Pairs of square-free arithmetic progressions in infinite words

Pascal Ochem*and Matthieu Rosenfeld†

keywords: Combinatorics on words; square-free words,

Context and subject: A *square* is a non-empty word of the form uu (e.g., *doudou*, *tata*, 00). A word is then said to be *square-free*, if it contains no factor that is a square. For instance, *paris* and *potato* are square-free, but not *patate* which contains the square *atat*. Thue introduced this notion in 1905. He proved that there exists an infinite square-free word over an alphabet of only 3 letters (see [1] for a translation in modern mathematical English). On the other hand, it is easy to verify that every word over a binary alphabet of length at least 4 contains a square. This subject and many of its variants and generalizations have since received a lot of attention in combinatorics on words.

Given an infinite word $w = w_0w_1\dots$ and an integer p, we write $[w]_p = w_0w_pw_{2p}w_{3p}$ the subsequence of w that is obtained by keeping every p-th letter. Harju [3] studied the following question:

Problem 1. *Given p, does there exist an infinite square-free word w over a ternary alphabet such that $[w]_p$ is square-free?*

Harju showed that this problem admits a positive solution for all $p \geq 3$. At the end of this article, he then asks the following more difficult question:

Problem 2. *Do there exist pairs (p, q) of relatively prime integers such that there exists a square-free word w over a ternary alphabet for which both $[w]_p$ and $[w]_q$ are square-free?*

In [2] Currie et Al. gave a positive answer to this question by proving that there exist at least two such pairs $(3, 11)$ and $(5, 6)$. They also mention that computer experiments suggest that this might not be possible for the pair $(5, 8)$. They then raise the question of characterizing the pairs (p, q) with this property. The aim of this internship is to solve this question.

For specific pairs of small integers, we hope to give explicit constructions based for instance on morphisms. Other techniques based on adapting the solution to Problem 4 from [2] (or even by improving the results from [4]) should allow us to solve the question positively for infinite families of pairs. Finally, for the negative results the use of clever backtracking should allow us to eliminate some pairs.

Some aspects of this subject will rely on purely combinatorial proofs. Others will require the use of computers to find specific constructions or conduct different kind of clever exhaustive verifications. Therefore, the student is expected to have some knowledge of theoretical computer science and mathematics as well as suitable programming skills.

*pascal.ochem@lirmm.fr
†matthieu.rosenfeld@lirmm.fr

Other details: This internship will take place at the LIRMM in Montpellier under the supervision of Pascal Ochem and Matthieu Rosenfeld. We have the possibility to fund the student if needed. At the end of the internship, we will discuss the possibility to start a Ph.D. on other topics related to combinatorics on words.

References

[1] Jean Berstel. Axel Thue's papers on repetitions in words a translation, 1995.

[2] James Currie, Tero Harju, Pascal Ochem, and Narad Rampersad. Some further results on square-free arithmetic progressions in infinite words. *Theoret. Comput. Sci.*, 799:140–148, December 2019.

[3] Tero Harju. On square-free arithmetic progressions in infinite words. *Theoret. Comput. Sci.*, 770:95–100, May 2019.

[4] Matthieu Rosenfeld. How far away must forced letters be so that squares are still avoidable? *Math. Comput.*, 89(326):3057–3071, November 2020.