# Logics and Algorithms for Graph Minors

#### Giannos Stamoulis

AIGCo team Laboratoire d'Informatique, de Robotique et de Microélectronique de Montpellier

#### Committee:

Anuj Dawar	revi
Marcin Pilipczuk	revi
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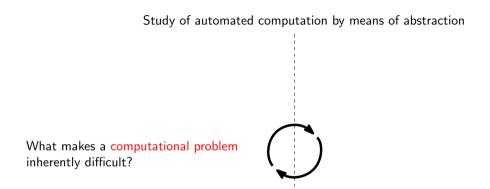
Amphithéâtre Jean Jacques Moreau, 12/12/2023

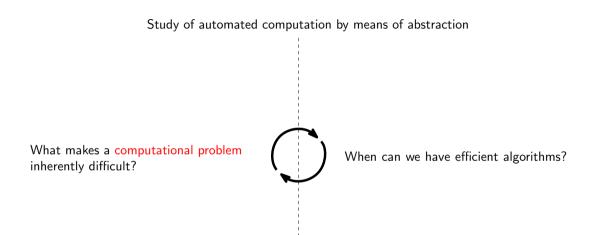


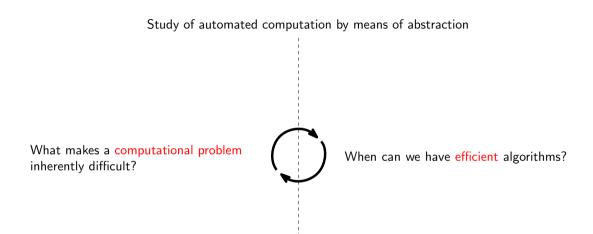
Study of automated computation by means of abstraction

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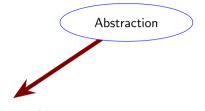
What makes a computational problem inherently difficult?



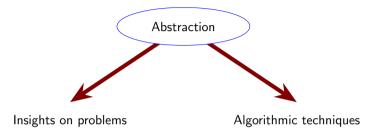


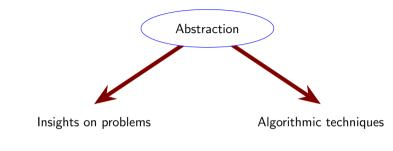




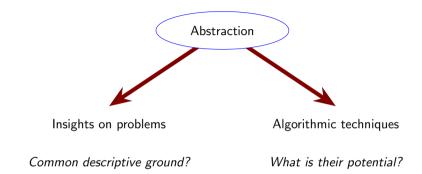


Insights on problems





Common descriptive ground?



• Model of abstraction: *Graphs* 

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- Model of abstraction: *Graphs*
- Decision problems: answered by YES or NO



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- Decision problems: answered by YES or NO



Given a graph G, does it have property X ?

- Model of abstraction: Graphs
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▷ How to describe a property?

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Given a graph G, does it have property X ?

 $\triangleright$  How to describe a property?  $\rightarrow$  Machine description

- Model of abstraction: Graphs
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Given a graph G, does it have property X ?

 $\triangleright$  How to describe a property?  $\rightarrow$  Machine description  $\rightarrow$  Logic

- Model of abstraction: Graphs
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Given a graph G, does it have property X ?

 $\label{eq:how to describe a property?} \begin{array}{l} \rightarrow \mbox{Machine description} \\ \rightarrow \mbox{Logic (abstract language to describe properties/problems)} \end{array}$ 

- Model of abstraction: Graphs
- Decision problems: answered by YES or NO



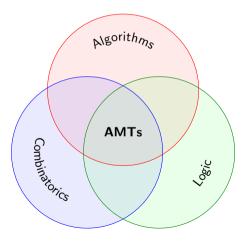
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 $\label{eq:how to describe a property?} \hspace{0.2cm} \rightarrow \hspace{0.2cm} \begin{array}{l} \text{Machine description} \\ \rightarrow \hspace{0.2cm} Logic \hspace{0.2cm} (\textit{abstract language to describe properties/problems}) \end{array}$ 

▷ How to use the structure of the graph to obtain efficient algorithms ?

### Algorithmic meta-theorems (AMTs):

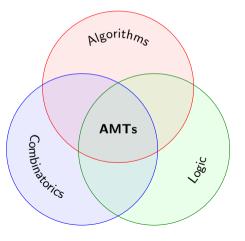
General mathematical conditions that allow the automatic derivation of efficient algorithms.



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Conditions: logical  $(\mathbf{C}_L)$  & combinatorial  $(\mathbf{C}_C)$ 

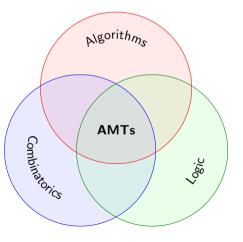


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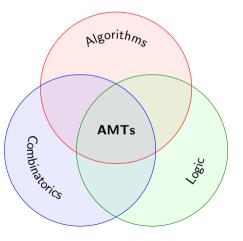
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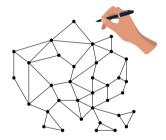
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"Algorithms that give algorithms"



When can a graph be drawn on the plane without crossings?

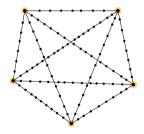
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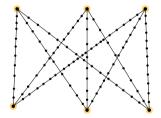


When can a graph be drawn on the plane without crossings?

#### Kuratowski-Pontryagin theorem (1930):

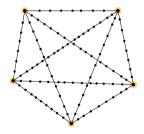
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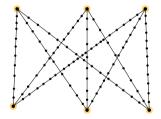




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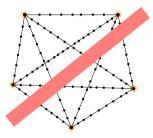
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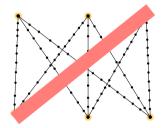




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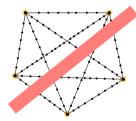
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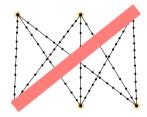




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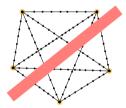
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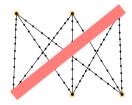




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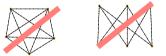
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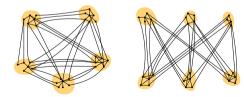
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G is planar  $\iff$  G does not contain  $K_5$  or  $K_{3,3}$  as a topological minor.



## Wagner's theorem (1937):



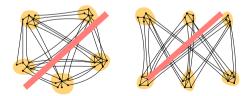
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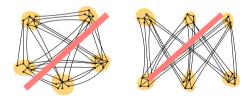
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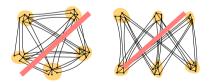
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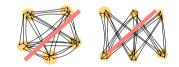
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# $\begin{array}{l} \label{eq:conjecture:} \mbox{Surface embeddability of graphs is characterized by a few obstructions.} \\ (minor-minimal graphs not satisfying $\mathcal{P}$) \end{array}$







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(maintained on minors)







No. Research und F.O. Broscon\*

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and here property A, then the an of minor-minimal graphs without property F in finite.	this first points point to make a path had, to page of headed to work 1 on terms has be		<ol> <li>Wagner unspected that if \$2, \$2,\$ and control anyone if they produe then their out a profil (1) is a back-flat is, is included to a more of</li> </ol>	We prove that the intermal billing specified of the intermal is a sub-the term paped with an extra the strandom limit is a final to a strategistic specific and the strategistic specific specific specific specific specific specific specific specific specific specific terms and any specific s		have a second second state to second a firm to second a second se
		All propher in this paper are finite, and more haven loops or multiple algor- P(2) and $E(3)$ shows the same of various and edges of the graph $S$ , respec- tively. A new decomposition of a graph $S$ is a graph $T, F$ , where $T$ is a true and $S' = (Z, +1) P(TE)$ is a dashy of advance of $P(S)$ , with the following	produce, these determined at a start of control of the start has a fixed by the start of the sta	a stream by participites register in a transmission. Was here associat consequences, for example, a singler that is if d' is a stream d'apply such that as another is	Let if be a graph doorse on a day, and he day writes of C on the boundary of the day by a second s	(a) a set of a set
i. brannerser	1. Internet the second se	F(0) and E(0) denote the arts of version and odges of the graph G. respec-	plane. We have to show in a barry paper that Wagner's conjectors is loss to present and the mathematical data approved the analysis for proof. I conversion	susceptor to a triver of another, and areas member of all a planet, then of a	The data description of a second state of the arc laws in the data description of C periods, is, and (i) a (i) (i) (i) (i) (ii) (iii) (ii	combinest for instance, it suffaces that if a constrained well "year op" for testion asing and a sufface in an inclusion managed that appent " that much a surragional
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hat if he a graph. (All graphs in this paper are finite, and must have longe	All graphs is this paper are finite, and may have large ar multiple signs. F(4)) and E(2) shows the new of various and signs of our graph (1), respectively. Let O be a gaph A. a reac-domenousline of $O$ is a handy $(E_i + 0, f)$ of solution of $F(2)$ , together with a low T with $V(T) = J$ , with			of a more of these and this constraining the barrant matched of property of only in a graph to plane strations when the strain. If the public has to		
or multiple silper series we since colorwise.) A supporter XX, of educity of F(G) (the series set of G) is a path-documposition of G if the following	(K) ( 0 f) of advanced F(C), supplier with a low T with F(T) = J, with	(W1) UDC//0 P078 = P305	6. Bothesecretes		1 house the	
		(W1) For every adjace of O there exists i C P(7) such that a ban head, such in S.	By a "graph" we shall mean incorpt when we my otherwise) in this	1. benessermen		1. homomorphics
(W1) Sir every edge e q <sup>2</sup> 6, arme X, (1 C/C/) contains built and effective.	(NU) VA.V., i 4.70 + P(H). (NU) Know with of the limit in such in some X. (1.4.7).		pager a finite, undimitted graph which may have longe and endings adges, A graph is a water of another if the first can be obtained from a subgraph	Graphs in this paper are finite, and may have loops and exclusive edges. A graph is a naive of another if the four can be obtained by contraction	The problem DBBORDT CODNETTED DTBTR is given a graph C and vegets $h_{ij}$ ,	Let G be a graph shown on a connected softeer E, and let A, v <sub>0</sub> , A <sub>0</sub> ,
(W) BUILDERSPECTORES	(WE) Know sign of it has both in such in some X, U it V. (WE) Know ( , U it V. ) for an abread of P from 1 in U dow X it X, C X.		A graph is a story of another if the new unit to orthograph of the moreal by adaptionization.	A graph is a same of sensitive $T$ that four such is efficient by constraining from a subgraph of the sense of the "successful" of a graph. This concept has been discussed as dischargence of the strain, such the the constraints on exclusions are difficult a given heats. A considerangement of $T_{12}$ of a graph K is a low $T$ equipher with a latit $p = c(L_{12}, L_{12}, C_{12}, $	perfected to furthermore, and 4, 12 ec/4,82, respectively. If it is a variable part	Approx, Ap. In mature of E. When one there A water-deposite paths joining A and A 11 (11) (11). supported by a particular that if the waters A.
We APTERPETENT AND A	K = K = K.	$L \cap L \in L$ .	of the moved by subprovinguidan. There is an over-proving architectural residuabilitations theorems in graph theory (i) as "which advant features", we must a neural anoming that	how a subgraph of the second.	of the input, this was shown in he JP complete by Karp [13, and Lowl-	$\alpha_i$ and $\alpha_i \in [1 \le i \le k \le m_{i}]$ comparisonly? It is a gluounble that if the mathematical $\alpha_i = \alpha_{i+1} + \alpha_{$
The parts width of G in the minimum value of $k \ge 0$ such that O has a path decomposition $x = -K$ , with $ U  \le k + 1$ of $ U  \le k \le n$ . $K$ is a solutor of $O \in E$ is an its relaxance bound $\sum k$ defining some version.	The solution of the true decomposition is $\max_i(2i) + 1 + i + i + j$ . The rescaled of $-2$ is the momentum $i + i$ for any data frame $i$ is the true dimension of which $i_i \to i_i$ and $\max_i (2i)$ is the intervalue $i = 0$ is the momentum $i = i - 0$ such	The width of the true dependencies is	many (4) as variable many merchin," to make a many among the a stand has a smoothed memory if and only if a has no minor homorphic	has been also used in other papers of this action, but for the randor's colu-	planar. However, we suspect that if 6 is planar and success, here 5 and	adequately, the pathy will cold. More precisely, if
N is a minor of U if all can be obtained from U by deleting some version	widd of it is the minimum or a 6 such that 6 has a transforming-solices of	395.04J - D	a graph has a specified property if and only if it has to minor instruction to a manihor of a construction observation at all graphs. Industry the most supporting of these to does a the stands of Andreasters (1) and Kinen.	variation we define it again him. A new decomposition (3, 2) of a graph 5 in	intertiminal to fix our notice bounded number $p$ of argume of $B_1$ (its problem is probleminally solvable; and its this paper we show that this is true if $p = 1$	14 d'is annualed.
and/or adjan, and/or econversing some adjan. Was many therman of this paper is the following:			most supporting of these to date a the sound of Auchdemony (12) and Edition, Monday, and Wang [22], that a graph map he andwedded in the projective			(6) many neuron discuss in 2 <sup>+</sup> hitteness into distinct members of (x <sub>1</sub> , x <sub>1</sub> ,, x <sub>n</sub> , x <sub>n</sub> ). An longer "langely" other is, here a longer members of points in members with the distance of (x).
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to the Period party of the set of the party with the stary graph with the miner isomorphic or if has party with 4, 10.	Restored, Non-Jone 1981	theorypeation of waith w. Thus, for mample, own and forms have near width 1 (or 8 in degenerate	not only in the set of encluded minory community characterized, but also	(ii) (for each slipe $c$ of G frace exists $c \in V(F)$ such that $c$ has both sails to: in and, in the case of a loop) in $F_{c}$	We also competition that IMM/IDMT CODEXCIDING INTERT is performingly include for general graphs (2.4.1 a fixed. This is true (2.4.1) for is represented as a second of the second of the second of the CODEXCIDING INVERS is performingly solvable if the gauge of 0 is	(iii) over sheed zero down is f which is not sub-homotyle also has keep "length," and
	28		Manual activity presents in 107 years and a 103 striket		conversionized provides to personnally schedul if the passe of it is	
7 Particly suggested to barienal listena Franksien datas MCR 0001000.	Contraction of the second s	* Research partially supported by MEP times MCR minimum		* Reserve partially supported to IRAF Grant IRCE EXCENT.	breaded todard, that is one of the motivations for the present paper, for the studie given have are basically beneats for use in [1]. We are	"Research periods supported to MM Grant MCMINISME
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proposely the graphs measurable to $F_{12}$ or $F_{23}$ . This is another way to rear Konstructive theorem. To generate a graph can be descen to $F_{12}$ and each $F_{23}$	ded game. This result appears to have a number of superant prescapes-	denote the graphs (Py) Pi, P, Pill, and (P, OP), P, OR, M, M, Pill, and (P, OP), Pill, Pil	is, even annohilation provide and care made the densing at least d	with finitely action investigation contaction components, called orders, and for wave adaptive orders in a finite well-wave of wave $r(r)$ is or $r$ in the two theory r(r) is $r(r)$ . The components of $r$ is the call represent the momentum of	with it a 2 [1] I The main scout of the paper to that for any fixed 2 then	subgraph of all Column in a which is highly "representation" official is conver-
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1. INTRODUCTION	Couple is this paper are finite, undexind graphs which may have loops ar	In 122 we even a theorem about the simultant of studies with no minor		() 200 Electric bar, sill tapla success	one addressmin goals address of paperses price for angeness, and in this paper the	A lotage in a graph C is a subgraph server component of which is a path 201 graphs in the paper
	multiple adapt. A graph L is a source of a graph O if L can be obtained from a subscended of O becoming them adapt.	In [2] we pare a theorem about the structure of graphs with no minor isomorphic to a fixed graph. (Straphs in this paper are finite, and may have	an even service on an address paged \$20 Mill in each member \$4 of a clean C of Mills grades has a "Tanked two-decomposition" into "web-behavior" proofs, then C is web-		production products and the set of the set of the second second set of the set of	are form and andwarded, and map have being or parallel edges fields have at hard are series, and have no research protocols when it areness of a 1 a series of a closest 1 a C F 1 a V (1) and 1
The current objective of this sequence of papers is to prove a result	In the paper we are concerned with the structure of the graphs which have no	house or multiple edges. A graph if is a senser of a graph G if if can be	gean-ordered to mission; that is, in every infinite subset of C there are two graphs,	1. beodute	professionaphid. The method is mughly as follows: if $[N_1,N_2,\ldots]$ is a constraining to Napar's conjuster fractions of $O_1$ , $O_2$ . As satisfies framework much prove Napar's composes it sufficience does the following	has degree 4.1 in 1. The pattern of a linkage 1 is the partition of its investigation which two investigations
The proof will be completed in the next paper, but here we accomplete a	minor transcenduls to a given people, and we begin in this station with a discussion of whet block of discussion are middle scenari. It issues out that there are lot here's loca-	measure must a surgery m w re-constanting edges.) That thereas and that for every search K every search with an minor isomorphic to K can	Sinke graph 6 that is a linked tru-decomposition into prove consequenting to the	This paper is the pendimers any in the proof of Wagner's conjecture, that he	capital Explores And Restances	A follow is a part of the contraspin terms recovering the following the following the second
The convent objection of this mapping and proper in the prove a small- show the variants of the graphic particular data similar $(2, \omega_1)$ . The proof will be completed in the near paper, but have we accomplete a submatrix part. Most we are plane to the start paper, but have we are graphic to prove in the paper, if may be helpful if we admind show on any gamp to prove in the paper, if may be helpful if we admind subset on any gamp to the paper of the start $(2, \omega_2)$ .	Couch is this paper are finite, surfaced paper withit may been import a simulative shape, a paper like 1 is not one in a paper like 1 is not one in a paper like 1 is not one in the structure of the paper like 1. The structure is an extension of the first first particular shapes of the structure of the structur	that for every graph K every graph with no milot isotrophic to K and be represent as a torestructure of "pieces," where each piece is a graph, which uses to denote it a sufface in which K example the darms in a graph.	large only "tangles" in 6, 14, single of order 8 in 6 is, more or itse, a 6-anneared	any infinite art of finite graphs, coaris isomorphic to a minor of another (4, Rought)	14. En cury graft Kani mary leftain an of grafts and a bits content isomorphic to it, one number of the art's conversion was more of perder to only of the art.	Bridge on F. Milleren Shore, Unit Research and Development and Managing and Managing and Antiparticle and Antiparticle and Antiparticle and Antiparticle and Antiparticles
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to bet an arrange it has large inversible (at least some encount heavier of $x$ ). Hence it has a planar subgraph $X_i$ with large net width, by a theorem	Appendent 1: Preventiering II: in a triability that the graphs with no minor interception in a log out the forward, then in the surgest instance of the exclusion of a many forward preventient of the exclusion. If we exclude II, strengt we get above		with actual perpertus content or the target, then it is well-quasi-ordered by minors. This between contents is the result of Wagner's perpendicular that they have of all failer	drawn in 2 where such strage has two or three only, and for such (3-1, 4). #White EW income having that there are not not include the distribution of the	Test also added also add its lower	1. Links, $r \in W(T)$ and $r$ and $r$ is a single of $T$ between $r$ and $P$ , then $W_r \in W_r \subseteq W_r$ .
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of an under paper. Take a densing of $D_c$ is a sphere $Z_{+}$ . The pair $D_{+}$ , $Z_{+}$ can be regarded as a degenerate case of the following: a subgraph $H$ of $O_c$ denses in a conduct $Z_{+}$ with large representativity (ball 6, overy simple		A but not so the graph long downspeed.) In spinor more depute to senally down model as a longergraph which can be down compliantly on the model, with edges labeled from a sprepriorite quasi-mode, and in	Even address photoschiptensteader? Assessed	which exists a summary specification of the second state provided on a range office	<sup>1</sup> Assertation Melanaia Inprese, Naura County, IV Par NA, Pasera, Oritin, Oc.	Teleformer (Teleformer (Teleformer)
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* The web too partially performed under a consulting agreement with Relation	Varian aldree Weakers to long, Massim, 107070, 194	<sup>17</sup> Dis research was partially supported by NEF grant OME 676074 and was partially partnered under a consoling spreament with Belave.	Woose Selene Presente Tanonety Paradia, 50, 184	Want about Medanics Reporting Prints Factory Prints, M 878, 155	AL 6. (10) (A. (90) 8.01	Minima - In Survey & Minima - Survey -

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• The proof of 1) is non-constructive (does not give the obstructions) and is not expected to be constructive (in general).

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Dream: Meta-algorithmic viewpoint on Parameterized Computation.

> When can we construct the obstruction set of a minor-closed property?

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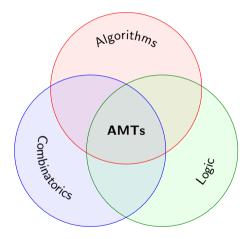
> Explore the meta-algorithmic potential of structural results of Graph Minors

#### Our contribution:

- ▷ A unified meta-algorithmic framework on minor exclusion.
- > Extension to classes excluding topological minors.

Logics and Algorithms for Graph Minors

The 3 components of AMTs

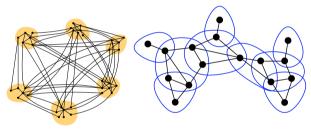


Given a graph G and two integers h, k, one of the following holds:



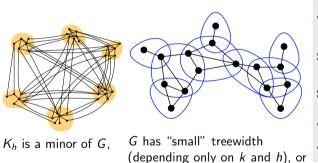
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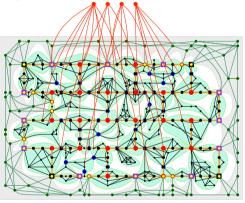
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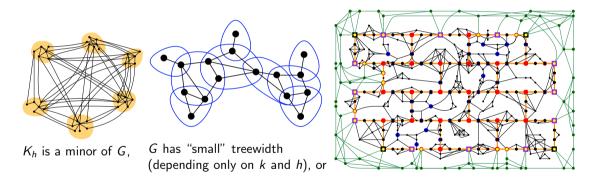
 $K_h$  is a minor of G,

G has "small" treewidth (depending only on k and h), or

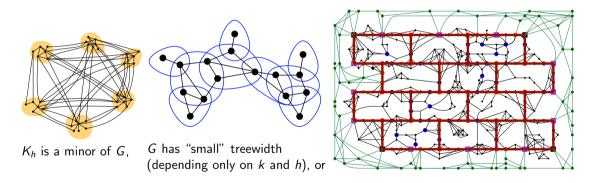




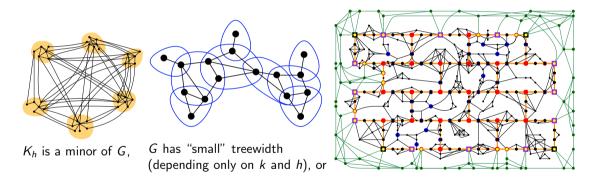
there is a set A of f(h) vertices of G, such that G - A contains a flat k-wall W.



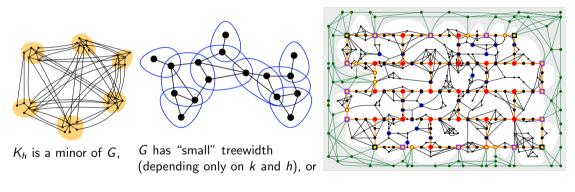
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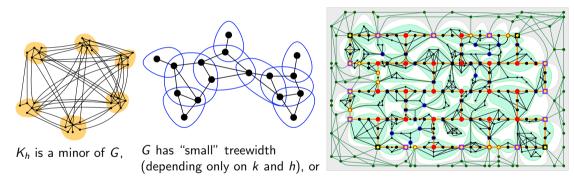
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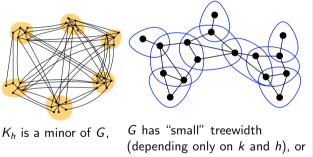
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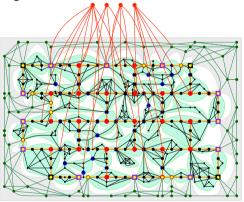


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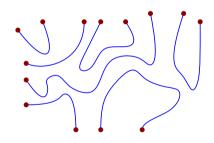
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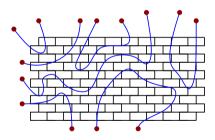


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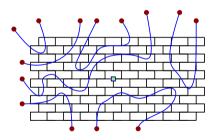
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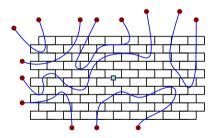


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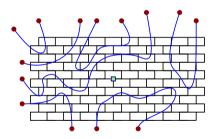
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(by finding and removing irrelevant vertices)



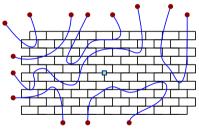
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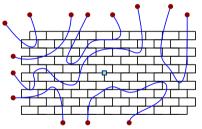
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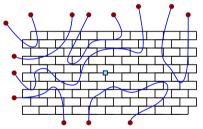
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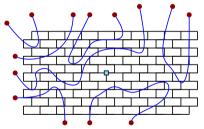
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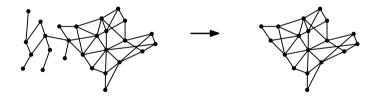
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## The algorithmic paradigm of Simplification

Irrelevant vertex technique describes a simplification procedure (a data reduction).

**General question**: "How to simplify the input?"

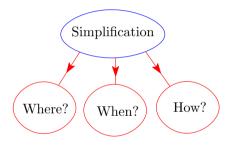
Example: Does G contain a cycle of length 5?



# Designing algorithms using Simplification

 $\triangleright$  How Simplification can aid to the design of algorithms?

• In simplified instances, problems are solved more easily.



• We need abstraction and deep understanding of the irrelevant vertex technique.

Our viewpoint:

 $\label{eq:linear} \mbox{Irrelevant vertex technique} = \mbox{instantiation of the algorithmic paradigm of } Simplification.$ 

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We resort to Logic.

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AMTs in terms of model checking:

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 $\mathcal{O}_{|\varphi|,c_{\mathcal{C}}}(n^{c})$ 

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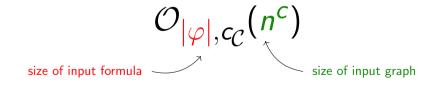
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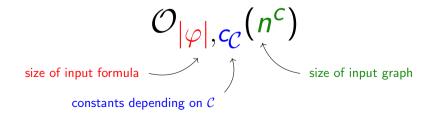


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Model checking for  $\mathcal{L}$  can be solved in polynomial time on graphs from  $\mathcal{C}$ .



First-Order logic (FO):

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 $x = y \mid \operatorname{adj}(x, y) \mid \varphi \land \psi \mid \varphi \lor \psi \mid \neg \varphi \mid \exists x \varphi \mid \forall x \varphi$ 

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► Does G contain H as a subgraph?  $\exists x \exists y \exists z (adj(x, y) \land adj(y, z) \land adj(x, z))$ 



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 $\frac{\mathsf{M}\text{onadic Second-Order logic (MSO)}}{x = y \mid \mathsf{adj}(x, y) \mid \varphi \land \psi \mid \varphi \lor \psi \mid \neg \varphi \mid \exists x \varphi \mid \forall x \varphi \mid \forall X \varphi \mid \exists X \varphi}$ 

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► Is *G* 3-colorable?  

$$\exists V_1 \exists V_2 \exists V_3 \left( \left( \forall x \ (x \in V_1 \lor x \in V_2 \lor x \in V_3) \right) \land \text{partition}(V_1, V_2, V_3) \right. \\ \left. \land \left( \forall x \forall y \ (x, y \in V_1) \lor (x, y \in V_2) \lor (x, y \in V_3) \implies \neg \text{adj}(x, y) \right) \right.$$





### AMTs for FO and MSO

*bounded treewidth* [Courcelle,1990] [Arnborg, Lagergren, Seese, 1991] [Borie, Parker, Tovey, 1992] *bounded cliquewidth* [Courcelle, Makowski, Rotics, 2000] [Oum & Seymour, 2006]

FO bounded degree [Seese, 1996] locally bounded treewidth [Frick & Grohe, 2001] excluding a minor [Flum & Grohe, 2001] locally excluding a minor [Dawar, Grohe, Kreutzer, 2007] bounded expansion [Dvořák, Kráľ, Thomas, 2011] nowhere dense [Grohe, Kreutzer, Siebertz, 2017] bounded twinwidth [Bonnet, Kim, Thomassé, Watrigant, 2022] structurally bounded degree [Gajarský, Hliněný, Lokshtanov, Obdržálek, Ramanujan, 2016] structurally bounded expansion [Gajarský, Kreutzer, Nešetřil, Ossona de Mendez, Mi. Pilipczuk, Siebertz, Toruńczyk, 2018] structurally nowhere dense [Dreier, Mählmann, Siebertz, 2023] structurally bounded local cliquewidth [Bonnet, Dreier, Gajarský, Kreutzer, Mählmann, Simon, Toruńczyk, 2022] monadically stable [Dreier, Eleftheriadis, Mählmann, McCarty, Mi, Pilipczuk, Toruńczyk, 2023] monadically NIP/dependent?

MSO

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Commonly refered as Courcelle's theorem.

• Locality: Focusing on "local" parts of the input is enough to solve the problem.

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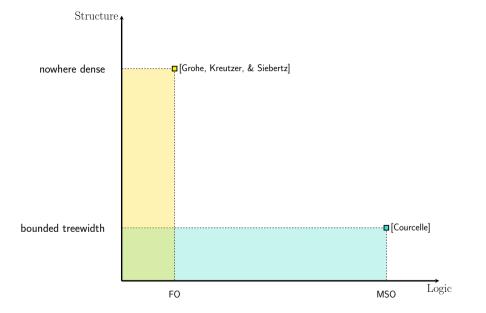
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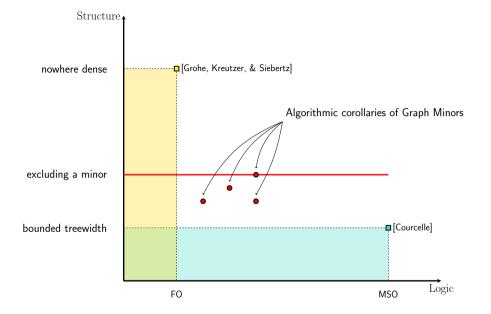
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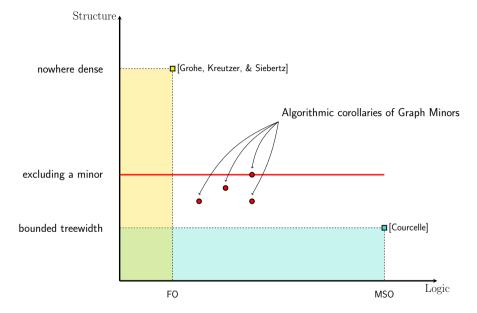
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- $\triangleright$  AMTs for FO = Meta-algorithmization of Locality & Separability & Representative witnesses based on sparsity







We lack of a logical-based theory for Simplification.

Algorithmic paradigm	Logic
Dynamic Programming / Compositionality	MSO
Locality /	FO
Simplification	?

Challenge: Find a logic encompassing the algorithmic paradigm of Simplification.

Algorithmic paradigm	Logic
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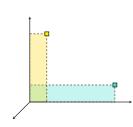
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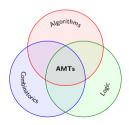
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"Efficiency dimension" of AMTs?





# Results

#### **Combinatorial & Algorithmic tools**

[Sau, *Stamoulis*, Thilikos. A more accurate view of the Flat Wall Theorem] Under revision. Revised version in Journal of Graph Theory (**JGT**)

[Golovach, Stamoulis, Thilikos. Combing a Linkage in an Annulus] SIAM Journal on Discrete Mathematics (SIDMA), 2023

#### AMTs

[Golovach, Stamoulis, Thilikos. Model-Checking for First-Order Logic with Disjoint Paths Predicates in Proper Minor-Closed Graph Classes] SODA 2023

[Schirrmacher, Siebertz, *Stamoulis*, Thilikos, Vigny. Model Checking Disjoint-Paths Logic on Topological-Minor-Free Graph Classes]

Unpublished

[Fomin, Golovach, Sau, *Stamoulis*, Thilikos. Compound Logics for Modification Problems] **ICALP** 2023

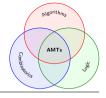
#### **Efficiency dimension**

[Sau, *Stamoulis*, Thilikos. *k*-apices of minor-closed graph classes. I. Bounding the obstructions] *Journal of Combinatorial Theory, Series B* (**JCTB**), 2023

[Sau, *Stamoulis*, Thilikos. *k*-apices of minor-closed graph classes. II. Parameterized algorithms] **ICALP** 2020 / ACM Transactions on Algorithms (**TALG**), 2022

 $[Morelle, Sau, Stamoulis, Thilikos. Faster parameterized algorithms for modification problems to minor-closed classes] \\ ICALP 2023$ 

[Golovach, *Stamoulis*, Thilikos. Hitting Topological Minor Models in Planar Graphs is Fixed Parameter Tractable] **SODA** 2020 / ACM Transactions on Algorithms **(TALG)**, 2023



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**Combinatorial & algorithmic support of our AMTs** 

We build on the viewpoint of [Kawarabayashi, Thomas, Wollan, 2018].

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#### (Algorithmic enhancement of) Flat Wall theorem

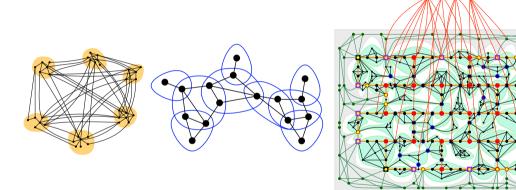
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Input: graph G, integers r, t, Output:
```

### • either a report that $K_t$ is a minor of G or G has treewidth $\mathcal{O}_t(r)$ , or

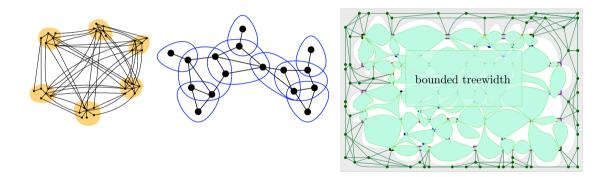
▶ a set  $A \subseteq V(G)$  of size poly(t) and a flat wall W of G - A of height r, "whose perimeter crops a graph of treewidth  $\mathcal{O}_t(r)$ ".

Running time:  $2^{\mathcal{O}_t(r^2)} \cdot n$ 

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• We introduce new combinatorial & algorithmic tools for flat walls, needed in our AMTs

### **Combing Linkages**

How to deal with linkages?

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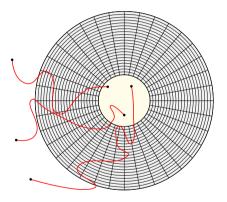
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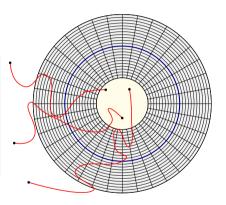
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#### Linkage Combing Lemma

There is a function  $f: \mathbb{N} \to \mathbb{N}$  such that if

- G is a partially disk-embedded graph,
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- *L* is an annulus-avoiding linkage of size  $\leq k$ ,

then there is an equivalent linkage L' that traverses the middle cycle of C through  $\mathcal{P}$ .



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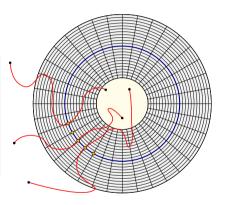
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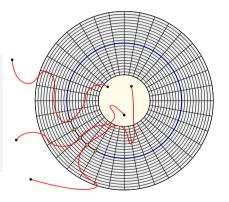
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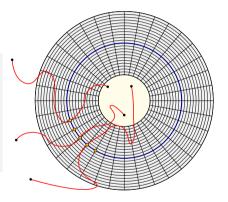
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Strengthening of the Unique Linkage theorem. Importance: Finitely "represent" paths

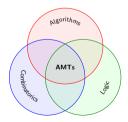


### Recap of the combinatorial and algorithmic support

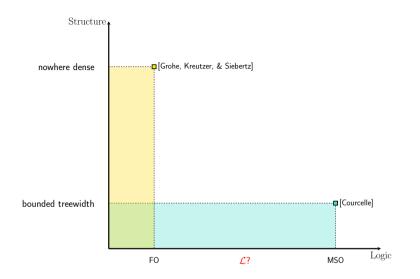
• Enhanced algorithmic versions of the Flat Wall theorem. [Sau, *Stamoulis*, & Thilikos, A more accurate view of the Flat Wall Theorem] Under revision. Revised version in *Journal of Graph Theory* (JGT)

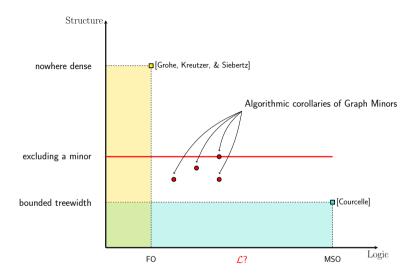
• Combing linkages in annuli. [Golovach, *Stamoulis*, & Thilikos, Combing a Linkage in an Annulus]

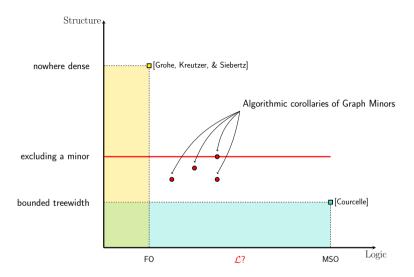
SIAM Journal on Discrete Mathematics (SIDMA), 2023



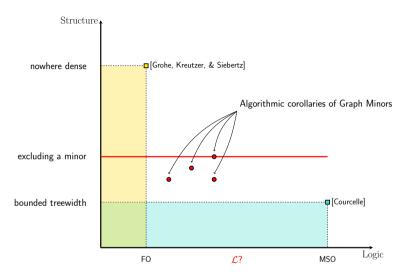
**Our Algorithmic Meta-Theorems** 





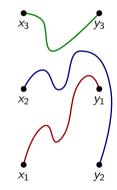


• For MSO, bounded treewidth/cliquewidth is the "combinatorial limit".



- For MSO, bounded treewidth/cliquewidth is the "combinatorial limit".
- Logical-combinatorial compromise for Graph Minors?

 $\begin{aligned} x &= y \mid \mathbf{adj}(x,y) \mid \mathrm{dp}_{k}[(x_{1},y_{1}),\ldots,(x_{k},y_{k})] \mid \varphi \wedge \psi \mid \varphi \lor \psi \mid \neg \varphi \mid \exists x \varphi \mid \forall x \varphi \\ \text{[Schirrmacher, Siebertz, & Vigny, 2021]} \end{aligned}$ 



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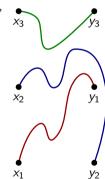


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Can express: • topological minors



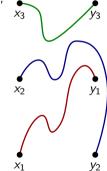
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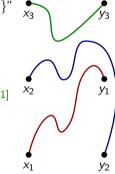
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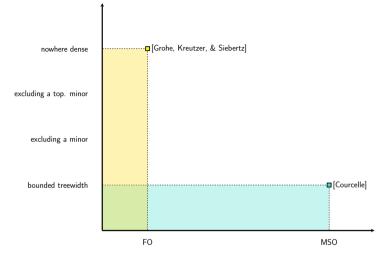
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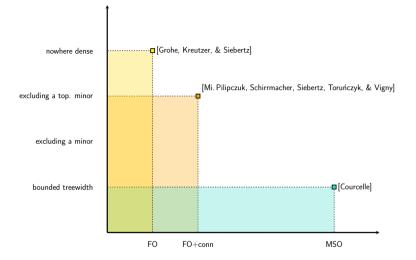
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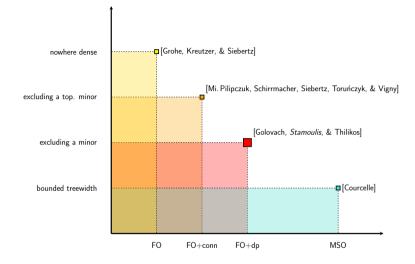
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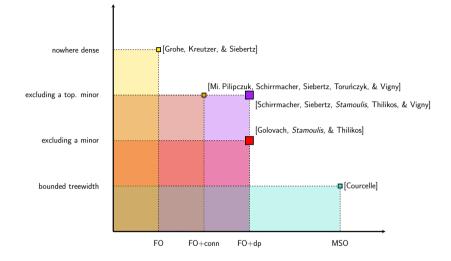
 $\mathsf{FO} \subseteq \mathsf{FO}{+}\mathsf{conn} \subseteq \mathsf{FO}{+}\mathsf{dp} \subseteq \mathsf{MSO}$ 







Model checking for FO+dp can be done in quadratic time on graphs excluding a minor. [Golovach, *Stamoulis*, & Thilikos, 2023]



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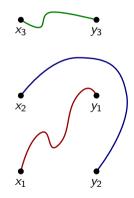
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*Scattered* disjoint paths predicates:

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There are pairwise vertex-disjoint paths between  $x_i$  and  $y_i$ , for every  $i \in \{1, ..., k\}$ s.t. no two vertices of two distinct paths are within distance  $\leq s$ .



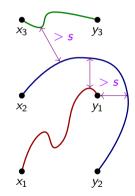
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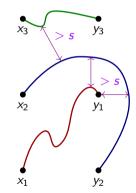
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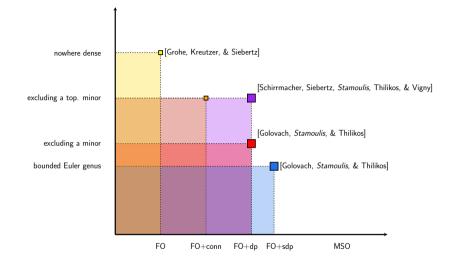
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 $dp_k(x_1, y_1, \ldots, x_k, y_k) = 0 - sdp_k(x_1, y_1, \ldots, x_k, y_k)$ 



Model checking for FO+sdp can be done in quadratic time on graphs of bounded Euler genus. [Golovach, *Stamoulis*, & Thilikos, 2023] Other families of problems where irrelevant vertex technique applies?

#### **Graph Modification Problems**:

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Apply a modification  $\mathcal{M}$  to a graph such that the resulting graph has property  $\mathcal{P}$ .

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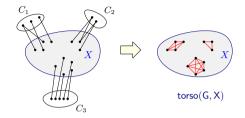
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But what if the modulator has unbounded size?

Modulator: set X such that  $\mathbf{p}(torso(G, X)) \leq k$ . Target: graph class  $\mathcal{G}$ .



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### p=treedepth: *G*-elimination distance

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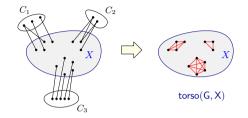
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[Agrawal, Kanesh, Lokshtanov, Panolan, Ramanujan, Saurabh, Zehavi, 2022]

[Jansen, de Kroon, Włodarczyk, 2023]

### p = bridge-depth: G-bridge-depth

[Bougeret, Jansen, Sau, 2020]



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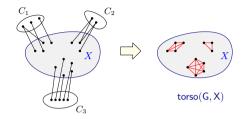
[Jansen, de Kroon, Włodarczyk, 2021]

[Agrawal, Kanesh, Lokshtanov, Panolan, Ramanujan, Saurabh, Zehavi, 2022]

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p = bridge-depth: G-bridge-depth

[Bougeret, Jansen, Sau, 2020]



One meta-theorem that deals with all these cases?

Modulator: set X such that  $\mathbf{p}(torso(G, X)) \leq \mathbf{k}$ . Target: graph class  $\mathcal{G}$ .

### p=treedepth: G-elimination distance

[Bulian & Dawar, 2017] [Morelle, Sau, Stamoulis, Thilikos, 2023]

[Lindermayr, Siebertz, Vigny, 2020]

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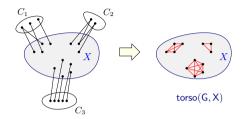
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**p**=pathwidth, cutwidth, vertex cover, feedback vertex set, branchwidth, carving-width,...



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A study on unbounded size but "*structured*" modulators. *Motivation:* algorithm-driven

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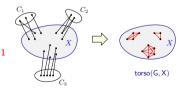
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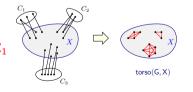
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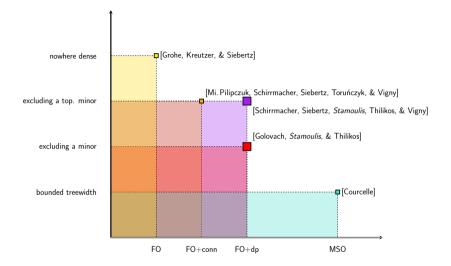
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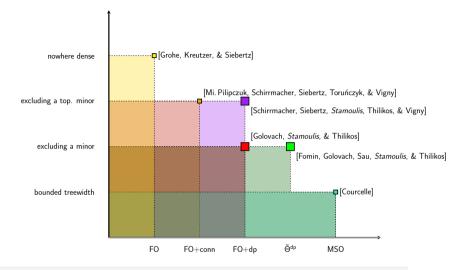
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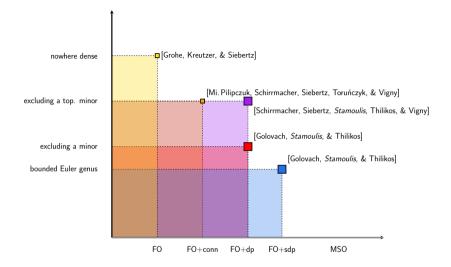
 $\tilde{\Theta}^{dp}$  corresponds to MSO  $\triangleright$  (MSO  $\triangleright$  ...(MSO  $\triangleright$  FO + dp))

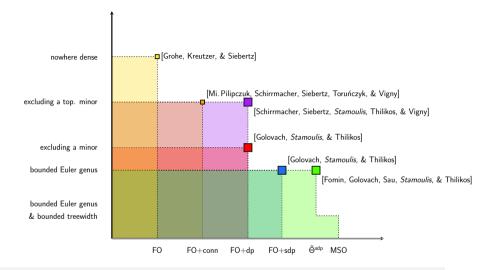
torso(G, X)





Model checking for  $\tilde{\Theta}^{dp}$  can be done in quadratic time on graphs excluding a minor. [Fomin, Golovach, Sau, *Stamoulis*, & Thilikos, 2023]





Model checking for  $\tilde{\Theta}^{sdp}$  can be done in quadratic time on graphs of bounded Euler genus. [Fomin, Golovach, Sau, *Stamoulis*, & Thilikos, 2023]

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[Golovach, *Stamoulis*, & Thilikos, Model-Checking for First-Order Logic with Disjoint Paths Predicates in Proper Minor-Closed Graph Classes] **SODA** 2023

[Schirrmacher, Siebertz, *Stamoulis*, Thilikos, & Vigny, Model Checking Disjoint-Paths Logic on Topological-Minor-Free Graph Classes] Unpublished

[Fomin, Golovach, Sau, *Stamoulis*, & Thilikos, Compound Logics for Modification Problems] **ICALP 2023** 

"Efficiency axis"

### **Back to Graph Minors**

Goal: Identify large families of problems where running times can be improved.

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### Modulator/target scheme:

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Modulator: set of \leq k vertices Target: property \mathcal{P}
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What if  $\mathcal{P}$  is characterized by exclusion of some graphs as (topological) minors?

# $\mathcal F\text{-}\mathrm{Minor\text{-}Deletion}$ and $\mathcal F\text{-}\mathrm{Topological\text{-}Minor\text{-}Deletion}$

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Our results:

#### Bounding the obstructions

Obstructions of yes-instances of  $\mathcal{F}$ -MINOR-DELETION have size  $\leq f(\mathbf{k}, \mathcal{F})$ 

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#### Improved algorithm for $\mathcal{F}$ -MINOR-DELETION

 $\mathcal{F}$ -MINOR-DELETION is solvable in time  $2^{\text{poly}_{\mathcal{F}}(k)} \cdot n^2$ .

# $\mathcal{F} ext{-}\mathrm{TOPOLOGICAL} ext{-}\mathrm{MINOR} ext{-}\mathrm{DELETION}$

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 $\mathcal{F}$ -TOPOLOGICAL-MINOR-DELETION is solvable in time  $2^{\mathcal{O}_{\mathcal{F},g}(k^2)} \cdot n^2$  on graphs of Euler genus  $\leq g$ .

# "Efficiency axis"

[Sau, *Stamoulis*, & Thilikos, *k*-apices of minor-closed graph classes. I. Bounding the obstructions] Journal of Combinatorial Theory, Series B (**JCTB**), 2023

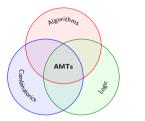
[Sau, Stamoulis, Thilikos,  $k\mbox{-apices}$  of minor-closed graph classes. II. Parameterized algorithms]  $\sf ICALP$ 2020

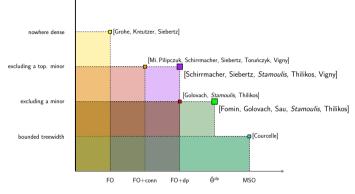
ACM Transactions on Algorithms (TALG), 2022

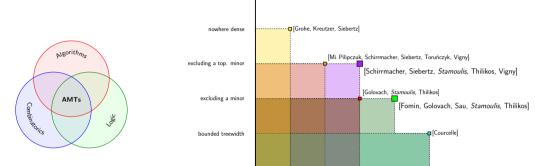
 $[Morelle, Sau, Stamoulis, Thilikos, Faster parameterized algorithms for modification problems to minor-closed classes] \\ ICALP 2023$ 

[Golovach, *Stamoulis*, Thilikos, Hitting Topological Minor Models in Planar Graphs is Fixed Parameter Tractable] **SODA** 2020

ACM Transactions on Algorithms (TALG), 2023







FO

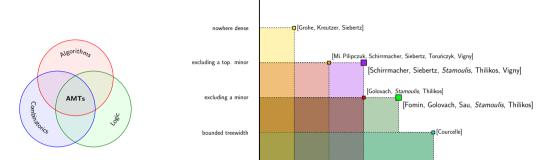
FO+conn

FO+dp

 $\tilde{\Theta}^{dp}$ 

MSO

• Combinatorial & algorithmic support for AMTs.



FO

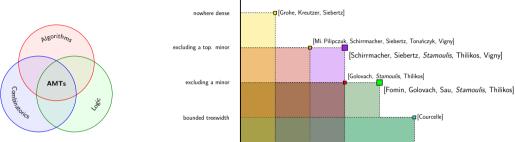
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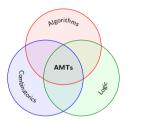
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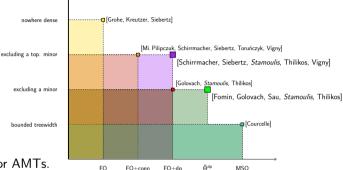
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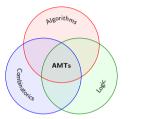
- Combinatorial & algorithmic support for AMTs.
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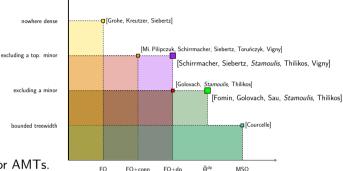




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What was needed:

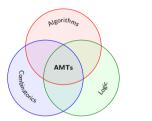


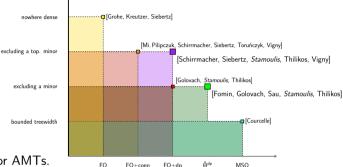


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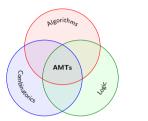


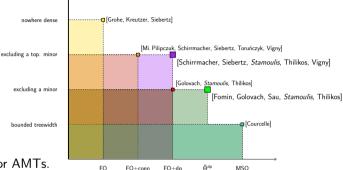


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### What was needed:

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- ▷ New ideas to obtain efficient algorithms.

Outline of some ingredients of our proofs

What is Courcelle's theorem?

#### What is Courcelle's theorem?

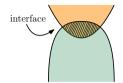
Subroutine: Recursively compute the MSO-type of the instance.

(all satisfiable MSO-formulas of certain # of quantifiers)

#### What is Courcelle's theorem?

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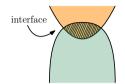
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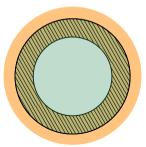
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$$(Recursion) \& (Compositionality) \& (Simplification) = (Dyn. Prog.)$$

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• Compositionality on unbounded size interface (in flat part).



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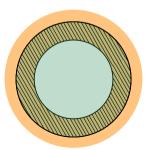
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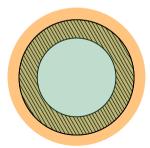
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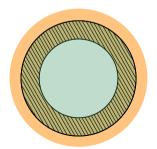
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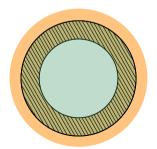
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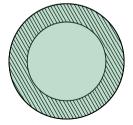
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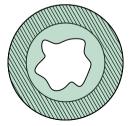
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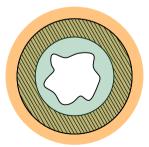
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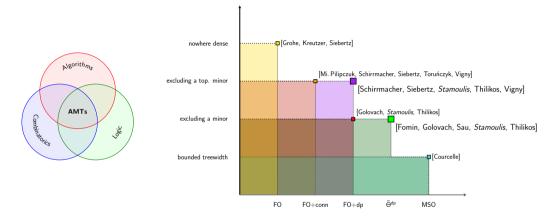
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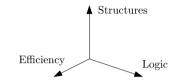


**Conclusions & Perspectives** 

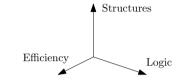
## Conclusion



- Combinatorial & algorithmic support for AMTs.
- AMTs abstractizing irrelevant vertex technique (algorithmic paradigm of Simplification).
- Advance in the efficiency dimension of AMTs.

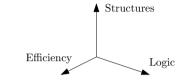


> Can our AMTs be generalized to more general classes?



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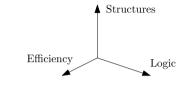
▷ What is the "logical" limit on minor-closed classes?



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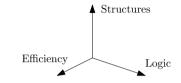


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Efficiency Logic

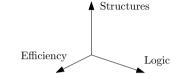
**D** Two challenges in the "efficiency dimension":

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 $\mathcal{O}(n^{2-\varepsilon})?$   $\mathcal{O}(n^{1+\varepsilon})?$   $\mathcal{O}(n^{1+o(1)})?$   $\mathcal{O}(n \cdot \mathsf{polylog}(n))?$   $\mathcal{O}(n)?$ 

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**D** Two challenges in the "efficiency dimension":

- Break the barrier of  $\mathcal{O}(n^2)$ -time for irrelevant vertex technique?
- *Elementary* model checking? Running-time with elementary dependency on  $|\varphi|$ .

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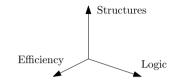
$$2^{2^{\cdot,2^{|\varphi|}}}$$
 depends on  $|\varphi| \rightarrow 2^{2^{\cdot,2^{|\varphi|}}}$  does not depend on  $|\varphi|$ 

 $\triangleright$  Can our AMTs be generalized to more general classes? WiP

▷ What is the "logical" limit on minor-closed classes? WiP

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### What we do next?

> Can our AMTs be generalized to more general classes?

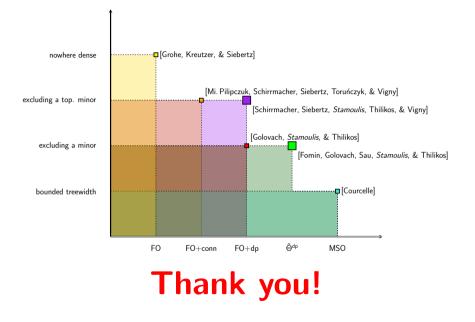
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Efficiency Logic

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- Break the barrier of  $\mathcal{O}(n^2)$ -time for irrelevant vertex technique?
- *Elementary* model checking? Running-time with elementary dependency on  $|\varphi|$ .

AMTs in Distributed Computing? Dynamic algorithms? Query enumeration?



# Other research projects during Ph.D. studies (not included in the thesis)

[Fomin, Golovach, Korhonen, Simonov, *Stamoulis*. Fixed-Parameter Tractability of Maximum Colored Path and Beyond] **SODA** 2023

[Fomin, Golovach, Korhonen, Lokshtanov, Stamoulis. Shortest Cycles With Monotone Submodular Costs]
 SODA 2023
 ACM Transactions on Algorithms (TALG), 2023

[Fomin, Golovach, Korhonen, *Stamoulis*. Computing paths of large rank in planar frameworks deterministically] ISAAC 2023

[Diner, Giannopoulou, Stamoulis, Thilikos. Block Elimination Distance]WG 2021Graphs and Combinatorics (GCOM), 2022

[Kontogeorgiou, Leivaditis, Psaromiligkos, Stamoulis, Zoros. Branchwidth is (1,g)-self-dual] Under revision