Additive cubes are avoidable over all but one set of four numbers

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Abstract

We show that for any set $\mathcal{A} \subseteq \mathbb{N}$ of size 4 such that $\mathcal{A}$ cannot be obtained by applying the same affine function to all the elements of $\{0, 1, 2, 3\}$, there is an infinite sequence of elements of $\mathcal{A}$ that contains no 3 consecutive blocs of same size and same sum. In fact, a rather simple argument allow us to replace $\mathbb{N}$ by $\mathbb{C}$ in the previous sentence, but the challenging part is to do it for integers (or rationals) alphabets. Cassaigne et Al. showed that this was the case for the set $\{0, 1, 3, 4\}$. The idea of their proof was used to show that this was also true for other alphabets. However, before the present paper it was known to be true only for finitely many alphabets (up to a trivial equivalence relation).

Keywords:
Abelian/additive equivalence, abelian/additive powers, combinatoric on words

1 Introduction

Let $k \geq 2$ be an integer and $(G, +)$ a semigroup. An *additive kth power* is a non empty word $w_1 \cdots w_k$ over $\mathcal{A} \subseteq G$ such for every $i \in \{2, \ldots, k\}$, $|w_i| = |w_1|$ and $\sum w_i = \sum w_1$ (where $\sum v$ denotes the sum of the letters in v seen as integers). It is a long standing question whether there is an infinite word w over a finite subset of $\mathbb{N}$ that avoids additive squares (additive 2nd powers) [4, 6]. One of the motivation is that a positive answer to this question would imply that additive squares are avoidable over any semigroup that contains some finitely generated infinite semigroup [6] (a simple application of Van der Waerden's theorem shows that additive powers are not avoidable over any other semigroup, see for example [3]). Cassaigne et al. [1] showed that there is an infinite word over the finite alphabet $\{0, 1, 3, 4\} \subseteq \mathbb{Z}$ without additive cubes (additive 3rd powers). Rao used this result to show that there are infinite words avoiding additive cubes over any alphabet $\{0, i, j\} \in \mathbb{N}^3$ with i and j co-prime, $i < j$ and $6 \leq j \leq 9$ (and he conjectured that the second condition can be replaced by $6 \leq j$) [7]. This motivates the following more general problem:

Problem 1.1. *Characterize the finite subsets of* $\mathbb{N}$ *over which additive cubes are avoidable.*

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It seems restrictive to use $\mathbb{N}$ instead of $\mathbb{R}$ (or $\mathbb{C}$), but solving Problem 1.1 for alphabets of the form $\{0, a_1, \ldots, a_m\} \in \mathbb{N}$ with the a_is being co-prime completely solves the problem for any finite alphabet over $\mathbb{C}$ (if the a_i are given by increasing order one can even add the condition that a_1 be smaller than $a_m - a_{m-1}$). For the sake of completeness, we give a short proof of this fact in the preliminaries.

If Rao's conjecture was to be true then the only remaining 3-letter alphabets over $\mathbb{C}$ to characterize, would be $\{0,1,2\}$, $\{0,1,3\}$, $\{0,1,4\}$ and $\{0,2,5\}$ (see [9, Section 2.2.2] for details). However, this conjecture was verified for only finitely many such alphabets (up to a trivial equivalence relation). Here we propose a rather simple twist on previously used ideas to show that additive cubes are avoidable over any alphabet $\mathcal{A} \subseteq \mathbb{N}$ of size 4 as long as $\mathcal{A}$ is not equivalent to $\{0,1,2,3\}$. This also implies that additive cubes are avoidable over any alphabet of numbers of size at least 5. Rao used the fact that additive cubes are avoidable over $\{0,1,3,4\}$ to show that they are avoidable over some 3-letter alphabets [7], so our result might also be an interesting milestone for solving Problem 1.1 for alphabets of size 3.

The paper is organized as follows. We first recall some notations and we give a simple equivalence relation such that for any two equivalent alphabets additive cubes are avoidable over the first if and only if they are avoidable over the second. Equipped with this equivalence relation we explain why it is enough to study alphabets in a particular form. Then we introduce the word $\mathbf{w_{a,b,c,d}}$ and we show that for all but finitely (up to our equivalence relation) many values of a, b, c, and d, $\mathbf{w_{a,b,c,d}}$ avoids additive cubes. Finally, using the literature for the remaining alphabets, we conclude that additive cubes are avoidable over all the remaining alphabets of size 4, but $\{0,1,2,3\}$. We leave the case of $\{0,1,2,3\}$ open.

2 Preliminaries

We use the standard notations introduced in Chapter 1 of [5]. In the rest of this article all our alphabets are finite sets of complex numbers. For the rest of this section let $\mathcal{A} \subseteq \mathbb{C}$ be such an alphabet. We denote by ε the empty word. Given a word $w \in \mathcal{A}^*$, we denote by $|w|$ the length of w and by $|w|_\alpha$ the number of occurrences of a letter $\alpha \in \mathcal{A}$ in w. Two words u and v are abelian equivalent, denoted by $u \simeq_{ab} v$ if $\psi(u) = \psi(v)$, *i.e.* if they are permutations of each other. They are additive equivalent, denoted by $u \simeq_{ad} v$, if $|u| = |v|$ and $\sum u = \sum v$, where $\sum v$ denotes the sum of the letters in v (since the letters are complex numbers). A word $uvw \in \mathcal{A}^*$ is an abelian cube (respectively an additive cube) if $u \simeq_{ab} v \simeq_{ab} w$ (respectively if $u \simeq_{ad} v \simeq_{ad} w$).

2.1 Alphabet from $\mathbb{N}$

For any function $h : \mathbb{C} \to \mathbb{C}$ and words w over $\mathcal{A} \subseteq \mathbb{C}$, $h(w)$ is the word obtained by replacing each letter of w by its image by h. We say that two alphabets $\mathcal{A}, \mathcal{A}' \subseteq \mathbb{C}$ of same size are *equivalent* if there is function $h : \mathcal{A} \to \mathcal{A}'$, such that for all $u, v \in \mathcal{A}^*$,

$$u \simeq_{ad} v \iff h(u) \simeq_{ad} h(v) \, .$$

In this subsection, we show that for any alphabet of complex numbers, we either already know if additive cubes are avoidable or it is equivalent to an alphabet of integers.

We start by giving sufficient conditions for two alphabets to be equivalent.

Lemma 2.1. *Let $a \in \mathbb{C}_{\neq 0}$, $b \in \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$ be the function such that for all x, $f(x) = ax + b$, then*

$$u \simeq_{ad} v \iff f(u) \simeq_{ad} f(v)\,.$$

Proof. By definition for any word w, $|w| = |f(w)|$. Thus for any $u, v \in \mathcal{A}^*$,

$$|u| = |v| \iff |f(u)| = |f(v)|\,.$$

For any $u, v \in \mathcal{A}^*$, such that $|u| = |v|$, the following are equivalent:

$$\begin{aligned}
&\sum u = \sum v, \\
\iff & a\sum u + |u|b = a\sum v + |u|b && \text{(since } a \neq 0) \\
\iff & a\sum u + |f(u)|b = a\sum v + |f(v)|b && \text{(since } |f(u)| = |u| = |v| = |f(v)|) \\
\iff & \sum f(u) = \sum f(v).
\end{aligned}$$

We deduce that

$$\Big(|u| = |v| \text{ and } \sum u = \sum v\Big) \iff \Big(|f(u)| = |f(v)| \text{ and } \sum f(u) = \sum f(v)\Big)$$

which concludes the proof. □

Recall that two complex numbers a and b are said to be rationally independent if there are no $(k_1, k_2) \in \mathbb{Z}$ such that $(k_1, k_2) \neq 0$ and $k_1 a + k_2 b = 0$.

Lemma 2.2. *Let $\mathcal{A} \subseteq \mathbb{C}$ then one of the following holds:*

1. *$|\mathcal{A}| \leq 2$, then additive cubes are not avoidable over $\mathcal{A}$,*
2. *$|\mathcal{A}| > 2$ and there are $a, b, c \in \mathcal{A}$, such that $b - a$ and $c - a$ are rationally independent, then additive cubes are avoidable over $\mathcal{A}$,*
3. *$|\mathcal{A}| > 2$ and for any pairwise different $a, b, c \in \mathcal{A}$, $b - a$ and $c - a$ are rationally dependent, then there is an alphabet $\mathcal{A}' = \{0, a_1, \ldots, a_m\} \in \mathbb{N}$ with the a_i co-primes such that $\mathcal{A}$ and $\mathcal{A}'$ are equivalent.*

Proof. Let us verify each statement one by one.

1. It is easy to verify that abelian cubes are not avoidable over 2 letters [2]. Since the addition is commutative on $\mathbb{C}$, additive cubes are not avoidable over any alphabet of complex numbers of size at most 2.

2. Since $b - a$ and $c - a$ are rationally independent, for any $k_1, k_2, k_3 \subseteq \mathbb{Z}$, if $0k_1 + (b - a)k_2 + (c - a)k_3 = 0$ then $k_2 = k_3 = 0$. Thus for any word $u, v \in \{0, b-a, c-a\}^*$, if $\sum u = \sum v$ then u has the same number of occurrences of $b - a$ (resp. $c - a$) than v; if moreover $|u| = |v|$ then they also have the same number of occurrences of 0. Thus, for any word $u, v \in \{0, b-a, c-a\}^*$, if $u \simeq_{ad} v$ then $u \simeq_{ab} v$ are abelian equivalent. From Lemma 2.1 (with $f : x \to x + a$), for any $u, v \in \{a, b, c\}^*$ such that $u \simeq_{ad} v$ then $u \simeq_{ab} v$ are abelian equivalent. Since abelian cubes are avoidable over 3 letters we deduce that additive cubes are avoidable over $\mathcal{A}$.

3. Let $\{b_1, \ldots, b_m\} = \mathcal{A}$ with $b_1 < b_2 \ldots < b_m$. For any i, $b_i - b_1$ and $b_2 - b_1$ are rationally dependent which implies $\frac{b_i - b_1}{b_2 - b_1} \in \mathbb{Q}_{>0}$. Thus there exists a positive $q \in \mathbb{N}$ such that for all i, $q\frac{b_i-b_1}{b_2-b_1} \in \mathbb{N}$ and the $q\frac{b_2-b_1}{b_2-b_1}, q\frac{b_3-b_1}{b_2-b_1}, \ldots, q\frac{b_m-b_1}{b_2-b_1}$ are co-prime. Finally, we apply Lemma 2.1 with $f : x \to q\frac{x-b_1}{b_2-b_1}$ and we get that the alphabet $\{0 = q\frac{b_1-b_1}{b_2-b_1}, q\frac{b_2-b_1}{b_2-b_1}, q\frac{b_3-b_1}{b_2-b_1}, \ldots, q\frac{b_m-b_1}{b_2-b_1}\}$ satisfies all the required conditions. □

Thus solving Problem 1.1 for alphabets of the form $\{0, a_1, \ldots, a_m\} \in \mathbb{N}$ with the a_is being co-prime completely solves the problem for any finite alphabet over $\mathbb{C}$. Notice that, in case 3., one can add the condition that $a_1 < a_m - a_{m-1}$, (if not one can apply $f : x \to a_m - x$) to this alphabet). One could also add that in the case $|\mathcal{A}| = 2$, one can avoid additive 4th powers (with an argument similar to 2. and the fact that abelian 4th powers are avoidable over 2 letters [2]).

3 The infinite word $\mathbf{W}_{a,b,c,d}$

For any real numbers $a, b, c, d \in \mathbb{R}$, let $\varphi_{a,b,c,d} : \{a, b, c, d\}^* \to \{a, b, c, d\}^*$ be the morphism such that

$$\varphi_{a,b,c,d}(a) = ac \quad ; \quad \varphi_{a,b,c,d}(b) = dc \quad ; \quad \varphi_{a,b,c,d}(c) = b \quad ; \quad \varphi_{a,b,c,d}(d) = ab$$

and $\mathbf{W}_{a,b,c,d} := \lim\limits_{n\to+\infty} \varphi^n_{a,b,c,d}(a)$ be the infinite fixed point of $\varphi_{a,b,c,d}$. Cassaigne et al. [1] showed in 2013 that $\mathbf{W}_{0,1,3,4}$ avoids additive cubes. In particular, it implies that $\mathbf{W}_{0,1,3,4}$ avoids abelian cubes, but since this properties does not depend on the choice of a, b, c, d we easily deduce the following Lemma.

Lemma 3.1. *For any pairwise distinct a, b, c, d the word $\mathbf{W}_{a,b,c,d}$ avoids abelian cubes.*

We define the *Parikh vector* Ψ as a map by

$$\begin{aligned} \Psi : \{a, b, c, d\}^* &\longrightarrow \mathbb{Z}^4 \\ w &\longmapsto {}^t\left(|w|_a \quad |w|_b \quad |w|_c \quad |w|_d\right). \end{aligned}$$

Let $M_\varphi = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ be the adjacency matrix of $\varphi_{a,b,c,d}$ and τ be the eigen vector of M_φ the closest to the following numerical approximation[1]

$$\tau \doteq \begin{pmatrix} 0.5788 - 0.5749i \\ -0.3219 + 0.2183i \\ -0.0690 + 0.6165i \\ -0.1662 - 0.6810i \end{pmatrix}.$$

[1]Remark that this is not an issue for us to use numerical approximation. Indeed, almost all the computations are numerically stable and if we start with good enough approximations, we get good enough approximations at the end (see footnote 2 for the only case where it matters that a coefficient is exactly 0). Moreover, it should be noted that all the eigenvalues of the matrix belong to an algebraic field extension of $\mathbb{Q}$ of degree 24 (according to Mathematica) and thus we could use the original proof of [1, Theorem 8] to get an exact value for C and only use exact computation on the rest of the article. However, exact computation in a field extension of degree 24 in Mathematica are great, but they are much harder to follow than numerical computations for a human being.

We borrow one more Theorem from [1].

Theorem 3.2 ([1, Theorem 8])**.** *There is a constant C such that for any two factors of $\mathbf{w}_{a,b,c,d}$ (not necessarily adjacent) u and v*

$$|\tau \cdot (\Psi(u) - \Psi(v))| < C,$$

where $2.175816 < C < 2.175817$.

Equipped with these two lemmas we easily deduce the following one.

Lemma 3.3. *For any $a, b, c, d \in \mathbb{R}$, let $M_{a,b,c,d} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \end{pmatrix}$. Suppose that $\mathbf{w}_{a,b,c,d}$ contains an additive cube, then there exists a vector $x \in \ker(M_{a,b,c,d}) \cap \mathbb{Z}^4 \setminus \{0\}$ such that $|\tau \cdot x| < C$, where C is given by Theorem 3.2.*

Proof. Let uvw be an additive cube factor of $\mathbf{w}_{a,b,c,d}$. By Lemma 3.1, uvw cannot be an abelian cube. Thus either $\Psi(u) \neq \Psi(v)$ or $\Psi(v) \neq \Psi(w)$. Without lose of generality, $\Psi(u) \neq \Psi(v)$. In this case, let $x = \Psi(u) - \Psi(v) \neq 0$. Since x is the difference of two Parikh vectors we get $x \in \mathbb{Z}^4$. Since uvw is an additive cube $|u| = |v|$ and $|u|_a a + |u|_b b + |u|_c c + |u|_d d = |v|_a a + |v|_b b + |v|_c c + |v|_d d$. This implies that $M_{a,b,c,d}(\Psi(u) - \Psi(v)) = 0$ which can be rewritten $x \in \ker(M_{a,b,c,d})$. We showed that $x \in \ker(M_{a,b,c,d}) \cap \mathbb{Z}^4 \setminus \{0\}$. By assumption u and v are two factors of $\mathbf{w}_{a,b,c,d}$ and by Theorem 3.2 we get

$$|\tau \cdot x| < C,$$

which concludes the proof. □

This really simple Lemma contains the main idea of the article. If we want to know for which choices of a, b, c and d, the word $\mathbf{w}_{a,b,c,d}$ avoids additive cubes, we only need to study the behavior of the lattice $\ker(M_{a,b,c,d}) \cap \mathbb{Z}^4 \setminus \{0\}$ which is tedious but only relies on simple arguments.

4 The case of $\mathbf{W}_{0,1,c,d}$

We study the lattice $\ker(M_{0,1,c,d}) \cap \mathbb{Z}^4 \setminus \{0\}$, for $c, d \in \mathbb{R}$, to show that in many cases additive cubes are avoidable over $\{0, 1, c, d\}$.

Theorem 4.1. *Let $c, d \in \mathbb{R}$. Suppose we have $d > c > 1$, $c \notin \{5/4, 4/3, 3/2, 2\}$ and $d \notin \{6-4c, 5-3c, 4-2c, 3-c, 2c-3, 2c-2, 2c-1, 3c-3, 2\}$ then $\mathbf{w}_{0,1,c,d}$ avoids additive cubes.*

Proof. From Lemma 3.3, it is enough to show that under our assumptions for any $x \in \ker(M_{0,1,c,d}) \cap \mathbb{Z}^4 \setminus \{0\}$, we get $|\tau \cdot x| \geq C$. Let us first express this set of vectors in a more convenient way. It is straight forward to check that if $\alpha = (c-1, -c, 1, 0)$ and $\beta = (d-1, -d, 0, 1)$, then α, β is a base of $\ker(M_{0,1,c,d})$. For any reals m and n, if $m\alpha + n\beta$ is an integral vector then $m \in \mathbb{N}$ (resp. $n \in \mathbb{N}$) because otherwise the third (resp. the fourth) coordinate is not an integer and $mc + nd \in \mathbb{Z}$ otherwise the first and second coordinates are not integers. We deduce that

$$\ker(M_{0,1,c,d}) \cap \mathbb{Z}^4 = \{m\alpha + n\beta | m, n \in \mathbb{Z}, mc + nd \in \mathbb{Z}\}.$$

Thus, we only need to show that, under our assumptions, for any $m, n \in \mathbb{Z}$ with $mc + nd \in \mathbb{Z}$ and $(m, n) \neq (0, 0)$, we get $|\tau \cdot (m\alpha + n\beta)| \geq C$.

Let us first show that this is the case if $n = 0$. In this case, $m \neq 0$, $|\tau \cdot m\alpha| = |m||\tau \cdot \alpha|$ and $mc \in \mathbb{Z}$. Numerical computation gives $f_0(c) := |\tau \cdot \alpha| \doteq \sqrt{1.83908 + c(-3.05698 + 1.44043c)}$. It is easily verified that the minimum of f_0 is reached at $c \doteq \frac{3.05698}{2 \times 1.44043} \doteq 1.06114$. Thus for any interval $x, y \in \mathbb{R}$ with $x < y$ and $1.06114 < y$ the minimum of f_0 over the interval $[x, y]$ is given by $f_0(\max\{1.06114, x\})$. We split this in few similar cases depending on the value of c.

- If $c > 2.85$ a simple computation gives $|\tau \cdot \alpha| > C$ and $|\tau \cdot m\alpha| > C$.
- If $c \in [1.9, 2.9] \setminus \{2\}$, a simple computation gives $|\tau \cdot \alpha| > \frac{C}{2}$. Moreover in this case the conditions $m \in \mathbb{Z}$ and $mc \in \mathbb{Z}$ imply $|m| \geq 2$ (since $c \notin \mathbb{Z}$) and $|\tau \cdot m\alpha| > C$.
- If $c \in [1.55, 1.95]$, a simple computation gives $|\tau \cdot \alpha| > \frac{C}{3}$. Moreover in this case the conditions $m \in \mathbb{Z}$ and $mc \in \mathbb{Z}$ imply $|m| \geq 3$ (since $2c \notin \mathbb{Z}$) and we get $|\tau \cdot m\alpha| > C$.
- If $c \in [1.3, 1.65] \setminus \{4/3, 3/2\}$, a simple computation gives $|\tau \cdot \alpha| > \frac{C}{4}$. Moreover in this case the conditions $m \in \mathbb{Z}$ and $mc \in \mathbb{Z}$ imply $|m| \geq 4$ (since $3c, 2c \notin \mathbb{Z}$) and we get $|\tau \cdot m\alpha| > C$.
- If $c \in]1, 1.35] \setminus \{5/4, 4/3\}$, a simple computation gives $|\tau \cdot \alpha| > \frac{C}{5}$. Moreover in this case the conditions $m \in \mathbb{Z}$ and $mc \in \mathbb{Z}$ imply $|m| \geq 4$ (since $4c, 3c, 2c \notin \mathbb{Z}$) and we get $|\tau \cdot m\alpha| > C$.

Let us now show that this is true if $|n| \geq 4$ and $m \in \mathbb{Z}$.

$$\begin{aligned} |m\tau \cdot \alpha + n\tau \cdot \beta| &= |n||\tau \cdot \alpha| \left| \frac{m}{n} + \frac{\tau \cdot \beta}{\tau \cdot \alpha} \right| \\ &\geq |n||\tau \cdot \alpha| \left| \operatorname{Im}\left(\frac{m}{n} + \frac{\tau \cdot \beta}{\tau \cdot \alpha} \right) \right| \\ &\geq |n||\tau \cdot \alpha| \left| \operatorname{Im}\left(\frac{\tau \cdot \beta}{\tau \cdot \alpha} \right) \right| \\ &\geq k|n|, \end{aligned} \tag{1}$$

where

$$k = |\tau \cdot \alpha| \left| \operatorname{Im}\left(\frac{\tau \cdot \beta}{\tau \cdot \alpha} \right) \right|.$$

The numerical computation of k gives:

$$\begin{aligned} k^2 \doteq & \frac{1}{c^2 - 2.12228\,c + 1.27676} \Big(0.217137\,d^2 + 0.533079\,dc + 0.327181c^2 \\ & + 0.217127\,d - 0.911556c + 0.634921 \Big) \\ k^2 - \left(\frac{C}{4}\right)^2 \doteq & \frac{1}{1.27676 - 2.12228c + c^2} \Big(0.257151 + 0.0312991c^2 \\ & + c(-0.283614 + 0.533079d) + (-0.742604 + 0.217137d)d \Big) \end{aligned}$$

The denominator is positive for any real c. Thus the sign is the same as the sign of the numerator. For a fixed d, the minimum of the numerator if reached for $c \doteq 0.00443843 - 0.00834245d < 0$ (since $d > 1$). Thus the numerator is an increasing function of c for $c > 0$ and in particular for fixed d and $1 \leq c < d$ the minimum is reached at $c = 1$ and is given by $0.00483619 + (-0.209525 + 0.217137d)d$ which is positive since $d > 1$. We conclude that $k > \frac{C}{4}$. We use equation (1) to get that if $|n| \geq 4$,

$$|m\tau \cdot \alpha + n\tau \cdot \beta| > C.$$

It remains to do the cases $|n| \in \{1, 2, 3\}$. Since multiplication by -1 does not change the absolute value, it is enough to take care of the cases $n \in \{1, 2, 3\}$. We treat each case in a similar way. Let us start with the case $n = 1$. We get numerically

$$\begin{aligned} P_{c,d,1}(m) &:= |\tau \cdot (m\alpha + \beta)|^2 - C^2 \\ &\doteq -4.16782 + 0.712407m - 1.17373cm + 1.83908m^2 - 3.05698cm^2 \\ &\quad + 1.44043c^2m^2 + (-1.17373 - 3.05698m + 2.88085cm)d + 1.44043d^2. \end{aligned}$$

$P_{c,d,1}(m)$ is a quadratic polynomial in d. Computing the discriminant yields $P_{c,d,1}(m) > 0, \forall c \in \mathbb{R}$ if and only if

$$\Delta_c(d) :\doteq 25.3914 + 3.07144m - 1.25108m^2 < 0.$$

This is a quadratic equation[2] in m and solving it yields

$$m \notin [-3.44178, 5.89681] \implies |\tau \cdot (m\alpha + \beta)| > C.$$

Thus we only need to check that, under our assumptions, for every $m \in \{5, 4, 3, 2, 1, 0, -1, -2, -3\}$ such that $mc + d \in \mathbb{Z}$, $P_{c,d,1}(m) > 0$. Let us detail the cases $m = -3$ and $m = 4$. Numerically, we get $P_{c,d,1}(-3) \doteq 10.2467 + 12.9638c^2 + c(-23.9917 - 8.64256d) + d(7.99723 + 1.44043d)$. This is a quadratic polynomial in d and we easily deduce that

$$P_{c,d,1}(-3) > 0 \iff d \in]-\infty, 3c - 3.54573[\cup]3c - 2.00625, \infty[^3$$

Thus in particular, since, by hypothesis, $d \neq 3c - 3$ then either $P_{c,d,1}(-3) > 0$ or $d \in [3c - 3.54573, 3c - 2.00625]$ and then $-3c + d \notin \mathbb{Z}$. The condition $P_{c,d,1}(4) > 0$ is equivalent to $d \in]6.1107 - 4c, \infty[$. Since $d > c > 1$ and $d \neq 6 - 4c$ then either $P_{c,d,1}(4) > 0$ or $d + 4c \notin \mathbb{Z}$. The other cases are rather similar, so we give for each of them the condition on the reals and the assumptions that allow us to conclude.

[2]Remark that this is no numerical coincidence that c does not appear. It can be formally verified by using the fact that $P_{c,d,1}(m)$ is of the form $(x + ym + z(d + cm))^2 + (x' + y'm + z'(d + cm))^2 - C^2$ with $, y, z \in \mathbb{R}$.

[3]As for the previous note, this is no numerical coincidence that there is no complicated squareroot involving c since c does not appear in the discriminant.

(1): an equivalent condition over d	A sufficient condition to get (1)
$P_{c,d,1}(5) > 0 \Leftrightarrow d \in]6.78141 - 5c, \infty[$	$d > c > 1$
$P_{c,d,1}(4) > 0 \Leftrightarrow d \in]6.1107 - 4c, \infty[$	$d > c > 1$ and $d \neq 6 - 4c$
$P_{c,d,1}(3) > 0 \Leftrightarrow d \in]5.26804 - 3c, \infty[$	$d > c > 1$ and $d \neq 5 - 3c$
$P_{c,d,1}(2) > 0 \Leftrightarrow d \in]4.31762 - 2c, \infty[$	$d > c > 1$ and $d \neq 4 - 2c$
$P_{c,d,1}(1) > 0 \Leftrightarrow d \in]3.27931 - c, \infty[$	$d > c > 1$ and $d \neq 3 - c$
$P_{c,d,1}(0) > 0 \Leftrightarrow d \notin [-1.34171, 2.15655]$	$d > c > 1$ and $d \neq 2$
$P_{c,d,1}(-1) > 0 \Leftrightarrow d \in]c + 0.939592, \infty[$	$d > c$
$P_{c,d,1}(-2) > 0$ $\Leftrightarrow d \notin [2c - 3.02493, 2c - 0.404774]$	$d \notin \{2c - 3, 2c - 2, 2c - 1\}$
$P_{c,d,1}(-3) > 0$ $\Leftrightarrow d \notin [3c - 3.54573, 3c - 2.00625]$	$d \neq 3c - 3$

The next case is $n = 2$ and we will treat it in a similar fashion. We get numerically

$$\begin{aligned} P_{c,d,2}(m) &:= |\tau \cdot (m\alpha + 2\beta)|^2 - C^2 \\ &\doteq -2.46898 + 1.42481m - 2.34745cm + 1.83908m^2 - 3.05698cm^2 \\ &\quad + 1.44043c^2m^2 + (-4.6949 - 6.11397m + 5.76171cm)d + 5.76171d^2. \end{aligned}$$

Computing the discriminant yields $P_{c,d,2}(m) > 0, \forall d \in \mathbb{R}$ if and only if

$$\Delta_c(d) :\doteq 78.9442 + (24.5715 - 5.00433m)m < 0.$$

This is a quadratic equation in m and solving it yields

$$m \notin [-2.21427, 7.12433] \implies |\tau \cdot (m\alpha + \beta)| > C.$$

Thus we only need to check that, under our assumptions, for every $m \in \{7, 6, 5, 4, 3, 2, 1, 0, -1, -2\}$ such that $mc + 2d \in \mathbb{Z}$, $P_{c,d,2}(m) > 0$. Each case is rather similar to the cases with $n = 1$, so we give for each of them the condition on the reals and the assumptions that allow us to conclude.

(1): an equivalent condition over d	A sufficient condition to get (1)
$P_{c,d,2}(7) > 0 \Leftrightarrow d \notin]3.91363 - 3.5c, 4.32919 - 3.5c[$	$d > c > 1$
$P_{c,d,2}(6) > 0 \Leftrightarrow d \notin]3.00088 - 3c, 4.1808 - 3c[$	$d > c > 1$
$P_{c,d,2}(5) > 0 \Leftrightarrow d \notin]2.30029 - 2.5c, 3.82024 - 2.5c[$	$d > c > 1$
$P_{c,d,2}(4) > 0 \Leftrightarrow d \notin]1.67431 - 2c, 3.38509 - 2c[$	$d > c > 1$
$P_{c,d,2}(3) > 0 \Leftrightarrow d \notin]1.09888 - 1.5c, 2.89938 - 1.5c[$	$d > c > 1$
$P_{c,d,2}(2) > 0 \Leftrightarrow d \notin]0.566427 - c, 2.3707 - c[$	$d > c > 1$
$P_{c,d,2}(1) > 0 \Leftrightarrow d \notin]0.0766769 - 0.5c, 1.79931 - 0.5c[$	$d > c > 1$
$P_{c,d,2}(0) > 0 \Leftrightarrow d \notin] - 0.363621, 1.17847[$	$d > c > 1$
$P_{c,d,2}(-1) > 0 \Leftrightarrow d \notin [-0.732884 + 0.5c, 0.486592 + 0.5c]$	$d > c$
$P_{c,d,2}(-2) > 0 \Leftrightarrow d \notin [-0.925154 + c, -0.382276 + c]$	$d > c$

The only remaining case is $n = 3$ and we will treat it in a similar fashion.

We get numerically

$$\begin{aligned}P_{c,d,3}(m) &:= |\tau \cdot (m\alpha + 2\beta)|^2 - C^2 \\ &\doteq 0.362434 + 2.13722m - 3.52118bm + 1.83908m^2 - 3.05698cm^2 \\ &+ 1.44043c^2m^2 + (-10.5635 - 9.17095m + 8.64256cm)d + 12.9638d^2\end{aligned}$$

Computing the discriminant yields $P_{c,d,3}(m) > 0, \forall d \in \mathbb{R}$ if and only if

$$\Delta_c(d) :\doteq 92.7941 + 82.929m - 11.2597m^2 < 0.$$

This is a quadratic equation in m and solving it yields

$$m \notin [-0.986756, 8.35184] \implies |\tau \cdot (m\alpha + \beta)| > C.$$

Thus we only need to check that, under our assumptions, for every $m \in \{8, 7, 6, 5, 4, 3, 2, 1, 0\}$ such that $mc + 3d \in \mathbb{Z}$, $P_{c,d,3}(m) > 0$. Each case is rather similar to the cases $n = 1, 2$, so we give for each of them the condition on the reals and the assumptions that allow us to conclude.

(1): an equivalent condition over d	**A sufficient condition to get (1)**
$P_{c,d,3}(8) > 0$ $\Leftrightarrow d \notin]3.00699 - 2.66667c, 3.46726 - 2.66667c[$	$d > c > 1$
$P_{c,d,3}(7) > 0$ $\Leftrightarrow d \notin]2.45816 - 2.33333c, 3.30867 - 2.33333c[$	$d > c > 1$
$P_{c,d,3}(6) > 0$ $\Leftrightarrow d \notin]2.00508 - 2c, 3.05432 - 2c[$	$d > c > 1$
$P_{c,d,3}(5) > 0$ $\Leftrightarrow d \notin]1.59624 - 1.66667c, 2.75573 - 1.66667c[$	$d > c > 1$
$P_{c,d,3}(4) > 0$ $\Leftrightarrow d \notin]1.21937 - 1.33333c, 2.42518 - 1.33333c[$	$d > c > 1$
$P_{c,d,3}(3) > 0$ $\Leftrightarrow d \notin]0.870753 - c, 2.06637 - c[$	$d > c > 1$
$P_{c,d,3}(2) > 0$ $\Leftrightarrow d \notin]0.551146 - 0.666667c, 1.67855 - 0.666667c[$	$d > c > 1$
$P_{c,d,3}(1) > 0$ $\Leftrightarrow d \notin]0.266517 - 0.333333c, 1.25575 - 0.333333c[$	$d > c > 1$
$P_{c,d,3}(0) > 0$ $\Leftrightarrow d \notin]0.0358908, 0.778955[$	$d > c > 1$

This concludes the proof. □

In fact, using one more symmetry we improve the previous result.

Theorem 4.2. *For any $(c, d) \in \mathbb{R}^2 \setminus \mathcal{F}$ additive cubes are avoidable over*

$\{0, 1, c, d\}$ *where*

$$\mathcal{F} = \left\{ \left(\frac{10}{9}, \frac{14}{9}\right), \left(\frac{9}{8}, \frac{3}{2}\right), \left(\frac{9}{8}, \frac{13}{8}\right), \left(\frac{8}{7}, \frac{10}{7}\right), \left(\frac{8}{7}, \frac{11}{7}\right), \left(\frac{8}{7}, \frac{12}{7}\right), \left(\frac{7}{6}, \frac{11}{6}\right), \left(\frac{7}{6}, \frac{3}{2}\right), \left(\frac{7}{6}, \frac{5}{3}\right), \left(\frac{6}{5}, \frac{8}{5}\right), \left(\frac{6}{5}, \frac{9}{5}\right), \left(\frac{6}{5}, 2\right), \left(\frac{5}{4}, \frac{7}{4}\right), \left(\frac{5}{4}, 2\right), \left(\frac{5}{4}, \frac{9}{4}\right), \left(\frac{5}{4}, \frac{5}{2}\right), \left(\frac{5}{4}, \frac{11}{4}\right), \left(\frac{5}{4}, 3\right), \left(\frac{5}{4}, \frac{13}{4}\right), \left(\frac{5}{4}, \frac{7}{2}\right), \left(\frac{4}{3}, 2\right), \left(\frac{4}{3}, \frac{7}{3}\right), \left(\frac{4}{3}, \frac{8}{3}\right), \left(\frac{4}{3}, 3\right), \left(\frac{4}{3}, \frac{10}{3}\right), \left(\frac{4}{3}, \frac{11}{3}\right), \left(\frac{4}{3}, 4\right), \left(\frac{3}{2}, \frac{5}{2}\right), \left(\frac{3}{2}, 3\right), \left(\frac{3}{2}, \frac{7}{2}\right), \left(\frac{3}{2}, 4\right), \left(\frac{3}{2}, \frac{9}{2}\right), \left(\frac{3}{2}, 5\right), (4, 5) \right\} \cup \left(\{(2, t), (t, 2t-2), (t, 2t-1), (t, 3t-3) : t \in \mathbb{R}\} \cap \{(c, d) : d > c > 1\}\right).$$

Proof. Let $\mathcal{X}$ be the following set of pairs of parametric equations.

$$\mathcal{X} = \{(5/4, t), (4/3, t), (3/2, t), (2, t), (t, 6-4t), (t, 5-3t), (t, 4-2t), (t, 3-t), (t, 2t-3), (t, 2t-2), (t, 2t-1), (t, 3t-3), (t, 2)\}.$$

For any pair $e = (x(t), y(t))$ of parametric equations, we denote by $\mathcal{C}(e)$ the associated parametric curve (that is the set of points define by $\{(x(t), y(t)) : t \in \mathbb{R}\}$).

By the previous theorem for any $c, d \in \mathbb{R}$ with $c > d > 1$ and $(c, d) \notin \bigcup_{e \in \mathcal{X}} \mathcal{C}(e)$ additive cubes are avoidable over $\{0, 1, c, d\}$. Moreover, for any $c, d \in \mathbb{R}$ with $d > c > 1$, the alphabet $\{0, 1, c, d\}$ is equivalent to the alphabet $\{0, 1, \frac{d-1}{d-c}, \frac{d}{d-c}\}$ (by the affine map $x \to \frac{d-x}{d-c}$). Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function such that for all x, y, $f(x, y) = \left(\frac{y-1}{y-x}, \frac{y}{y-x}\right)$. We deduce that for any $c, d \in \mathbb{R}$ with $d > c > 1$ and $(c, d) \notin \bigcup_{e \in \mathcal{X}} \mathcal{C}(f \circ e)$ additive cubes are avoidable over $\{0, 1, c, d\}$. Let

$$\mathcal{F} = \left(\bigcup_{e \in \mathcal{X}} \mathcal{C}(f \circ e)\right) \cap \left(\bigcup_{e \in \mathcal{X}} \mathcal{C}(e)\right) \cap \{(c, d) : d > c > 1\}.$$

We finally get that for any $c, d \in \mathbb{R}$ with $d > c > 1$ and $(c, d) \notin \mathcal{F}$ additive cubes are avoidable over $\{0, 1, c, d\}$. Let us now precisely compute $\mathcal{F}$. First one easily computes

$$\mathcal{C}(\{f \circ e : e \in \mathcal{X}\}) = \mathcal{C}\left(\left\{(t, 6t-4), (t, 5t-3), (t, 4t-2), (t, 3t-1), (t, \frac{3}{2}t-1), (t, 2(t-1)), (2, t), (t, 3(t-1)), (t, 2t), (t, 5t-4), (t, 4t-3), (t, 3t-2), (t, 2t-1)\right\}\right).$$

We get the set from the theorem by simply computing the intersection of the two sets (this is done by solving the 13^2 equations). $\square$

5 The case of $\mathbf{W}_{1,0,c,d}$

We do almost exactly the same proof as in the previous section to show the following result.

Theorem 5.1. *Let $c, d \in \mathbb{R}$. Suppose we have $d > c > 1$, $d \notin \{2, c+1, c+2, 2c+2, 2c+1, 2c, 3c, 3c+1, 1+\frac{c}{2}, \frac{1}{2}+c\}$ then $\mathbf{w}_{1,0,c,d}$ avoids additive cubes.*

Proof. Following the proof of Theorem 4.1, we only need to show that, under our assumptions, for any $m, n \in \mathbb{Z}$ with $mc+nd \in \mathbb{Z}$, $|\tau \cdot (m\alpha+n\beta)| > C$, where $\alpha = (-c, c-1, 1, 0)$ and $\beta = (-d, d-1, 0, 1)$.

Let us first show that this is the case if $n = 0$. This time we only need to do 2 sub-cases:

- If $c > 1.71$ a simple computation gives $|\tau \cdot \alpha| > C$ and $|\tau \cdot m\alpha| > C$.
- If $c \in]1, 2[$, a simple computation gives $|\tau \cdot \alpha| > \frac{C}{2}$. Moreover in this case the conditions $m \in \mathbb{Z}$ and $mc \in \mathbb{Z}$ imply $|m| \geq 2$ (since $c \notin \mathbb{Z}$) and we get $|\tau \cdot m\alpha| \geq C$.

Let us now show that this is true if $|n| \geq 4$ and $m \in \mathbb{Z}$. The same computation gives:

$$|m\tau \cdot \alpha + n\tau \cdot \beta| \geq k|n|, \tag{2}$$

where

$$k = |\tau \cdot \alpha| \left| \mathrm{Im}\left(\frac{\tau \cdot \beta}{\tau \cdot \alpha}\right) \right|.$$

The exact same approach as in the previous proof can be used to verify that $k^2 - (\frac{C}{4})^2 > 0$ for any $d > c > 1$. This gives with equation (2) that if $|n| \geq 4$,

$$|m\tau \cdot \alpha + n\tau \cdot \beta| > C.$$

It remains to do the cases $|n| \in \{1, 2, 3\}$ but it is enough to take care of the cases $n \in \{1, 2, 3\}$ as previously. We start with the case $n = 1$. Once again $P_{c,d,1}(m) := |\tau \cdot (m\alpha + \beta)|^2 - C^2$ is a quadratic polynomial in d. Computing its discriminant yields: $P_{c,d,1}(m) > 0, \forall c \in \mathbb{R}$ if and only if

$$\Delta_c(d) :\doteq 25.3914 + 3.07144m - 1.25108m^2 < 0.$$

This is a quadratic equation in m and solving it yields

$$m \notin [-3.44178, 5.89681] \implies |\tau \cdot (m\alpha + \beta)| > C$$

(the conditions on m happen to be exactly the same). Thus we only need to check that, under our assumptions, for every $m \in \{5, 4, 3, 2, 1, 0, -1, -2, -3\}$ such that $mc + d \in \mathbb{Z}$, $P_{c,d,1}(m) > 0$.

All the cases are similar to what we did in the previous proof, so we give for each of them the condition on the reals and the assumptions that allow us to conclude.

(1): an equivalent condition over d	A sufficient condition to get (1)
$P_{c,d,1}(5) > 0 \Leftrightarrow d \notin]-0.781405 - 5c, 1.35518 - 5c[$	$d > c > 1$
$P_{c,d,1}(4) > 0 \Leftrightarrow d \notin]-1.1107 - 4c, 1.80675 - 4c[$	$d > c > 1$
$P_{c,d,1}(3) > 0 \Leftrightarrow d \notin]-1.26804 - 3c, 2.08636 - 3c[$	$d > c > 1$
$P_{c,d,1}(2) > 0 \Leftrightarrow d \notin]-1.31762 - 2c, 2.25822 - 2c[$	$d > c > 1$
$P_{c,d,1}(1) > 0 \Leftrightarrow d \notin]-1.27931 - c, d > 2.34218 - c[$	$d > c > 1$
$P_{c,d,1}(0) > 0 \Leftrightarrow d \notin [-1.15655, 2.34171]$	$d > c > 1$ and $d \neq 2$
$P_{c,d,1}(-1) > 0 \Leftrightarrow d \notin]-0.939592 + c, 2.24702 + c[$	$d > c$ and $d \notin \{c+1, c+2\}$
$P_{c,d,1}(-2) > 0 \Leftrightarrow d \notin]-0.595226 + 2c, 2.02493 + 2c[$	$d \notin \{2c+2, 2c+1, 2c\}$
$P_{c,d,1}(-3) > 0 \Leftrightarrow d \notin]0.00625218 + 3c, 1.54573 + 3c)[$	$d \notin \{3c, 3c+1\}$ [4]

The next case is $n = 2$ and we treat it in a similar fashion. Once again we easily verify that the only interesting cases are $m \notin [-2.21427, 7.12433]$. Thus we only need to check that, under our assumptions, for every $m \in \{7, 6, 5, 4, 3, 2, 1, 0, -1, -2\}$ such that $mc + 2d \in \mathbb{Z}$, $P_{c,d,2}(m) > 0$. Each case is rather similar to the cases with $n = 1$, so we give for each of them the condition on the reals and the assumptions that allow us to conclude.

(1): an equivalent condition over d	A sufficient condition to get (1)
$P_{c,d,2}(7) > 0 \Leftrightarrow d \notin]0.170814 - 3.5c, 0.586373 - 3.5c[$	$d > c > 1$
$P_{c,d,2}(6) > 0 \Leftrightarrow d \notin]-0.180798 - 3c, 0.999122 - 3c[$	$d > c > 1$
$P_{c,d,2}(5) > 0 \Leftrightarrow d \notin]-0.320242 - 2.5c, 1.19971 - 2.5c[$	$d > c > 1$
$P_{c,d,2}(4) > 0 \Leftrightarrow d \notin]-0.385091 - 2.c, 1.32569 - 2.c[$	$d > c > 1$
$P_{c,d,2}(3) > 0 \Leftrightarrow d \notin]-0.399384 - 1.5c, 1.40112 - 1.5c[$	$d > c > 1$
$P_{c,d,2}(2) > 0 \Leftrightarrow d \notin]-0.370696 - c, 1.43357 - c[$	$d > c > 1$
$P_{c,d,2}(1) > 0 \Leftrightarrow d \notin]-0.299307 - 0.5c, 1.42332 - 0.5c[$	$d > c > 1$
$P_{c,d,2}(0) > 0 \Leftrightarrow d \notin]-0.178467, 1.36362[$	$d > 1$
$P_{c,d,2}(-1) > 0 \Leftrightarrow d \notin]0.0134084 + 0.5c, 1.23288 + 0.5c[$	$d > c$ and $d \neq 1 + \frac{c}{2}$
$P_{c,d,2}(-2) > 0 \Leftrightarrow d \notin]0.382276 + c, 0.925154 + c[$	$d > c$ and $d \neq \frac{1}{2} + c$

The only remaining case is $n = 3$ and we will treat it in a similar fashion. We once again compute the determinant of $P_{c,d,3}(m)$ seen a a polynomial in d and we easily deduce that

$$m \notin [-0.986756, 8.35184] \implies |\tau \cdot (m\alpha + \beta)| > C$$

Thus we only need to check that, under our assumptions, for every $m \in \{8, 7, 6, 5, 4, 3, 2, 1, 0\}$ such that $mc + 3d \in \mathbb{Z}$, $P_{c,d,3}(m) > 0$. By solving each of the corresponding 9 equations, we deduce these inequalities. This concludes the proof. □

We could improve this result with the same approach as the one we used in Theorem 4.2, but we already have a strong enough result for our purpose.

6 The remaining alphabets

Using Theorem 4.2 and Theorem 5.1 we conclude:

Theorem 6.1. *For any* $(c,d) \in \mathbb{R}^2 \setminus \mathcal{F}$ *additive cubes are avoidable over* $\{0,1,c,d\}$ *where*

$$\mathcal{F} = \Big\{ \left(\frac{10}{9},\frac{14}{9}\right), \left(\frac{9}{8},\frac{13}{8}\right), \left(\frac{8}{7},\frac{11}{7}\right), \left(\frac{7}{6},\frac{5}{3}\right), \left(\frac{6}{5},\frac{8}{5}\right), \left(\frac{6}{5},2\right), \left(\frac{5}{4},\frac{7}{4}\right), \left(\frac{5}{4},2\right), \left(\frac{5}{4},\frac{9}{4}\right), \left(\frac{5}{4},\frac{5}{2}\right), \left(\frac{5}{4},\frac{13}{4}\right), \left(\frac{5}{4},\frac{7}{2}\right), \left(\frac{4}{3},2\right), \left(\frac{4}{3},\frac{7}{3}\right), \left(\frac{4}{3},\frac{8}{3}\right), \left(\frac{4}{3},\frac{10}{3}\right), \left(\frac{4}{3},\frac{11}{3}\right), \left(\frac{3}{2},\frac{5}{2}\right), \left(\frac{3}{2},3\right), \left(\frac{3}{2},\frac{7}{2}\right), (4,5), \left(\frac{4}{3},\frac{5}{3}\right), \left(\frac{3}{2},2\right), \left(\frac{8}{5},\frac{9}{5}\right), \left(\frac{5}{3},2\right), \left(\frac{7}{4},\frac{9}{4}\right), \left(2,\frac{5}{2}\right), (2,3), (2,4), (2,5), \left(\frac{5}{2},3\right), \left(\frac{5}{2},\frac{9}{2}\right), (3,4), (3,5), (3,6), (4,6), (4,9) \Big\}$$

Proof. This set is obtained by taking the intersection of the sets of forbidden pairs from Theorem 4.2 and Theorem 5.1. □

In order to study the remaining alphabets let us recall the following results from the literature.

Theorem 6.2 ([7])**.** *Additive cubes are avoidable over any of the following alphabets:*

$$\{0,1,5\}, \{0,1,6\}, \{0,1,7\}, \{0,2,7\}, \{0,3,7\},$$
$$\{0,1,8\}, \{0,3,8\}, \{0,1,9\}, \{0,2,9\}, \{0,4,9\}.$$

Theorem 6.3 ([8, Theorem 9])**.** *Additive cubes are avoidable over any of the following alphabets:*

$$\{0,2,3,6\}, \{0,1,2,4\}, \{0,2,3,5\}.$$

We will use the fact that almost all the remaining alphabets contain an alphabet equivalent to an alphabet from Theorem 6.2 or Theorem 6.3 to give our main result.

Theorem 6.4. *For any rational numbers* c *and* d *with* $c < d$ *and* $(c,d) \neq (2,3)$ *additive cubes are avoidable over* $\{0,1,c,d\}$.

Proof. $\left\{0,1,\frac{10}{9},\frac{14}{9}\right\}$ contains an alphabet equivalent to $\{0,1,5\}$ (apply $x \to 9x-9$ to $\left\{1,\frac{10}{9},\frac{14}{9}\right\}$). We deduce from Theorem 6.2 that additive cubes are avoidable over both alphabet. We do the same thing for other alphabets and we provide for each of them the alphabet from Theorem 6.2 or from Theorem 6.3 in table 1. This concludes the proof. □

$\left(\frac{10}{9},\frac{14}{9}\right),\left(\frac{5}{4},\frac{7}{2}\right),\left(\frac{4}{3},\frac{5}{3}\right),(4,5),\left(2,\frac{5}{2}\right),(2,5),(3,5)$	$\{0,1,5\}$
$\left(\frac{6}{5},\frac{8}{5}\right),\left(\frac{6}{5},2\right),\left(\frac{5}{4},\frac{5}{2}\right),\left(\frac{5}{3},2\right),\left(\frac{5}{2},3\right),(3,6),(4,6)$	$\{0,1,6\}$
$\left(\frac{7}{6},\frac{5}{3}\right),\left(\frac{4}{3},\frac{10}{3}\right)$	$\{0,1,7\}$
$\left(\frac{3}{2},\frac{7}{2}\right)$	$\{0,2,7\}$
$\left(\frac{5}{4},\frac{7}{4}\right),\left(\frac{4}{3},\frac{7}{3}\right)$	$\{0,3,7\}$
$\left(\frac{8}{7},\frac{11}{7}\right),\left(\frac{4}{3},\frac{11}{3}\right),\left(\frac{3}{2},\frac{5}{2}\right)$	$\{0,1,8\}$
$\left(\frac{5}{4},2\right),\left(\frac{4}{3},\frac{8}{3}\right)$	$\{0,3,8\}$
$\left(\frac{9}{8},\frac{13}{8}\right),\left(\frac{5}{4},\frac{13}{4}\right),\left(\frac{8}{5},\frac{9}{5}\right)$	$\{0,1,9\}$
$\left(\frac{5}{2},\frac{9}{2}\right)$	$\{0,2,9\}$
$\left(\frac{5}{4},\frac{9}{4}\right),\left(\frac{7}{4},\frac{9}{4}\right),(4,9)$	$\{0,4,9\}$
$\left(\frac{4}{3},2\right),\left(\frac{3}{2},3\right)$	$\{0,2,3,6\}$
$\left(\frac{3}{2},2\right),(2,4)$	$\{0,1,2,4\}$
$(3,4)$	$\{0,1,3,4\}$

Table 1: Every remaining alphabet contains an alphabet equivalent to an alphabet from theorems 6.2 or 6.3

We can reformulate this result in the terms of question 1.1.

Corollary 6.5. *Let $\mathcal{A} \subset \mathbb{C}$ be an alphabet with $|\mathcal{A}| \geq 4$. If $\mathcal{A}$ is not equivalent to $\{0,1,2,3\}$ then additive cubes are avoidable over $\mathcal{A}$.*

Remark that we showed that for all but finitely many integral alphabets of size 4 the word $\mathbf{W}_{a,b,c,d}$ can be used to avoid additive cubes. This is probably not the only morphic word with this property. Indeed, as long as the adjacency matrix of a morphism has at most 2 eigenvalues of norm at least 1, we can deduce something similar to Theorem 3.2 (see [8] for details). If the word also avoids abelian cubes, we can show something similar to Lemma 3.3. In general the conditions of this Lemma should be strong enough to study the lattice in a similar way than what we did.

Let us conclude by restating 3 remaining related open questions. First this is natural to ask wether additive cubes are avoidable over the only remaining alphabet.

Question 1. *Are additive cubes avoidable over $\{0,1,2,3\}$?*

We do not dare conjecture anything. In one hand, it would be surprising that this is the only alphabet (up to equivalence) over which additive cubes cannot be avoided. On the other hand, given an alphabet of size 4 other than this one it is really easy to find a construction that avoids additive cubes with a simple computer program (although it is much harder to prove that the construction is correct), while for $\{0,1,2,3\}$ running the same program much longer (say 100000 times longer) does not provide any candidate construction.

It seems that additive cubes are avoidable over most alphabets of size 3 and our result might help to show that.

Question 2. *Can we characterize the sets of integers of size 3 over which additive cubes are avoidable?*

Finally, we still don't know whether additive square are avoidable over any finite subset of $\mathbb{Z}$.

Question 3. *Is there any finite alphabet of integers over which additive squares are avoidable?*

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