*c.f.* (e6)).

This concludes the proof of Property 17 and so the proof of Theorem 1.

## 6 Conclusion

A maximum outerplanar graph on $n$ vertices having $2n-3$ edges and a planar graph on $n$ vertices having at most $3n-6$ edges, it could be that every planar graph contains $p$ outerplanar subgraphs such that each edge belongs to $q$ of them for some $p$ and $q$ verifying $\frac{3}{2} \leq \frac{p}{q} \leq 2$. For the case of bipartite planar graphs, since they have at most $2n-4$ edges, the integers $p$ and $q$ could be such that $1 \leq \frac{p}{q} \leq 2$. The bipartite planar graphs are so sparse that they are the union of two trees [16], which is two graphs trivially outerplanar. However Theorem 1 is optimal even for bipartite planar graphs.

**Theorem 18 ([8] p. 58)** *For any integers $p$ and $q$ with $\frac{p}{q} < 2$, the bipartite planar graph $K_{2,2p+1}$ has no $p$ outerplanar subgraphs covering each edge $q$ times.*

The proofs of Property 13, Property 14 and Property 17 being constructive, one could design an algorithm $\mathcal{A}$ with input a planar graph and with output two outerplanar graphs covering it. A planar graph $G$ having at most $3|V(G)|-6$ edges, we can construct a triangulation $T$ containing $G$ in linear time (*i.e.* $O(|V(G)|)$). Furthermore Richards [17] showed how to decompose a triangulation $T$ into 4-connected triangulations in linear time. Using convenient data structures it makes no doubt that $\mathcal{A}$ could be linear.

The decomposition technique used to prove Property 13 and Property 14 seems to be very ad hoc. Surprisingly, exactly the same decomposition technique allowed the author and J. Chalopin [2] to prove the following conjecture of Scheinerman [19].

**Conjecture 19** *Every planar graph is the intersection graph of a set of segments in the plane.*

In such intersection model of a graph, the vertex set is a set of segments and the edge set corresponds to the pairs of intersecting segments.

In [9] S. Gravier and C. Payan gave a reformulation of the Four Colour Theorem. In this reformulation they consider the outerplanar graph induced by the edges on and inside (resp. on and outside) a hamiltonian cycle in 4-connected triangulations. Theorem 1 implies a restriction on the graphs considered in this reformulation. These graphs are such that we can assign each edge on the hamiltonian cycle to one of the two graphs and obtain two $S$-free graphs.

It is shown in [5] that every graph embeddable on a surface $\mathbb{S}$ is coverable by two graphs with bounded tree-width. Can Theorem 1 be generalized to others surfaces? For each surface $\mathbb{S}$, a graph is *outer*-$\mathbb{S}$ if it admits an embedding on $\mathbb{S}$ with no crossing edges and such that all the vertices are incident to the same face, the *outer-face*. We propose the following conjecture.

**Conjecture 20** *Every graph embeddable on $\mathbb{S}$ is coverable by two outer-$\mathbb{S}$ graphs.*

This conjecture holds for 5-connected toroidal graphs (*i.e.* embeddable on the torus). Indeed Brunet and Richter [1] showed that 5-connected toroidal triangulations have a hamiltonian cycle separating the torus into two connected regions. Taking the edges of $C$ and the edges in one of the region we obtain an outer-toroidal graph. Indeed, all the vertices are incident to the same face, the face bounded by $C$. Another family of embedded graphs is known to be hamiltonian, the family of 4-connected projective-planar graphs [20]. However this result does not imply our conjecture for this family of graphs since the hamiltonian cycles obtained do not necessarily separate the projective plane into two connected regions. Note that Conjecture 20 could not be much strengthened (contradicting a conjecture proposed by the author [8]). Let $\mathbb{S}_g$ denotes the oriented surface of genus $g$.

**Theorem 21** *For every $g \geq 24$ such that $n \equiv 0 \pmod{12}$, there exists a graph embeddable in $\mathbb{S}_g$ that does not admit any edge partition into an outer-$\mathbb{S}_{g_1}$ graph and an outer-$\mathbb{S}_{g_2}$ graph when $g_1 + g_2 \leq \frac{5}{3}g$.*

**PROOF.**

**Claim 22** *Consider an outer-$\mathbb{S}_g$ graph $G$ with $n$ vertices, $m$ edges and $f$ faces. For every outer-$\mathbb{S}_g$ graph $G^+$ such that $G = G^+ \setminus V_2$, for some stable set $V_2 \subseteq \{v \in V(G^+) | d_{G^+}(v) = 2\}$, we have $|V_2| \leq 3n + 6g - m - 3$.*

Given any outer-$\mathbb{S}_g$ graph $G$, let $G^+$ and $V_2$ be such that $|V_2|$ is maximized. This clearly implies that $G^+$ is connected and thus around its outer-face we have a facial walk $W_o = (v_1, v_2, \ldots, v_l)$ of length $l$. Now let $G^*$ be the multigraph embedded in $\mathbb{S}_g$, obtained from $G^+$ by adding a new vertex $x$ incident to each occurrence in $W_o$. According to the construction, $G^*$ has $n^* = n + |V_2| + 1$ vertices and $m^* = m + 2|V_2| + l$ edges. Although $G^*$ is a multigraph, Euler's formula apply and since all its faces have length at least three we have that $m^* \leq 3n^* + 6(g - 1)$. Furthermore, since $G^+$ is an outer-$\mathbb{S}_g$ graph and since $V_2$ is a stable set, all the vertices in $V_2$ appear in $W_o$ and none of them are consecutive in this walk. Thus $2|V_2| \leq l$ and $m + 4|V_2| \leq m^* \leq 3(n + |V_2| + 1) + 6(g - 1)$ which implies the claim.

The Map Color Theorem says that a complete graph on $n$ vertices has an embedding in $\mathbb{S}_g$ if and only if $g \geq \frac{1}{12}(n-3)(n-4)$ [18]. For any $n > 0$ such that $n \equiv 0 \pmod{12}$ we have that $K_n$ has an embedding in $\mathbb{S}_g$, for $g = \frac{1}{12}(n-3)(n-4)$, that is a triangulation of $\mathbb{S}_g$. For these couples $(n, g)$, let $K_n^*$ be a stellation corresponding to a given embedding of $K_n$ in $\mathbb{S}_g$. This means that given this embedding of $K_n$ in $\mathbb{S}_g$ we add a vertex in every face and we link it to the three vertices incident to this face. Let $V_3$ in $K_n^*$ be the maximum stable set with vertices of degree 3. Note that since $K_n$ triangulates $\mathbb{S}_g$, $K_n$ has $\frac{2}{3}|E(K_n)|$ triangular faces and thus $|V_3| = \frac{1}{3}n(n-1)$.

Note that if we add pendent vertices in an outer-$\mathbb{S}_g$ graph this graph remains outer-$\mathbb{S}_g$. So given an edge partition of $K_n^*$ into an outer-$\mathbb{S}_{g_1}$ graph $H_1$ and an outer-$\mathbb{S}_{g_2}$ graph $H_2$, we can consider that every vertex of degree three in $K_n^*$ has degree one in $H_i$, for $i \in \{1,2\}$, and degree two in $H_{3-i}$. Since $|V_3| = \frac{1}{3}n(n-1)$, and since by Claim 22 $H_i$ has at most $3n+6g_i-|E(H_i)|-3$ vertices of degree two from $V_3$ (the remaining vertices of $V_3$ have degree one in $H_i$), $g_1$ and $g_2$ should be such that $6n+6(g_1+g_2)-m-6 \geq \frac{1}{3}n(n-1)$. Since this inequality does not hold when $\frac{5}{3}g \geq g_1+g_2$ and $n \geq 24$, this concludes the proof of the theorem.

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