Logics and Algorithms for Graph Minors

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AlGCo team

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Amphithéâtre Jean Jacques Moreau, 12/12/2023

Computation as a mathematical subject

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Study of automated computation by means of abstraction

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What makes a computational problem
inherently difficult?

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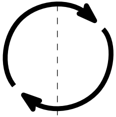

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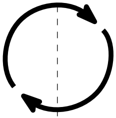

When can we have efficient algorithms?

Computation as a mathematical subject

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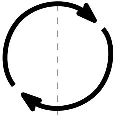
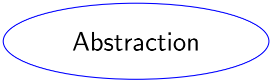


When can we have efficient algorithms?

The power of abstraction



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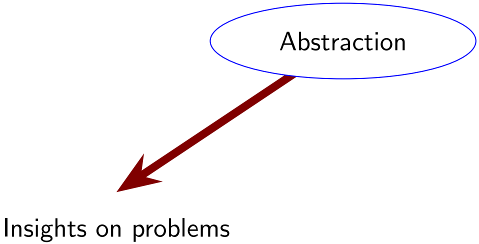

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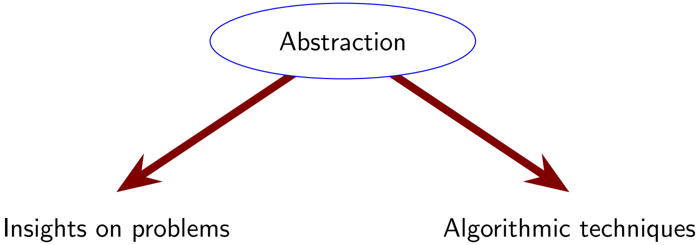

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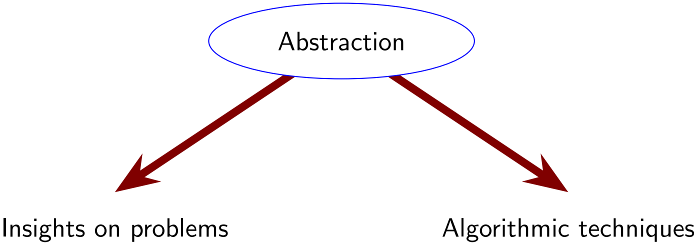

Common descriptive ground?

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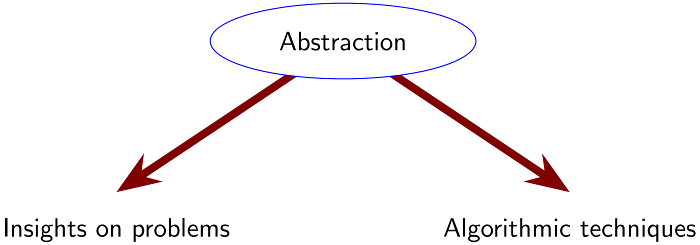

Common descriptive ground?

What is their potential?

Graphs and algorithms

- Model of abstraction: *Graphs*

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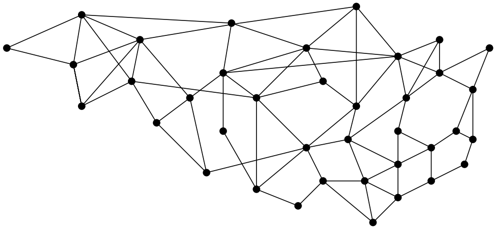

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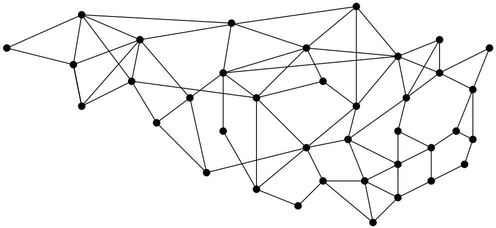

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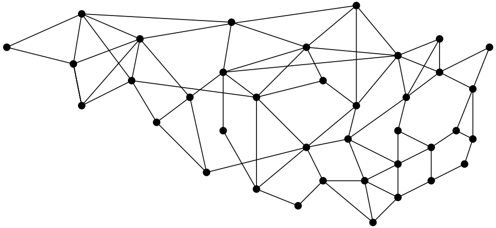

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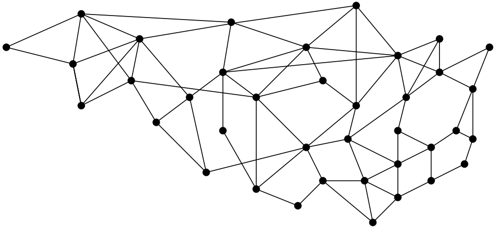
Graphs and algorithms



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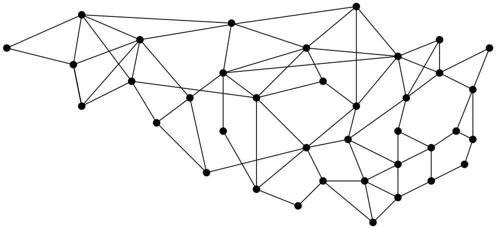

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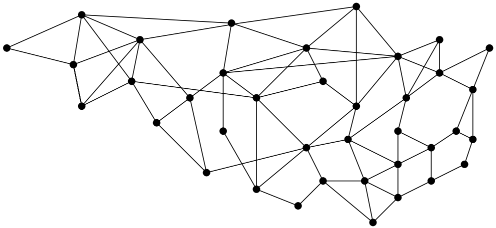

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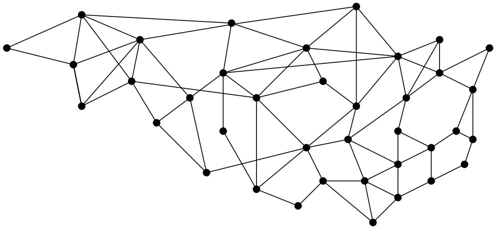
Graphs and algorithms and logic



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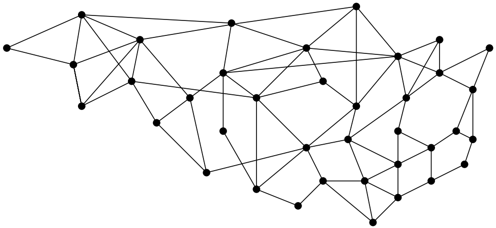

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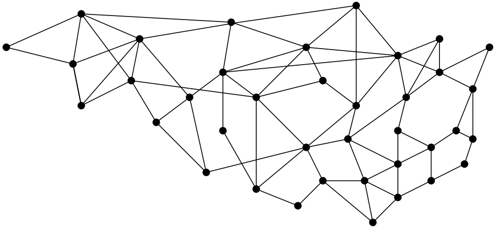

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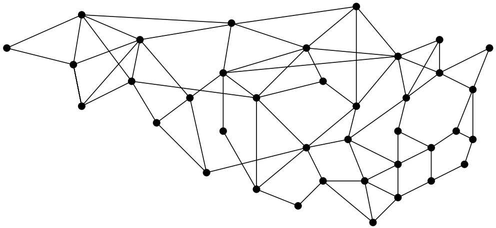

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▷ How to use the structure of the graph to obtain efficient algorithms ?

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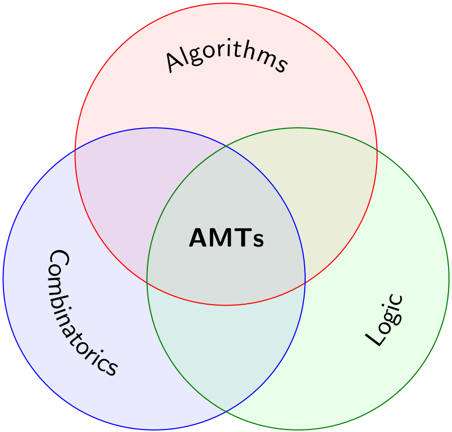
Algorithmic meta-theorems (**AMTs**):
General mathematical conditions that allow the automatic derivation of efficient algorithms.



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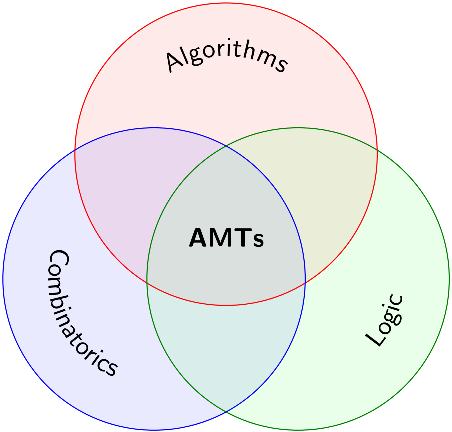

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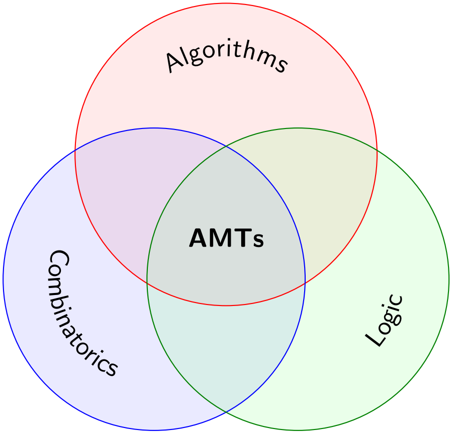

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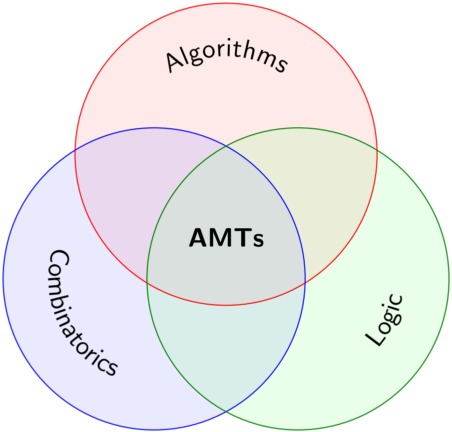

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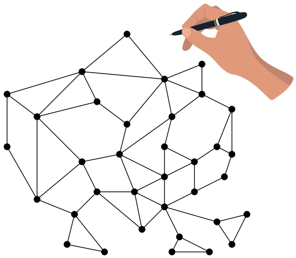

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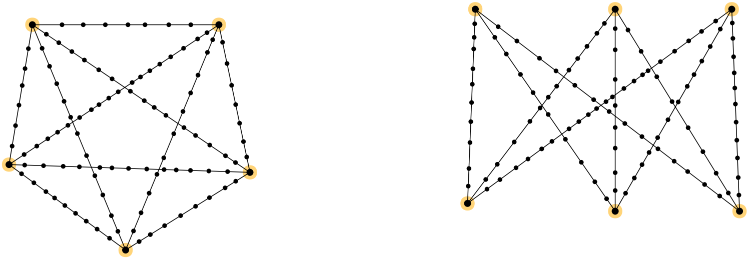
Kuratowski-Pontryagin theorem (1930):

G is planar $\iff$ G does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph.



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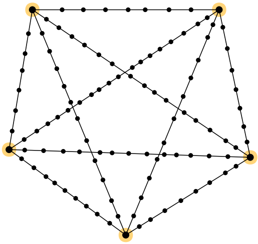

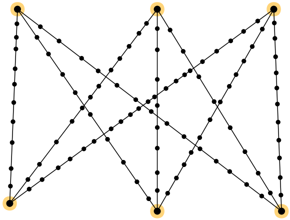

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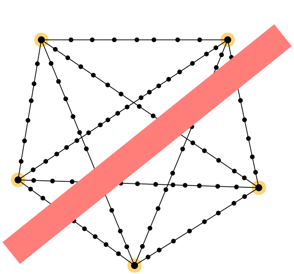

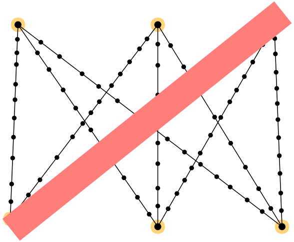

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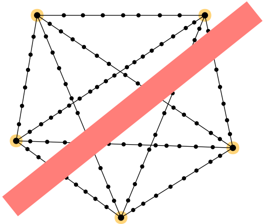

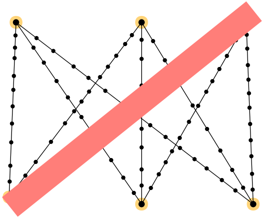

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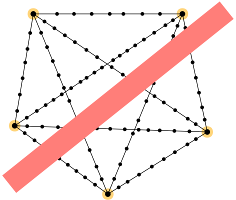

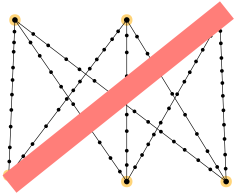

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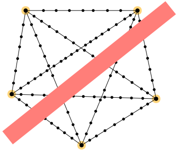

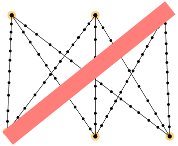

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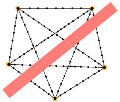

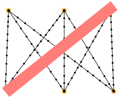

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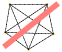

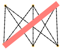

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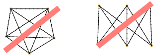

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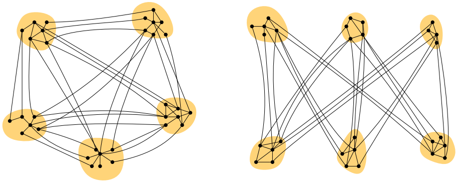

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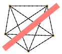

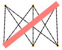

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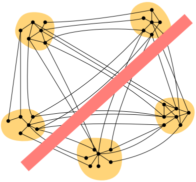

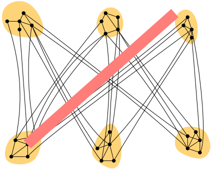

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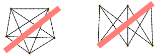

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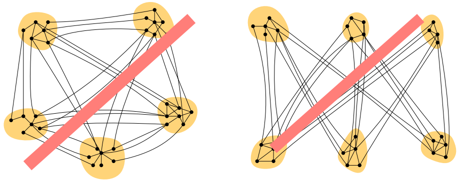

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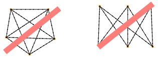

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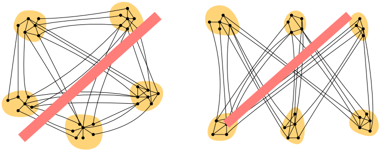

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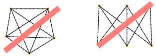

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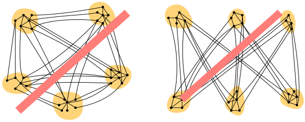

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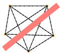

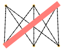

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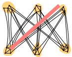

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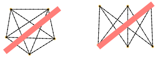

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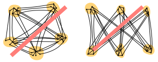

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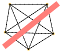
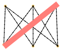

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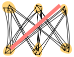

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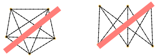

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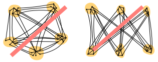

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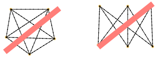

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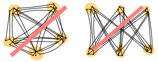

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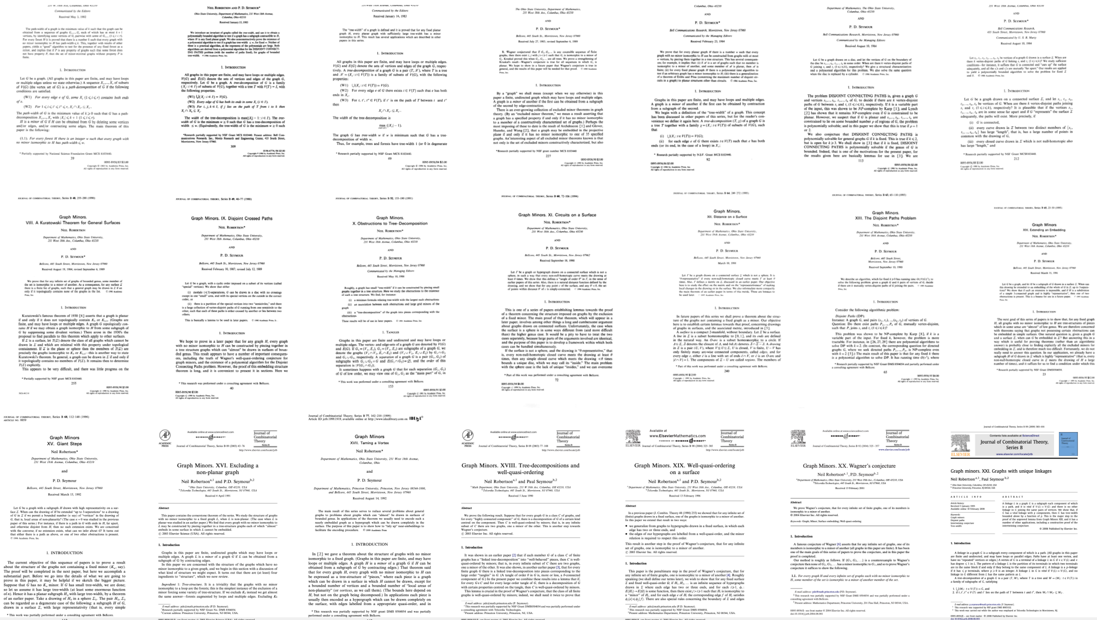
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- The proof of 1) is non-constructive (does not give the obstructions) and is not expected to be constructive (in general).

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Study of auxiliary measure *conditioning* the computational complexity of problems.

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`Dream`: Meta-algorithmic viewpoint on Parameterized Computation.

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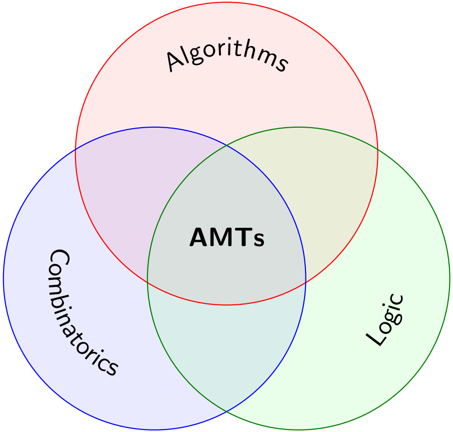
Our contribution:

- A unified meta-algorithmic framework on minor exclusion.
- Extension to classes excluding topological minors.

Logics and Algorithms for Graph Minors

The 3 components of AMTs



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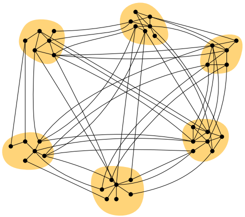

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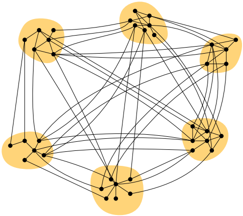

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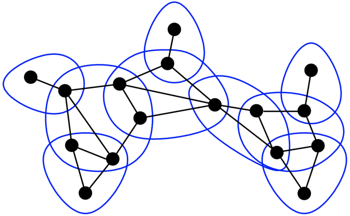

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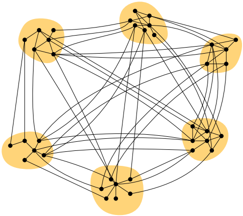

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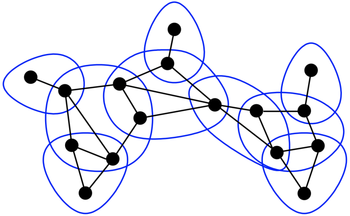

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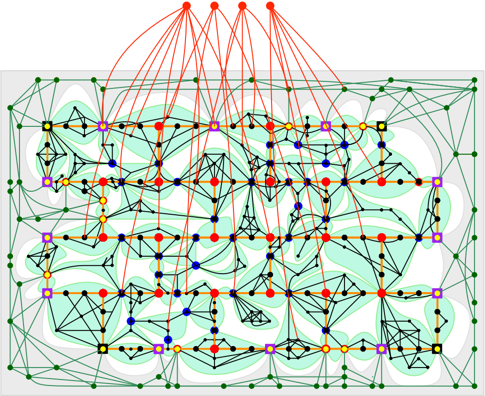

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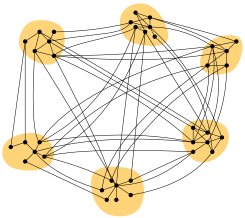

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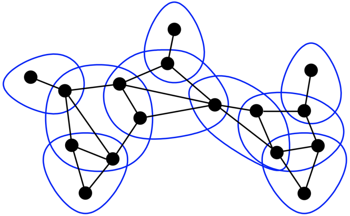

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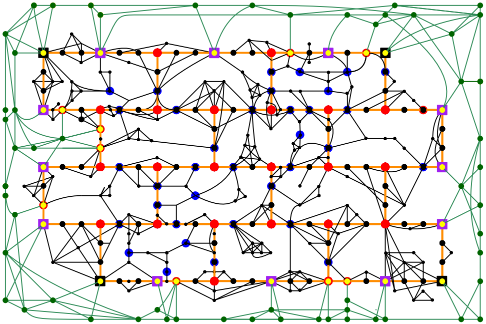

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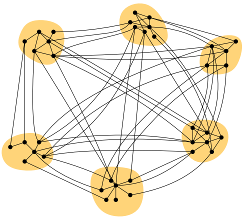

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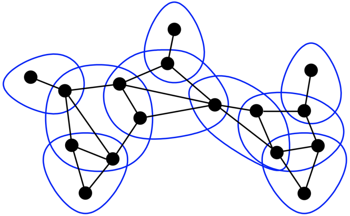

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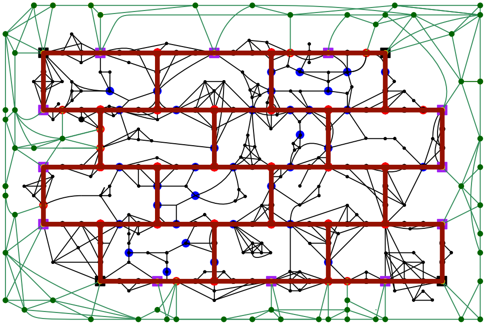

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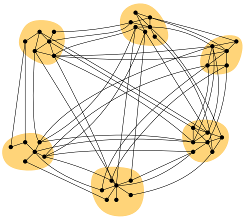

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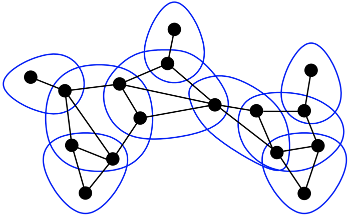

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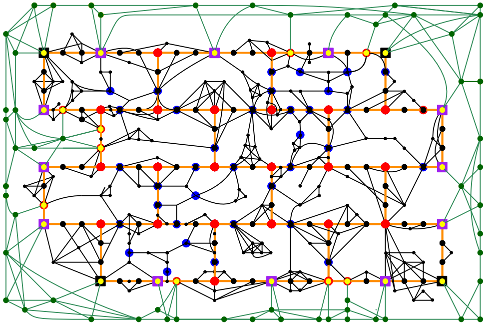

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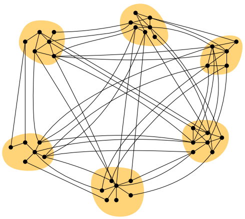

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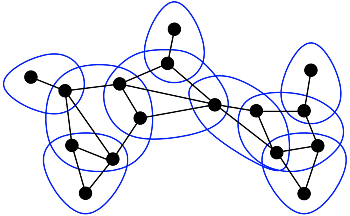

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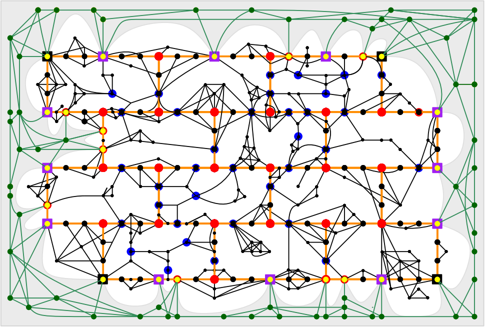

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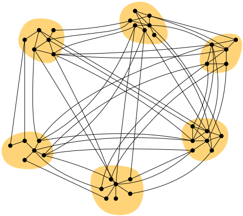

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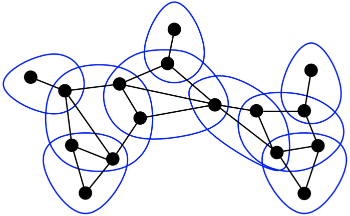

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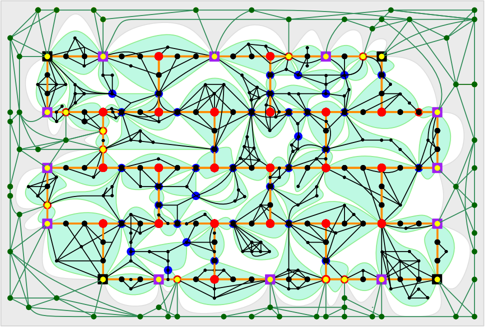

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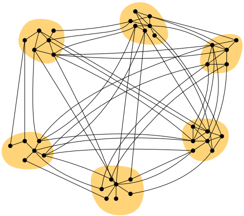

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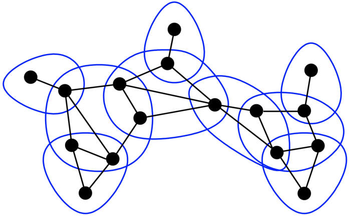

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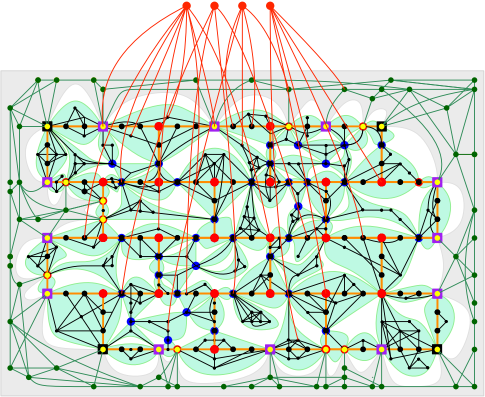

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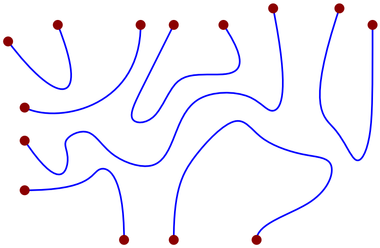

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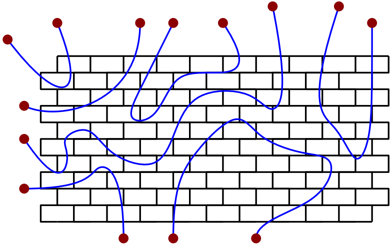

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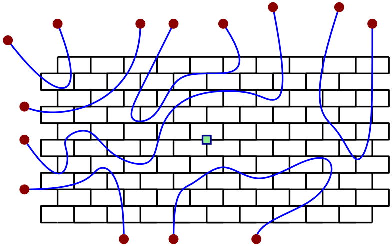

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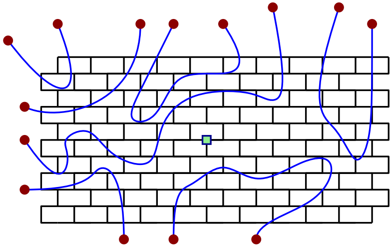

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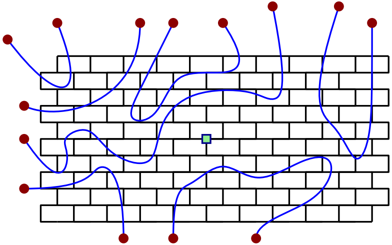

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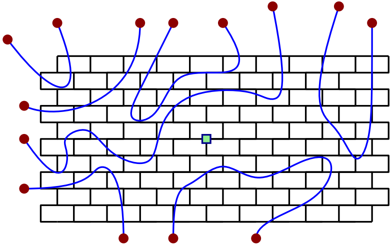

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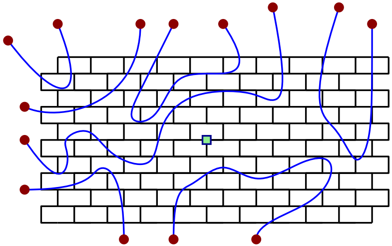

▷ Why irrelevant vertices are *irrelevant*?

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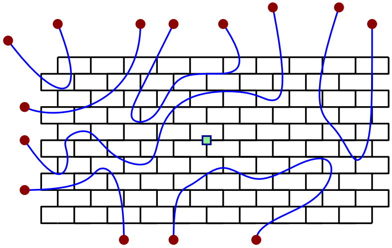
(by finding and removing irrelevant vertices)

▷ More than 50 papers using this technique.



▷ Why irrelevant vertices are *irrelevant*? *Unique Linkage theorem*

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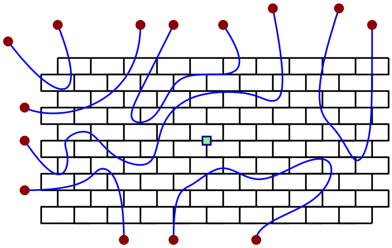

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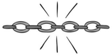
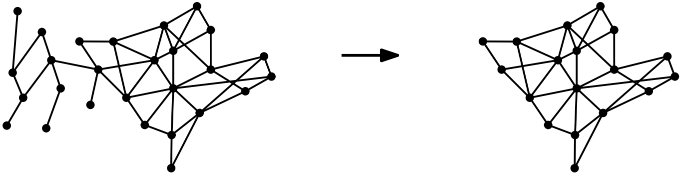
Flat Wall theorem  Irrelevant vertex technique

The algorithmic paradigm of Simplification

Irrelevant vertex technique describes a simplification procedure (a **data reduction**).

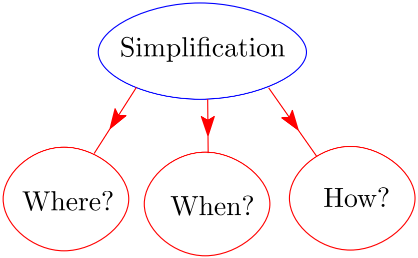
General question: *"How to simplify the input?"*

Example: Does G contain a cycle of length 5?



Designing algorithms using Simplification

- ▷ How Simplification can aid to the design of algorithms?
- In simplified instances, problems are solved more easily.



- We need abstraction and deep understanding of the irrelevant vertex technique.

Meta-algorithmization of the irrelevant vertex technique

Our viewpoint:
Irrelevant vertex technique = instantiation of the **algorithmic paradigm of** Simplification.

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We resort to Logic.

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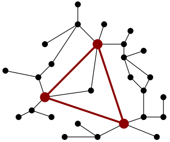

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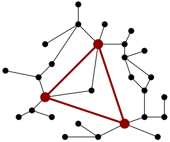

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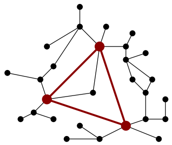

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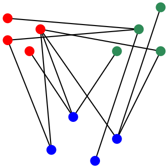

- Is G 3-*colorable*?

$$\exists V_1 \exists V_2 \exists V_3 \Big(\big(\forall x \ (x \in V_1 \vee x \in V_2 \vee x \in V_3)\big) \wedge \text{partition}(V_1, V_2, V_3)$$

$$\wedge \big(\forall x \forall y \ (x, y \in V_1) \vee (x, y \in V_2) \vee (x, y \in V_3) \implies \neg\mathbf{adj}(x, y)\big)\Big)$$

AMTs for FO and MSO

bounded treewidth [Courcelle,1990] [Arnborg, Lagergren, Seese, 1991] [Borie, Parker, Tovey, 1992]
bounded cliquewidth [Courcelle, Makowski, Rotics, 2000] [Oum & Seymour, 2006] **MSO**

---

bounded degree [Seese, 1996] **FO**
locally bounded treewidth [Frick & Grohe, 2001]
excluding a minor [Flum & Grohe, 2001]
locally excluding a minor [Dawar, Grohe, Kreutzer, 2007]
bounded expansion [Dvořák, Kráľ, Thomas, 2011]
nowhere dense [Grohe, Kreutzer, Siebertz, 2017]
bounded twinwidth [Bonnet, Kim, Thomassé, Watrigant, 2022]
structurally bounded degree [Gajarský, Hliněný, Lokshtanov, Obdržálek, Ramanujan, 2016]
structurally bounded expansion [Gajarský, Kreutzer, Nešetřil, Ossona de Mendez, Mi. Pilipczuk, Siebertz, Toruńczyk, 2018]
structurally nowhere dense [Dreier, Mählmann, Siebertz, 2023]
structurally bounded local cliquewidth [Bonnet, Dreier, Gajarský, Kreutzer, Mählmann, Simon, Toruńczyk, 2022]
monadically stable [Dreier, Eleftheriadis, Mählmann, McCarty, Mi. Pilipczuk, Toruńczyk, 2023]
monadically NIP/dependent ?

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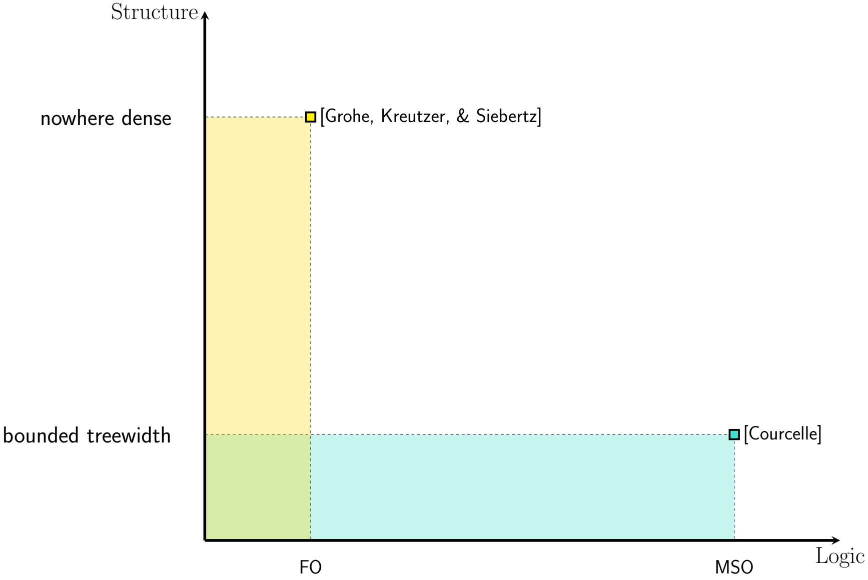
▷ AMTs for **FO** = Meta-algorithmization of Locality & Separability & Representative witnesses
based on sparsity


Structure
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FO
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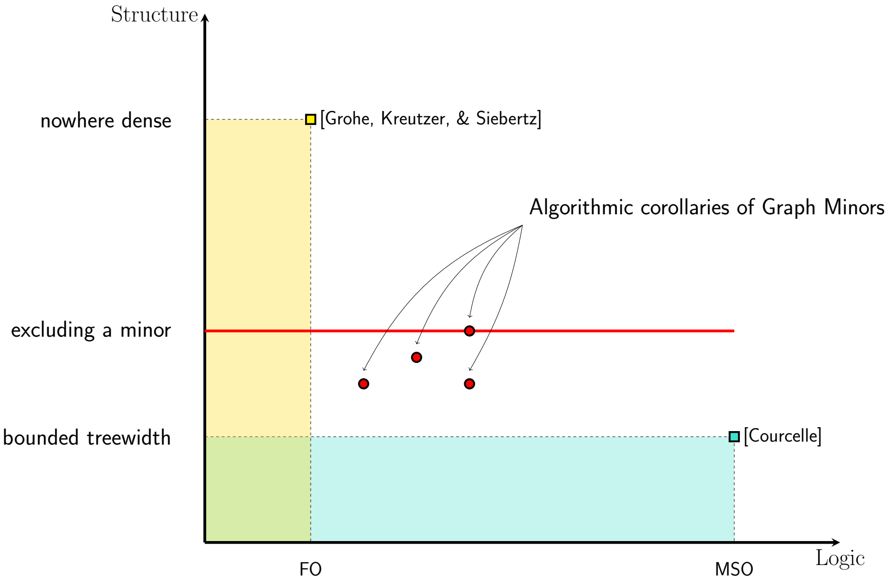
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Algorithmic corollaries of Graph Minors
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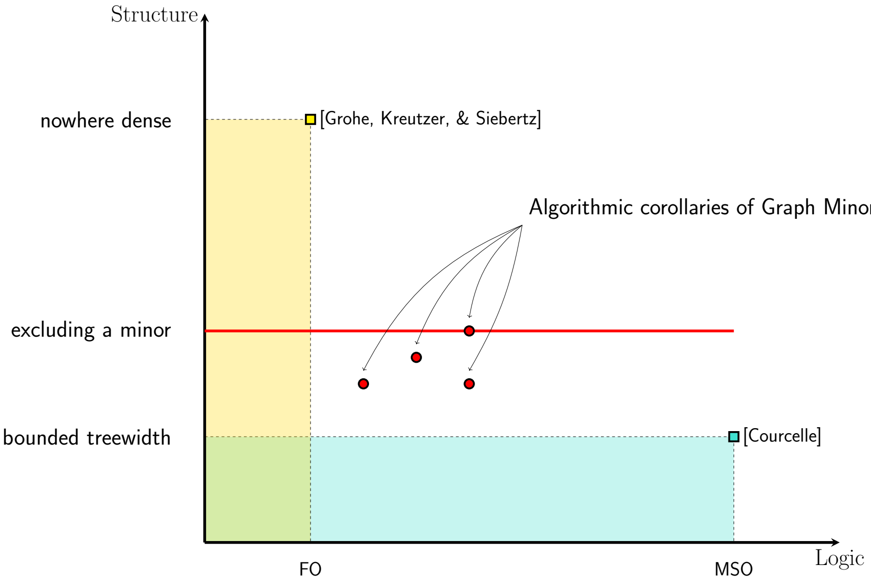

We lack of a logical-based theory for Simplification.

Logics and Algorithms for Graph Minors

Algorithmic paradigm	*Logic*
Dynamic Programming / Compositionality	MSO
Locality / ...	FO
Simplification	?

Challenge: Find a logic encompassing the algorithmic paradigm of Simplification.

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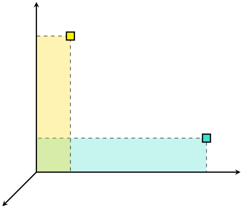
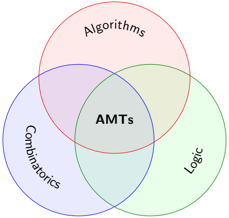
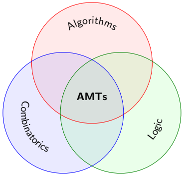
▷ "**Efficiency dimension**" of AMTs?





Results



Combinatorial & Algorithmic tools

[Sau, *Stamoulis*, Thilikos. A more accurate view of the Flat Wall Theorem]
Under revision. Revised version in Journal of Graph Theory (**JGT**)

[Golovach, *Stamoulis*, Thilikos. Combing a Linkage in an Annulus]
*SIAM Journal on Discrete Mathematics (**SIDMA**), 2023*

AMTs

[Golovach, *Stamoulis*, Thilikos. Model-Checking for First-Order Logic with Disjoint Paths Predicates in Proper Minor-Closed Graph Classes]
SODA 2023

[Schirrmacher, Siebertz, *Stamoulis*, Thilikos, Vigny. Model Checking Disjoint-Paths Logic on Topological-Minor-Free Graph Classes]
Unpublished
[Fomin, Golovach, Sau, *Stamoulis*, Thilikos. Compound Logics for Modification Problems]
ICALP 2023

Efficiency dimension

[Sau, *Stamoulis*, Thilikos. k-apices of minor-closed graph classes. I. Bounding the obstructions]
*Journal of Combinatorial Theory, Series B (**JCTB**), 2023*

[Sau, *Stamoulis*, Thilikos. k-apices of minor-closed graph classes. II. Parameterized algorithms]
ICALP 2020 / ACM Transactions on Algorithms (**TALG**), 2022

[Morelle, Sau, *Stamoulis*, Thilikos. Faster parameterized algorithms for modification problems to minor-closed classes]
ICALP 2023

[Golovach, *Stamoulis*, Thilikos. Hitting Topological Minor Models in Planar Graphs is Fixed Parameter Tractable]
SODA 2020 / ACM Transactions on Algorithms (**TALG**), 2023

Combinatorial & algorithmic support of our AMTs

Enhanced algorithmic versions of the Flat Wall theorem

We build on the viewpoint of [Kawarabayashi, Thomas, Wollan, 2018].

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(Algorithmic enhancement of) Flat Wall theorem

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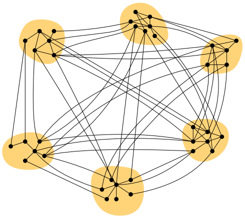

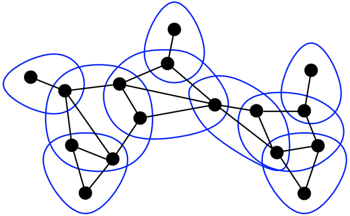

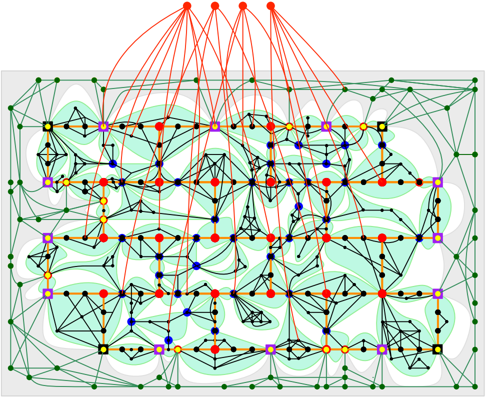

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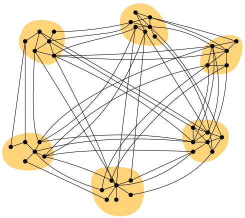

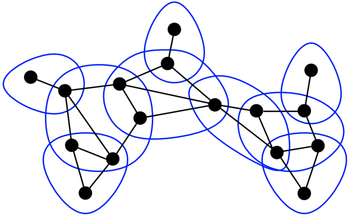

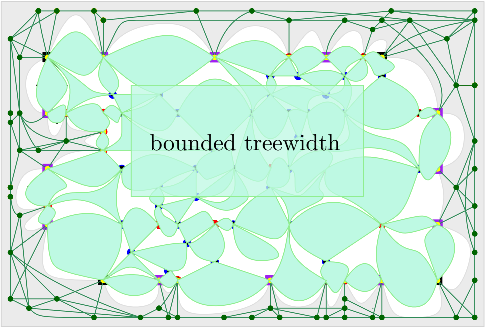

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- We introduce new combinatorial & algorithmic tools for flat walls, needed in our AMTs

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How to deal with linkages?

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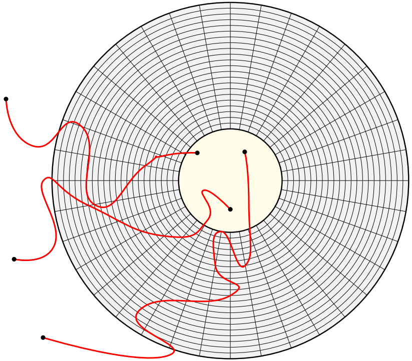

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▷ Avoiding a vertex is not enough! **We need to comb!**

Linkage Combing Lemma

There is a function $f : \mathbb{N} \to \mathbb{N}$ such that if

- G is a partially disk-embedded graph,
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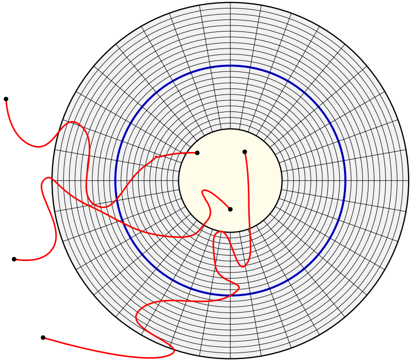
then there is an equivalent linkage L' that traverses the middle cycle of $\mathcal{C}$ through $\mathcal{P}$.



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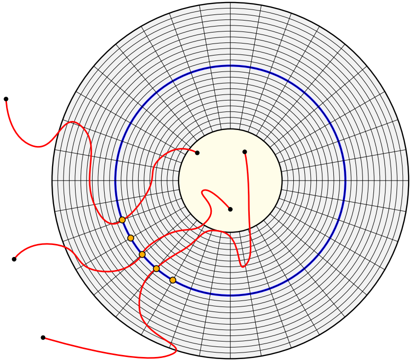

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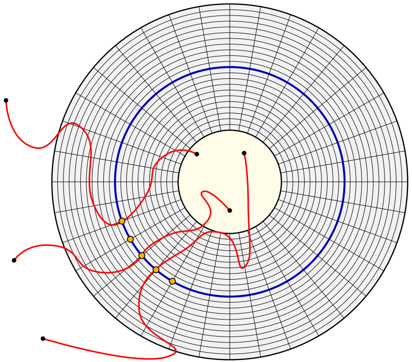

Strengthening of the *Unique Linkage theorem*.

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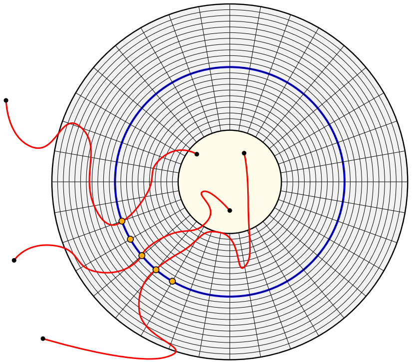


Strengthening of the *Unique Linkage theorem*.

Importance: *Finitely "represent" paths*

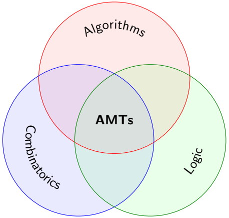
Recap of the combinatorial and algorithmic support

- Enhanced algorithmic versions of the Flat Wall theorem.

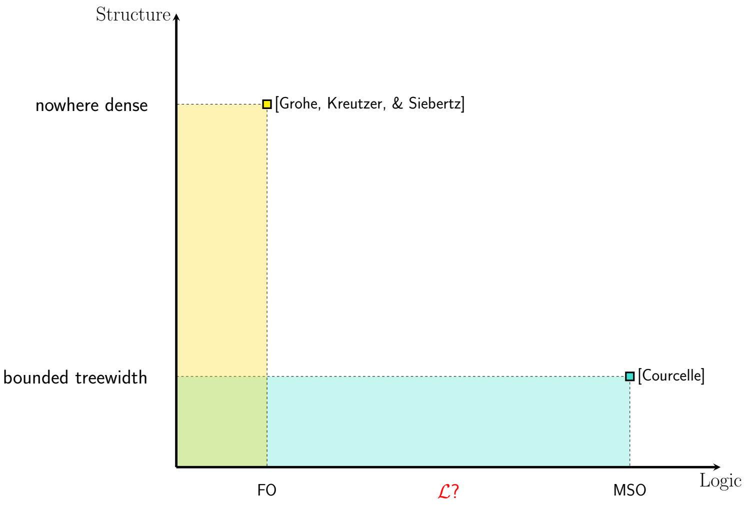
[Sau, *Stamoulis*, & Thilikos, A more accurate view of the Flat Wall Theorem]
Under revision. Revised version in *Journal of Graph Theory* (**JGT**)

- Combing linkages in annuli.

[Golovach, *Stamoulis*, & Thilikos, Combing a Linkage in an Annulus]
SIAM Journal on Discrete Mathematics (**SIDMA**), 2023



Our Algorithmic Meta-Theorems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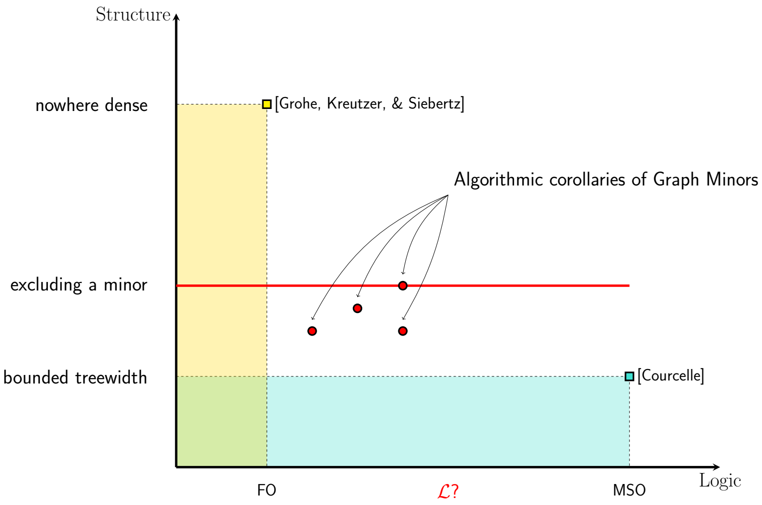

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Algorithmic corollaries of Graph Minors
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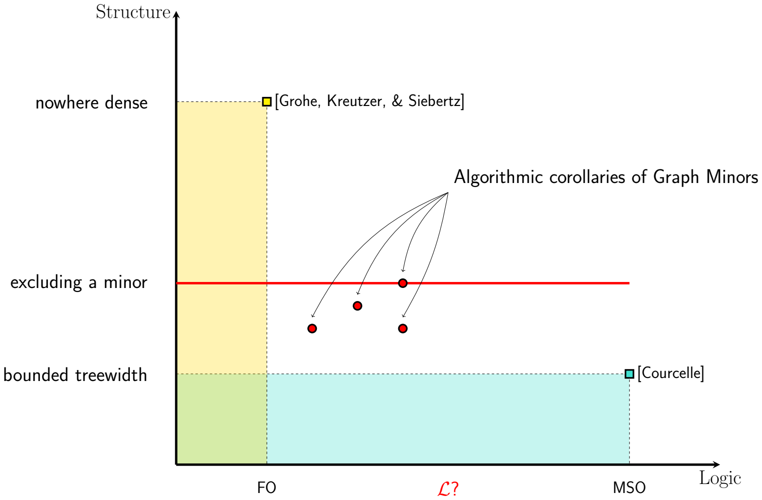

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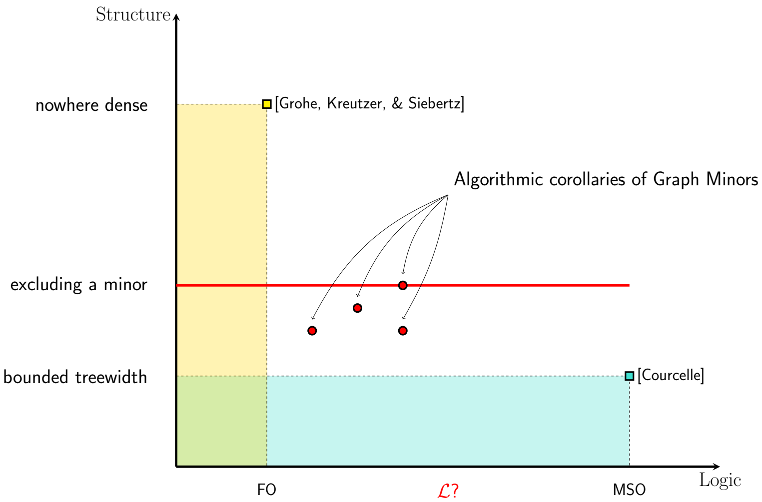

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Disjoint-paths logic (FO+dp)

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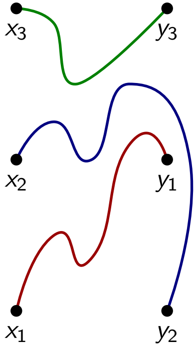

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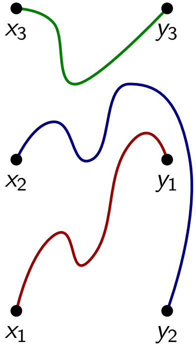

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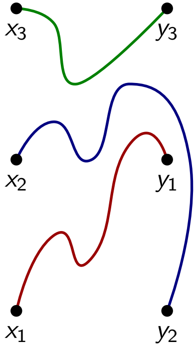

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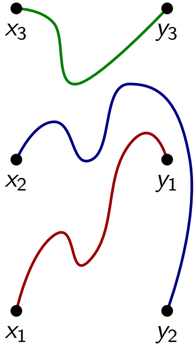

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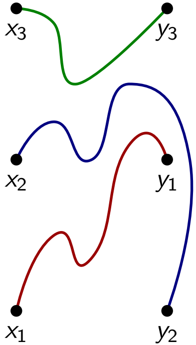

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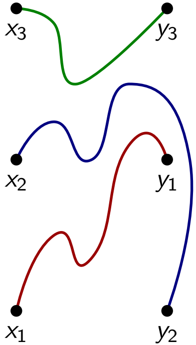

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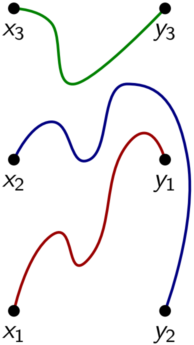

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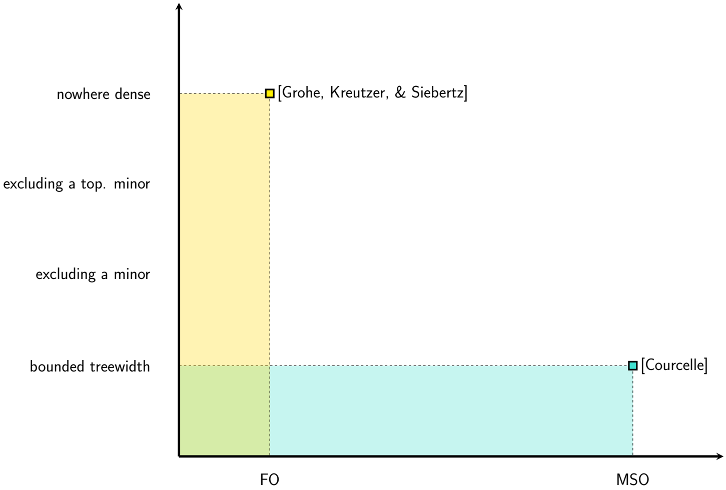
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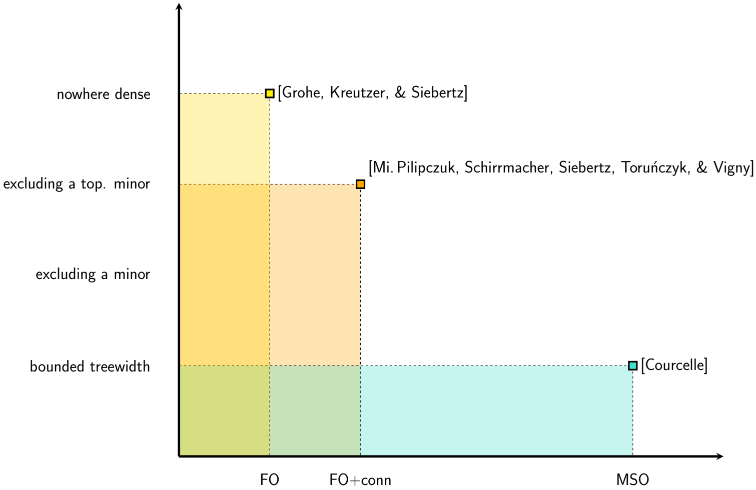
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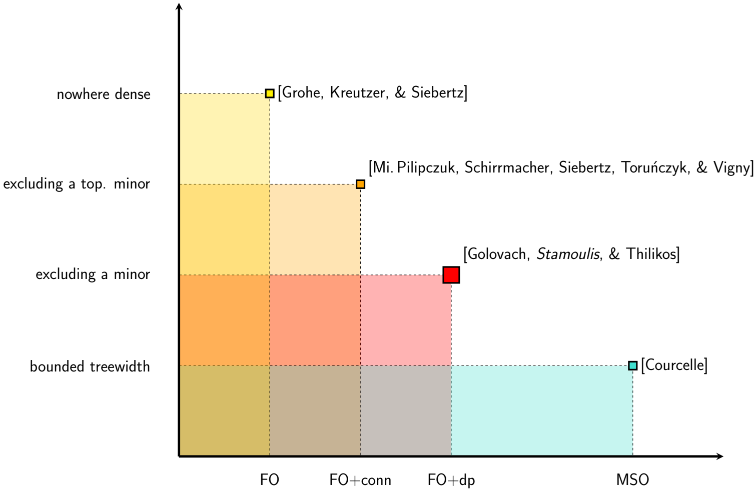

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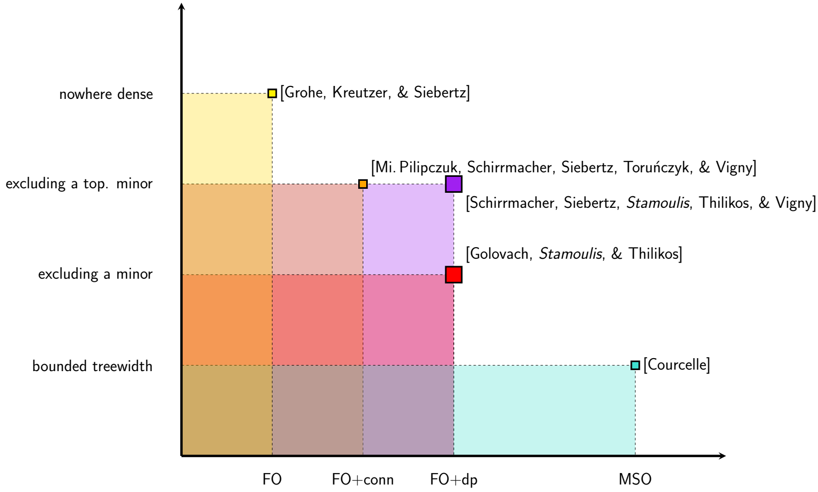

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Scattered disjoint paths predicates:

$s\text{-sdp}_k(x_1, y_1, \ldots, x_k, y_k)$

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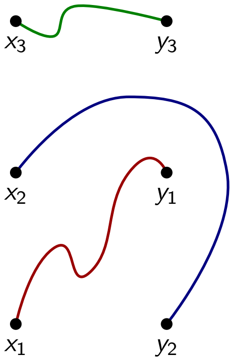

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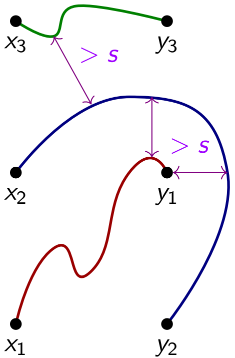

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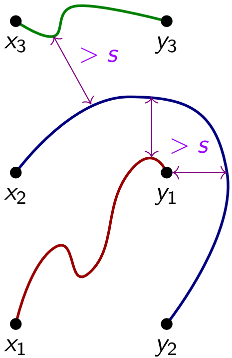

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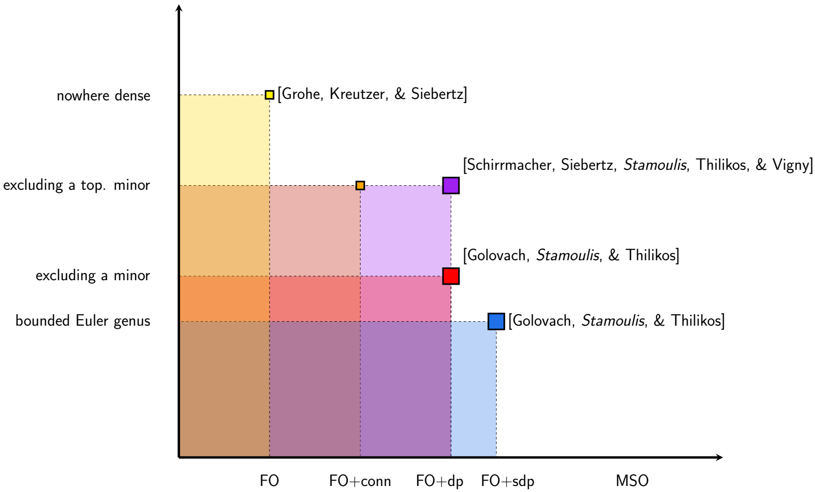

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Other families of problems where irrelevant vertex technique applies?

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Apply a modification $\mathcal{M}$ to a graph such that the resulting graph has property $\mathcal{P}$.

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$\hookrightarrow$ **modulator/target scheme**.

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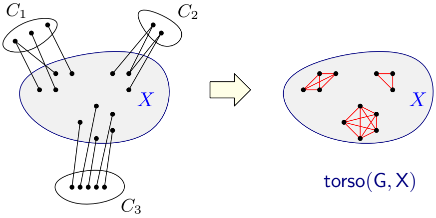

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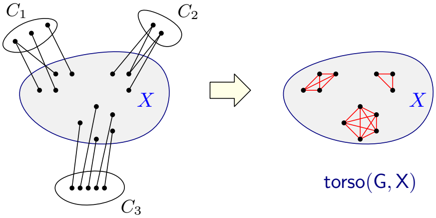

p=treedepth: $\mathcal{G}$-elimination distance

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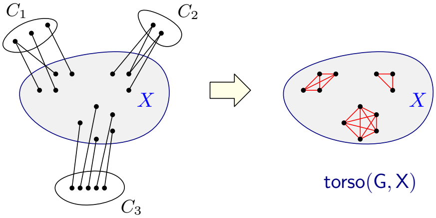

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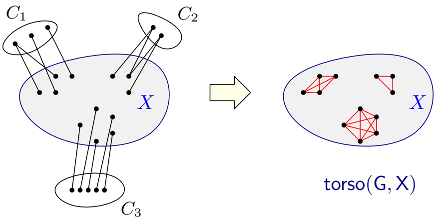

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[Bougeret, Jansen, Sau, 2020]

One meta-theorem that deals with all these cases?

p=pathwidth, cutwidth, vertex cover, feedback vertex set, branchwidth, carving-width,...

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A study on unbounded size but "*structured*" modulators.

Motivation: algorithm-driven

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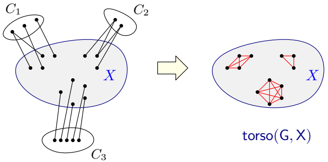

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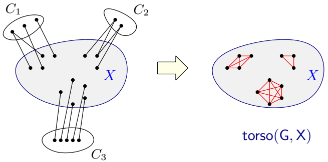

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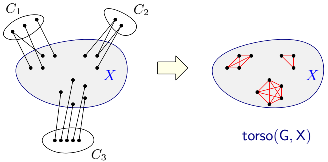
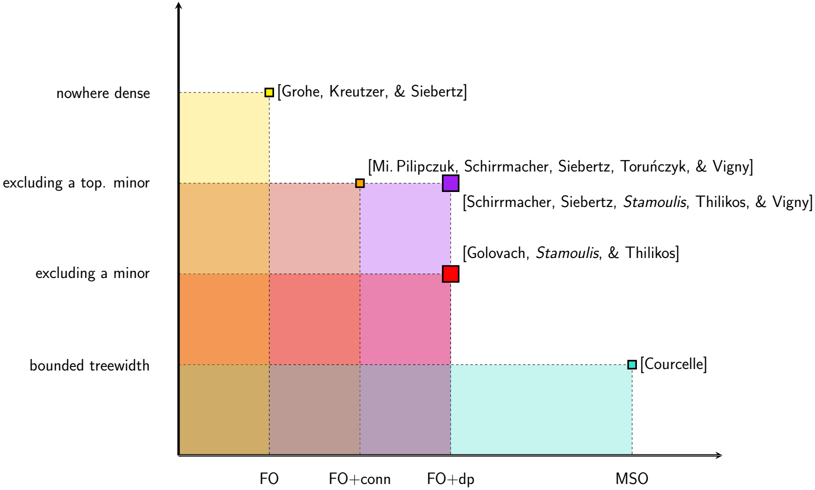


$\tilde{\Theta}^{\mathsf{dp}}$ corresponds to MSO $\triangleright$ (MSO $\triangleright$ ...(MSO $\triangleright$ FO + dp))


nowhere dense
excluding a top. minor
excluding a minor
bounded treewidth
FO
FO+conn
FO+dp
MSO
[Grohe, Kreutzer, & Siebertz]
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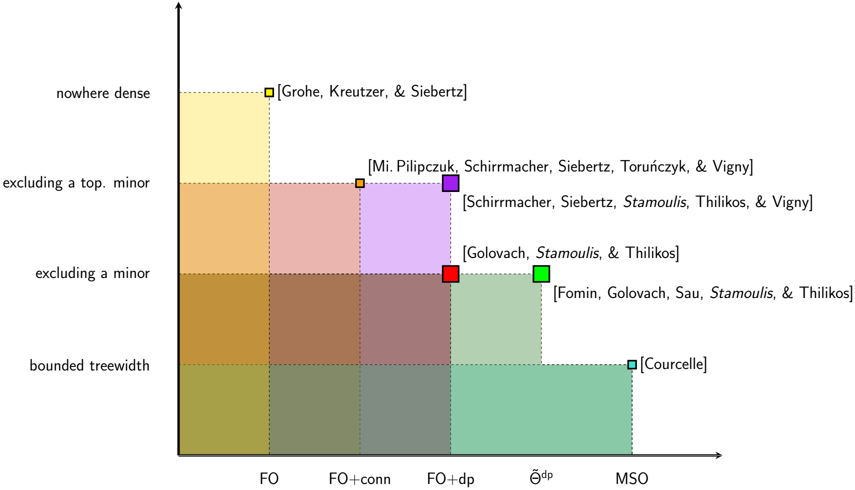
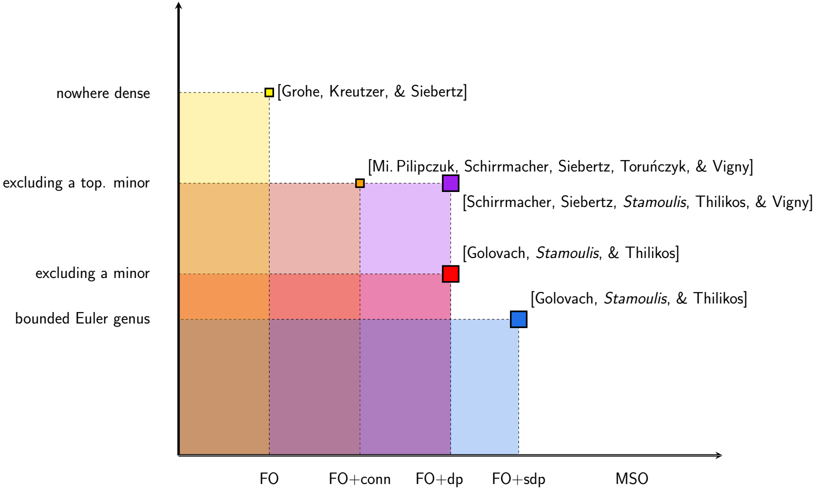


Model checking for $\tilde{\Theta}^{dp}$ can be done in quadratic time on graphs excluding a minor.

[Fomin, Golovach, Sau, *Stamoulis*, & Thilikos, 2023]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FO
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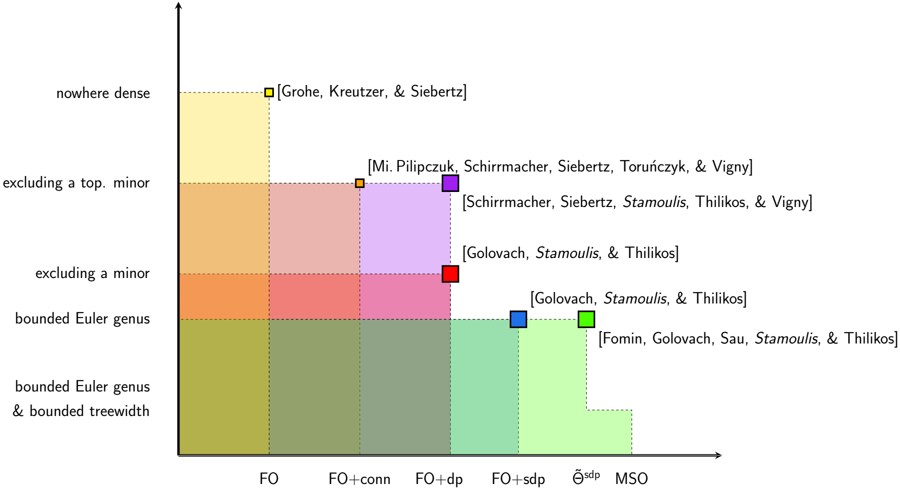
FO+sdp
MSO



Model checking for $\tilde{\Theta}^{\mathsf{sdp}}$ can be done in quadratic time on graphs of bounded Euler genus.

[Fomin, Golovach, Sau, *Stamoulis*, & Thilikos, 2023]

Our AMTs

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[Golovach, *Stamoulis*, & Thilikos, Model-Checking for First-Order Logic with Disjoint Paths Predicates in Proper Minor-Closed Graph Classes]
SODA 2023

[Schirrmacher, Siebertz, *Stamoulis*, Thilikos, & Vigny, Model Checking Disjoint-Paths Logic on Topological-Minor-Free Graph Classes]
Unpublished

[Fomin, Golovach, Sau, *Stamoulis*, & Thilikos, Compound Logics for Modification Problems]
ICALP 2023

“Efficiency axis”

Back to Graph Minors

Goal: Identify large families of problems where running times can be improved.

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What if $\mathcal{P}$ is characterized by exclusion of some graphs as (topological) minors?

$\mathcal{F}$-Minor-deletion **and** $\mathcal{F}$-Topological-Minor-deletion

For **finite** set of graphs $\mathcal{F}$:

$\mathcal{F}$-Minor-deletion and $\mathcal{F}$-Topological-Minor-deletion

For **finite** set of graphs $\mathcal{F}$:

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Delete $\leqslant k$ vertices such that the obtained graph does not contain any $F \in \mathcal{F}$ as a minor.

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[Robertson & Seymour, GM I – GM XXII]: $\mathcal{F}$-Minor-deletion is solvable (non-constructively).

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Our results:

Bounding the obstructions

Obstructions of yes-instances of $\mathcal{F}$-Minor-deletion have size $\leqslant f(k, \mathcal{F})$

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Obstructions of yes-instances of $\mathcal{F}$-Minor-deletion have size $\leqslant f(k, \mathcal{F}) = 2^{2^{2^{2^{poly_{\mathcal{F}}(k)}}}}$

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Improved algorithm for $\mathcal{F}$-Minor-deletion

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$\mathcal{F}$-Topological-Minor-Deletion is solvable in time $2^{\mathcal{O}_{\mathcal{F},g}(k^2)} \cdot n^2$ on graphs of Euler genus $\leqslant g$.

“Efficiency axis”

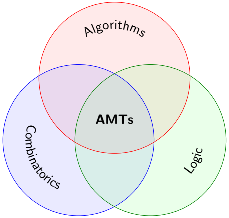
[Sau, *Stamoulis*, & Thilikos, *k*-apices of minor-closed graph classes. I. Bounding the obstructions]
Journal of Combinatorial Theory, Series B (**JCTB**), 2023

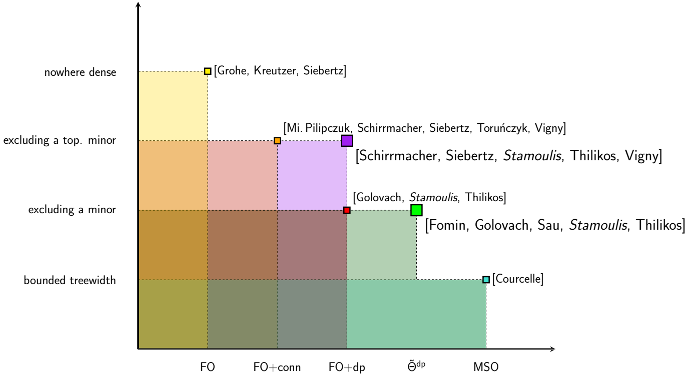
[Sau, *Stamoulis*, Thilikos, *k*-apices of minor-closed graph classes. II. Parameterized algorithms]
ICALP 2020
ACM Transactions on Algorithms (**TALG**), 2022

[Morelle, Sau, *Stamoulis*, Thilikos, Faster parameterized algorithms for modification problems to minor-closed classes]
ICALP 2023

[Golovach, *Stamoulis*, Thilikos, Hitting Topological Minor Models in Planar Graphs is Fixed Parameter Tractable]
SODA 2020
ACM Transactions on Algorithms (**TALG**), 2023

Recap of results of the thesis





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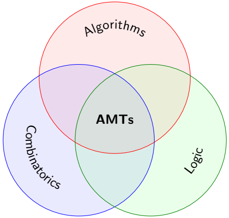

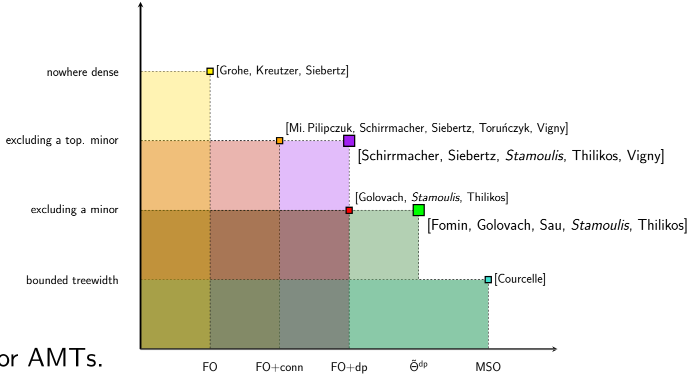

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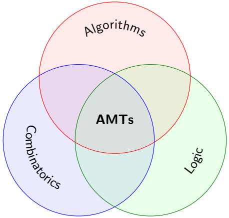

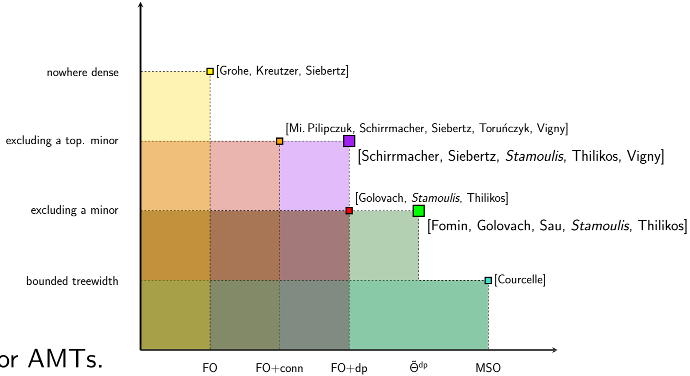

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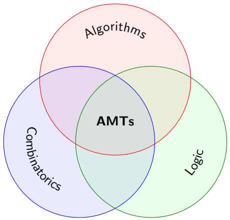

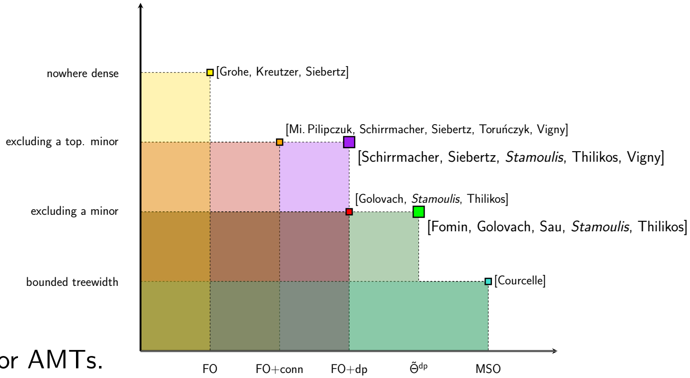

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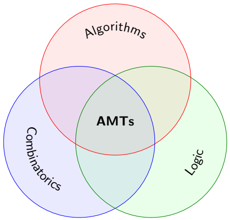

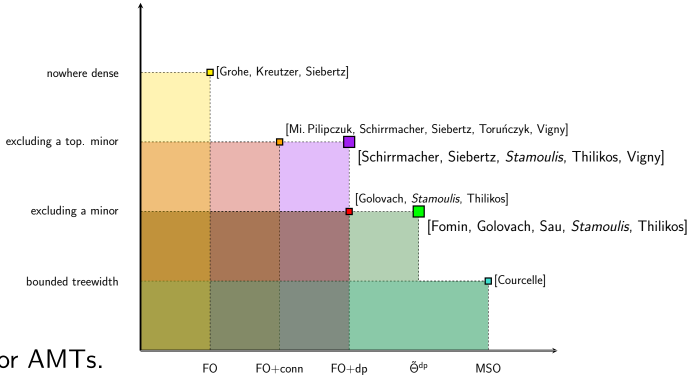

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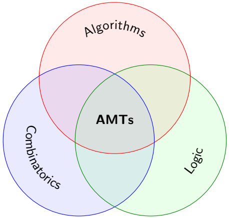

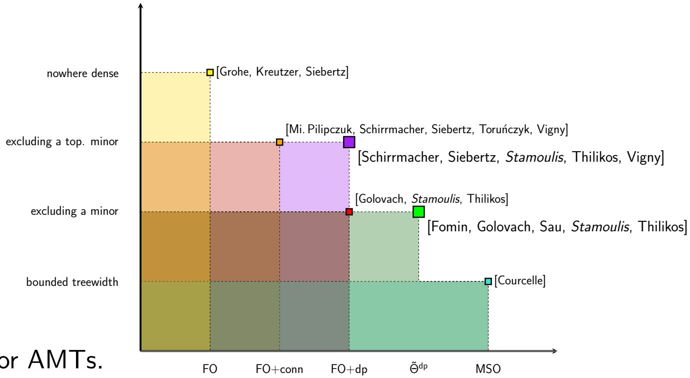

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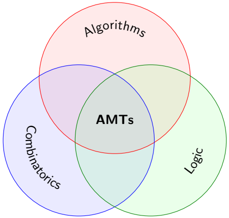

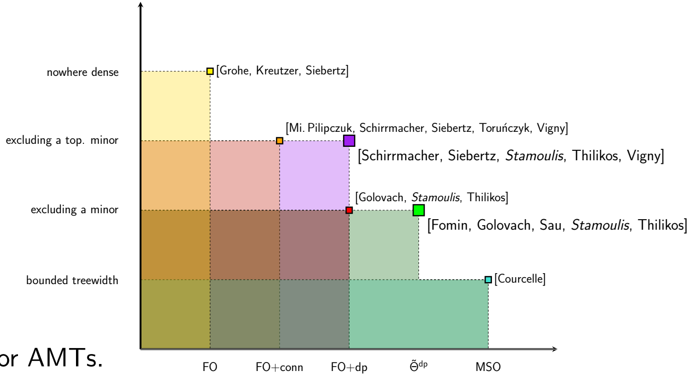

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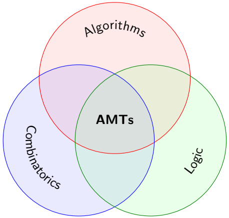

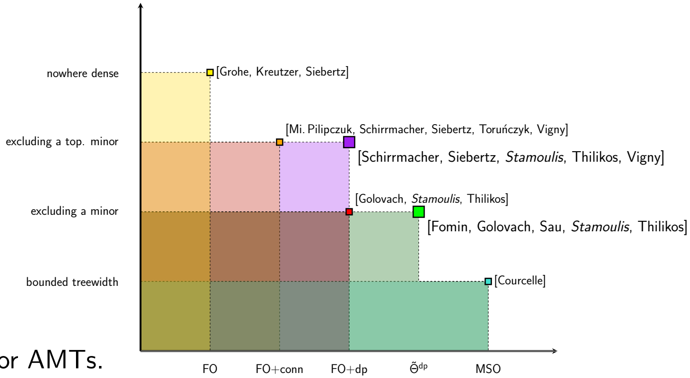

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▷ New ideas to obtain efficient algorithms.

Outline of some ingredients of our proofs

How to meta-algorithmize Simplification?

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What is Courcelle's theorem?

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Subroutine: Recursively compute the MSO-type of the instance.

(all satisfiable MSO-formulas of certain # of quantifiers)

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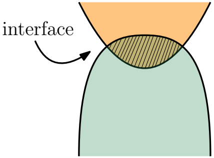

- Compositionality of MSO on small size interface. [Feferman-Vaught theorem]

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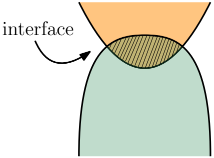

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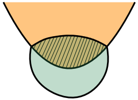

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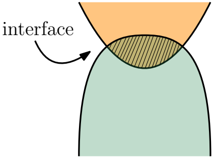

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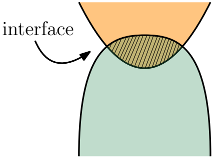

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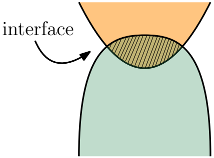

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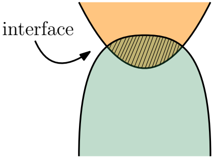

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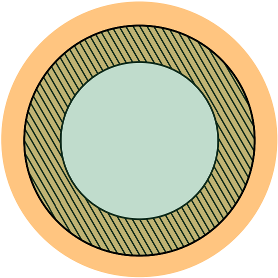

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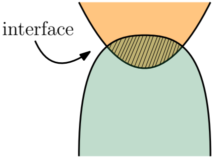

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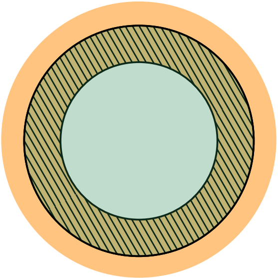
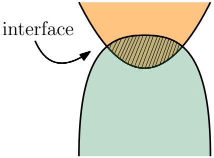


How to meta-algorithmize Simplification?

What is Courcelle's theorem?

Subroutine: Recursively compute the MSO-type of the instance.



- Compositionality of MSO on small size interface. [Feferman-Vaught theorem]

▷ *Simplification:* Reduction to type-representative of small size.

Recursion & Compositionality & Simplification = Dyn. Prog.

For our AMTs: *"Local-to-Global" approach*

Idea: Simplify flat parts of the input.

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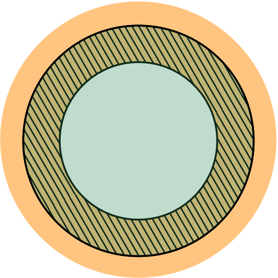

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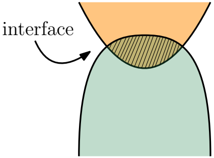

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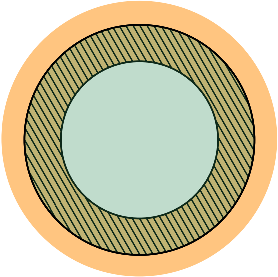

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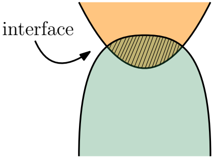

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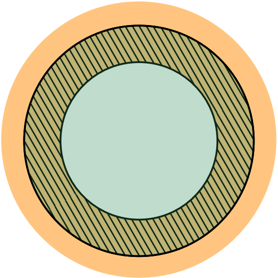

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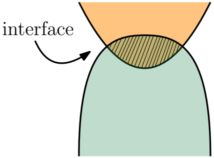

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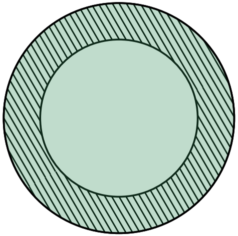

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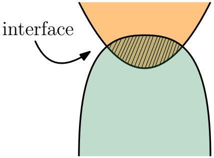

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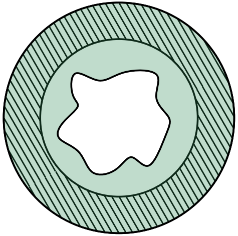

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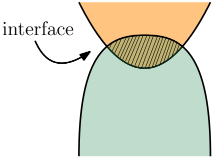

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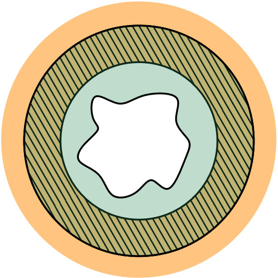

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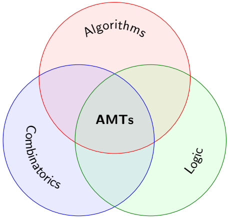
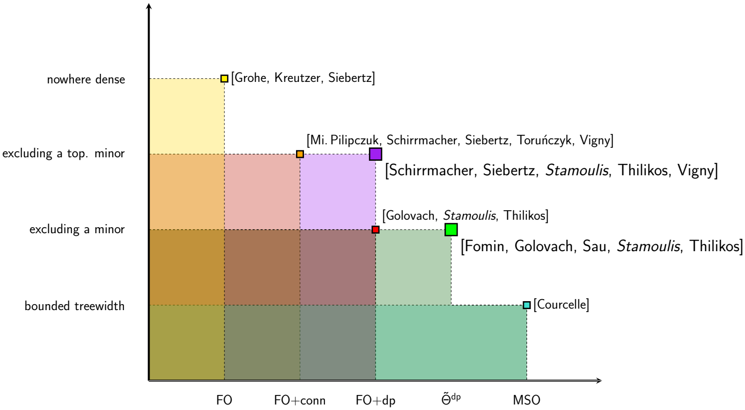
Conclusions & Perspectives

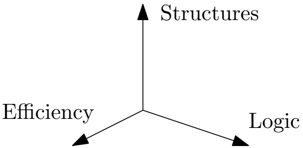
Conclusion





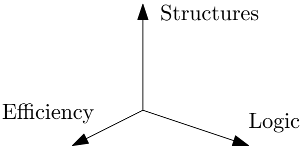
- Combinatorial & algorithmic support for AMTs.
- AMTs abstractizing irrelevant vertex technique (*algorithmic paradigm of* Simplification).
- Advance in the efficiency dimension of AMTs.

What to do next?



What to do next?

- Can our AMTs be generalized to more general classes?



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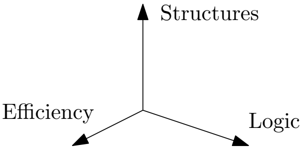
- What is the "logical" limit on minor-closed classes?



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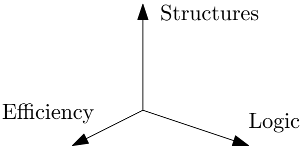

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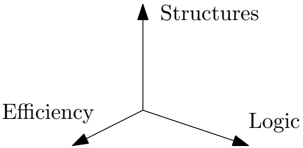

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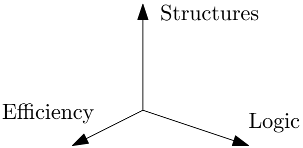

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$$\mathcal{O}(n^{2-\varepsilon})? \quad \mathcal{O}(n^{1+\varepsilon})? \quad \mathcal{O}(n^{1+o(1)})? \quad \mathcal{O}(n \cdot \mathrm{polylog}(n))? \quad \mathcal{O}(n)?$$

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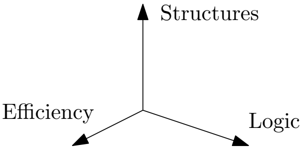

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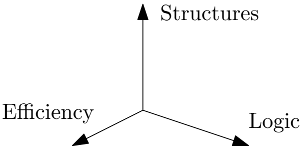

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$$\left.2^{2^{\cdot^{\cdot^{2^{|\varphi|}}}}}\right\}\text{ depends on } |\varphi| \quad \to \quad \left.2^{2^{\cdot^{\cdot^{2^{|\varphi|}}}}}\right\}\textbf{ does not}\text{ depend on } |\varphi|$$

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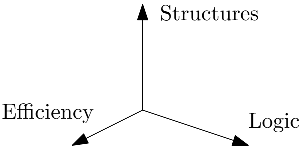

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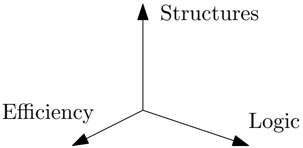

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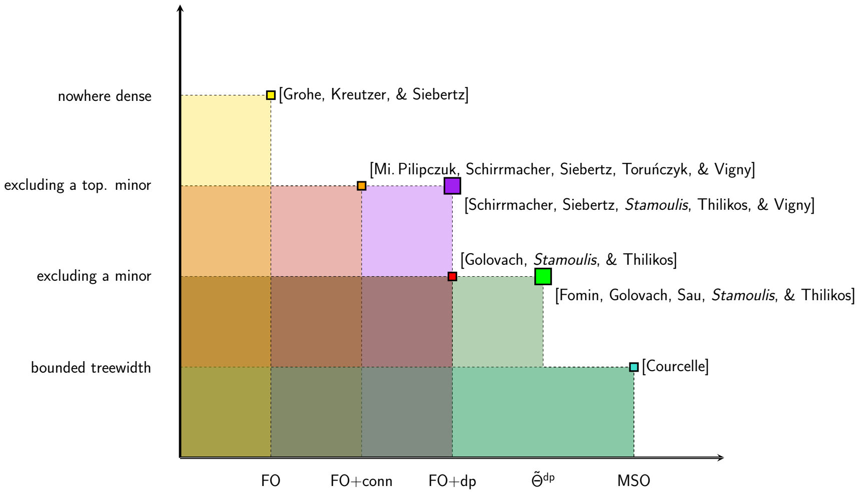
AMTs in Distributed Computing? Dynamic algorithms? Query enumeration?


nowhere dense
excluding a top. minor
excluding a minor
bounded treewidth
[Grohe, Kreutzer, & Siebertz]
[Mi. Pilipczuk, Schirrmacher, Siebertz, Toruńczyk, & Vigny]
[Schirrmacher, Siebertz, *Stamoulis*, Thilikos, & Vigny]
[Golovach, *Stamoulis*, & Thilikos]
[Fomin, Golovach, Sau, *Stamoulis*, & Thilikos]
[Courcelle]
FO
FO+conn
FO+dp
$\tilde{\Theta}^{\mathsf{dp}}$
MSO

Thank you!

Other research projects during Ph.D. studies (not included in the thesis)

[Fomin, Golovach, Korhonen, Simonov, *Stamoulis*. Fixed-Parameter Tractability of Maximum Colored Path and Beyond]

SODA 2023

[Fomin, Golovach, Korhonen, Lokshtanov, *Stamoulis*. Shortest Cycles With Monotone Submodular Costs]

SODA 2023

ACM Transactions on Algorithms (**TALG**), 2023

[Fomin, Golovach, Korhonen, *Stamoulis*. Computing paths of large rank in planar frameworks deterministically]

ISAAC 2023

[Diner, Giannopoulou, *Stamoulis*, Thilikos. Block Elimination Distance]

WG 2021

Graphs and Combinatorics (**GCOM**), 2022

[Kontogeorgiou, Leivaditis, Psaromiligkos, *Stamoulis*, Zoros. Branchwidth is (1,g)-self-dual]

Under revision