On the Interval Number of Special Graphs

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Abstract: The interval number of a graph G is the least natural number t such that G is the intersection graph of sets, each of which is the union of at most t intervals, denoted by $i(G)$. Griggs and West showed that $i(G) \leq \lceil \frac{1}{2}(d+1) \rceil$. We describe the extremal graphs for that inequality when d is even. For three special perfect graph classes we give bounds on the interval number in terms of the independence number. Finally, we show that a graph needs to contain large complete bipartite induced subgraphs in order to have interval number larger than the random graph on the same number of vertices. J Graph Theory 46: 241–253, 2004

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1. INTRODUCTION

One way to represent a graph G is the intersection representation. That is, one assigns a set to each vertex of G, such that v is adjacent to w if and only if their assigned sets meet. A *t-interval representation* is such an assignment, in which each set consists of at most t closed intervals. The *interval number* of G, denoted by $i(G)$, is the least integer t for which a t-representation of G exists.

Furthermore, a representation is *displayed* if each set in the representation has an open interval disjoint from the other sets. Such an interval is called a *displayed segment*. We wish to establish bounds on the interval number of a graph G if certain subgraphs are forbidden in G. Theorem 1.3 is a small improvement of Theorem 1.1, if the graphs have some additional structure, while the other theorems give bounds on the interval number if certain graphs are not present as induced subgraphs.

A. The Degree Bound

Theorem 1.1 [8]. *If G is a graph with maximum degree d, then $i(G) \leq \lceil \frac{1}{2}(d+1) \rceil$.*

This upper bound is sharp, as it was shown by Griggs and West in [8]. We spell out their result in a little different, but equivalent form. A representation is *depth-two*, if every point of the line is covered by at most two intervals.

Theorem 1.2 [8]. *If a graph G has a depth-two t interval representation, then $t \geq \lceil (e(G)+1)/v(G) \rceil$.*

Thus the equality is attained for example by d-regular, triangle-free graphs, since the interval representations of those are always depth-two. The original proof of Theorem 1.1 was greatly simplified in [11] and [18]. However, the following statement of [11] turned out to be false.

Claim 1.1 [11]. *If a graph G with maximal degree d has no d-regular, K_3-free component, then $i(G) \leq \lceil \frac{1}{2} d \rceil$.*

In the next section, we give a counterexample that motivated the following definitions. First of all, let us call a graph G *even* if all its degrees are even.

If x is a cut-vertex of G, and the vertices of $G - x$ can be partitioned into two non-empty sets A and B such that G has no edges joining A and B, and x has an even number of neighbors in each of A and B, then the operation of replacing G with two subgraphs of G induced by $A \cup \{x\}$ and $B \cup \{x\}$ is called an *even split* of G, and $\{x\}$ is the *pivot vertex* of the split. An *even decomposition* of G is a set of graphs which can be obtained from G by repeated even splits.

A connected graph H is *rich* if it contains a triangle T_H that after deleting the edges of T_H, the remaining graph has at most two non-trivial components. Note that the triangle T_H is not necessarily unique.

Finally for an even graph G a *rich decomposition* is such an even decomposition in which the arising components (blocks) are all rich. With these notions we can spell out the correct characterization of extremal graphs.

Theorem 1.3. *Let G be a connected graph of maximal degree $d = 2k$. Then G has a k-interval representation if and only if either G is not $2k$-regular or G has a rich decomposition.*

B. Small Forbidden Subgraphs

Most of the classical results on the interval number were upper bounds for a general graph G, in terms of some monotone increasing function of G. (See for example [2,8,12,15]. It is also possible to derive bounds that are smaller for denser graphs. See in [3] that $i(G) \leq \lceil \sqrt{e(\bar{G})}/2 \rceil + o(n)$, where n and $e(\bar{G})$ are the number of vertices and edges of the complement of G, respectively.

Such bounds have also been studied for special families of graphs, like as chordal graphs, where the interval number is bounded by a function of the clique number. We continue this direction for other classes of perfect graphs using the independence number.

We refer to [4] as a vast collection of facts on special graphs, but also try to spell out some facts about them that we use.

A graph G is *triangulated* or *chordal* if for $n \geq 4$ it does not contain an induced C_n, a circuit of length n. A special class of chordal graphs is the family of the *split* graphs, where the vertex set of a split graph can be partitioned into two parts, an independent set and a clique.

A graph G is a *comparability* graph if there is a partial order $\mathcal{P}$ on its vertices such that vertices u and v form an edge in G if and only if u and v are comparable in $\mathcal{P}$. Let $\omega(G)$ and $\alpha(G)$ denote the clique number and the independence number of G, respectively. For a graph G, let $\chi(G)$ be the *chromatic number* of G, that is the smallest number k such that the vertex set of G can be partitioned into k independent sets.

It was shown in [1] that $i(G) \leq \lceil \omega(G)/2 \rceil + 1$ for a chordal graph G. An even better result can be found in [13], that $i(G) \leq (1 + o(1))\omega(G)/\log_2 \omega(G)$. We prove the following bounds on the interval number in terms of the independence number for graphs in special families of perfect graphs. These results share the flavor that the bound is smaller for denser graphs.

Theorem 1.4. *If G is a graph having no 4-cycle as an induced subgraph, then $i(G) \leq \lceil \chi(\bar{G})/2 \rceil$.*

Corollary 1.1. *If G is a chordal graph, then $i(G) \leq \lceil \alpha(G)/2 \rceil$.*

Theorem 1.5. *If G is a split graph, then $i(G) \leq (1 + o(1))\alpha(G)/\log_2 \alpha(G)$, and there is a sequence of split graphs G_k such that $\alpha(G_k) = k$ and $i(G_k) > (1/2 + o(1))k/\log_2 k$, where k goes to infinity.*

Theorem 1.6. *If G is a comparability graph, then* $i(G) \leq \alpha(G)$.

For the complete bipartite graph $K_{m,n}$, Trotter and Harary [17] proved that $i(K_{m,n}) = \lceil (mn+1)/(m+n) \rceil$. For complete multipartite graphs (see [9]), $i(K_{n_1,n_2,\ldots}) = i(K_{n_1,n_2}) + 1$ (except for certain values), where $n_1 \geq n_2 \geq \cdots$. One checks that in both cases $i(G) \leq \lceil (\alpha(G)+1)/2 \rceil$.

Note that $K_{m,n}$ and the complete multipartite graphs are comparability graphs. We believe a common generalization of these results, that is

Conjecture 1.1. *If G is a comparability graph, then* $i(G) \leq \lceil (\alpha(G)+1)/2 \rceil$.

C. Large Forbidden Complete Bipartite Graphs

One may ask, what induced subgraphs force high interval number. For the random graph $G_{n,1/2}$, $i(G_{n,1/2}) = (1/2 + o(1))n/\log_2 n$ holds almost surely [14]. If G contains the complete bipartite graph $K_{k,k}$ as an induced subgraph, then $i(G) \geq \lceil (k+1)/2 \rceil$. The following result roughly states that big induced complete bipartite graphs are required for high interval number.

Theorem 1.7. *Let k be a positive integer. If a graph G does not contain* $K_{k,k}$ *as an induced subgraph, then* $i(G) \leq (1+o(1))n/\log_2 n$, *where n is the order of G.*

Remark. We do not have a matching lower bound here. Standard use of the probabilistic method (see e.g. [6]) shows the existence of a bipartite $K_{r,r}$-free graph G which has $2n$ vertices and $n^{2-2/r}$ edges. Applying the formula $i(G) \geq \lceil (e(G)+1)/v(G) \rceil$ (see [8]) to G, which is triangle-free, we get that $i(G) = \Omega(n^{1-2/r})$.

2. COUNTEREXAMPLES TO CLAIM 1

Let H and Q be two triangle-free connected graphs satisfying that there is a $h \in V(H)$ of degree $2k-2$, a $q \in V(Q)$ of degree 2, and all the other vertices are of degree $2k$. (It is easy to see that such graphs exists for $k \geq 2$). To construct a counterexample G for Claim 1, we glue together $2k-2$ copies of H and one copy of Q as follows. Let $h_1, h_2, \ldots, h_{2k-2}$ denote the "h" vertices in the $2k-2$ copies of H. To get G, we take the copies, and connect for $1 \leq i \leq k-1$ the vertices h_{2i-1} with h_{2i}, and both with q. Clearly, G is a $2k$-regular graph, therefore $i(G) \leq k+1$. We prove that in fact $i(G) = k+1$.

So let us suppose on contrary that G has a k-representation. By Theorem 1.2 the representation must contain three intersecting intervals, since $e(G) = kv(G)$, which can be realized only if the vertices of the intervals span a triangle. In G the vertex set of each triangle is in the form h_{2i-1}, h_{2i}, q. Without loss of generality, we may assume that the intervals assigned to h_1, h_2, and q have common intersection, and denote the three intersecting intervals by I_{h_1}, I_{h_2}, and I_q. It is easy to check that the union of two of these intervals must contain the third one. If $I_{h_1} \subset I_{h_2} \cup I_q$, then our representation must have a depth-two sub-representation $\mathcal{I}_1$ for H_1, using at most k intervals for vertices different from h_1, and at most $k-1$ intervals for h_1, since I_{h_2} and I_q isolate I_{h_1} from I_{H_1}. But applying Theorem 1.2 to H_1 leads to a contradiction, and similarly, the cases of $I_{h_2} \subset I_{h_1} \cup I_q$ and $I_q \subset I_{h_1} \cup I_{h_2}$ are also impossible.

3. PROOFS

For the sake of completeness we repeat the proof of Theorem 1 and the first part that of Theorem 1 given in [11]. Note, that the statement of Theorem 1.1 has been slightly rephrased in order to make its connection to the stronger version of Theorem 1.3 more apparent.

Theorem 3.1 [modified version of Theorem 1.1]**.** *There is a displayed interval representation for the graph G such that at most* $\lceil (d(v)+1)/2 \rceil$ *intervals are assigned to each vertex v, where* $d(v)$ *designates the degree of the vertex v.*

A. Proof of Theorem 3.1

A *trail W* in G is a sequence of vertices $W = \{v_1, v_2, \ldots, v_l\}$ such that there is an edge between v_i and v_{i+1} for each $i = 1, 2, \ldots, t-1$, and all these edges are distinct. Let us partition the edges of G into the minimum number of edge-disjoint trails $\{W_i\}_{i=1}^{j}$. Now represent the trail $W_i = (v_1^i, v_2^i, \ldots, v_{t(i)}^i)$ for $1 \le i \le j$, assigning an I_p^i interval to the vertex v_p^i such that two intervals intersect if and only if the corresponding vertices are consecutive in the trail W_i.

This procedure leads to a displayed interval representation of G. Since a vertex v can be an endvertex of the trails at most twice, if v is represented by k intervals, then $d(v) \ge 2(k-2)+2 = 2k-2$. Hence,

$$\left\lceil \frac{1}{2}(d(v)+1) \right\rceil \ge \left\lceil \frac{1}{2}(2k-2+1) \right\rceil = \left\lceil k - \frac{1}{2} \right\rceil = k. \qquad \blacksquare$$

B. Proof of Theorem 1.3

First, we show that if a connected graph G has maximum degree $2k$ and $i(G) = k+1$, then G must be $2k$-regular. Let G be a graph with $i(G) = k+1$ and maximal degree $2k$. Consider the set of all partitions of the edge set of G into minimum number of edge-disjoint trails. Let us choose among these partitions a partition $\{W_i\}_{i=1}^{j}$ which also minimizes the size of the set Q of vertices occurring $k+1$ times in the walks $\{W_i\}_{i=1}^{j}$. The interval-representation of G is the same as in the proof of Theorem 1.1.:

If $Q = \emptyset$, we are done. For an $x \in Q$ there exists a $p \in \{1, \ldots, j\}$ such that $x = v_1^p, x = v_{t(p)}^p$ and $x \notin W_l$ for all $l \neq p$. The last statement follows from the minimality of j, since in case of $x = v_s^l \in W_l$ we could replace the trails W_p and W_l by the trail

$$W^* = (v_1^l, v_2^l, \ldots, v_s^l, v_2^p, \ldots, v_{t(p)}^p, v_{s+1}^l, \ldots, v_{t(l)}^l).$$

For any vertex $y = v_s^p \neq x$ from W_p, we can transform the trail W_p into the trail

$$W_p^* = (v_s^p, v_{s-1}^p, \ldots, v_1^p, v_{t(p)-1}^p, v_{t(p)-2}^p, \ldots, v_s^p).$$

That is, by the minimality of Q, y occurs in the trails $\{W_i\}_{i \neq p} \cup \{W_p^*\}$ $k+1$ times. Then again, all neighbors of y are in W_p. That is the vertex set of W_p is a $2k$-regular component of G. But G is connected and $2k$-regular, hence $V(G) = W_p$.

In order to finish the proof of Theorem 1.3, we need to show that for a $2k$-regular graph G the assumptions $i(G) \leq k$ and G has a rich decomposition are equivalent. We state Theorem 1.3 in a stronger form:

Stronger version of Theorem 1.3. *If G is an even graph, then there is a interval representation of G assigning at most $d(v)/2$ intervals to each vertex v if and only if G has a rich decomposition.*

First, we assume that G has a rich decomposition. It is enough to prove the statement for the blocks of the rich decomposition of G, since putting together their representation we get the stated representation for G.

Having a rich component H, we can represent the edges of the triangle T_H by using three overlapping intervals, one for each vertices of T_H. However, it is crucial to put such two intervals to the two sides of that sub-representation which would fall into different non-trivial components after deleting the edges of T_H. Now to represent the non-trivial components of $H \backslash T_H$, we use the ideas of Theorem 1.1. Since such a component is an even graph, it has an Eulerian circuit. We make a trail out of that circuit by starting it and finishing it with a vertex of T_H. Then we represent these (at most two) trails, and identify one of their end interval with the appropriate interval representing T_H.

To prove the other direction, we assume that G is connected and that G has a representation assigning at most $d(v)/2$ intervals to each vertex v. Then there is a point which is the element of at least three intervals. (Imagine that we draw the intervals one by one. If there is no such a point, then every new interval may represent at most two units of the degree sum of G (or with other words, at most one edge). Since the first drawn interval does not represent any edges, there would not be enough edges represented.) We call these three intervals I_x, I_y, and I_z, and the vertices that associated to those are x, y and z, respectively. (Note that $T_G := \{x, y, z\}$ is a triangle.)

If the deletion of the edges of the triangle T_G leaves one or two connected non-trivial components, then the whole G itself is a rich component.

Let us examine the case when deleting the edges of T_G leaves three non-trivial components. Let us observe that one of those intervals is in the union of the other two; say $I_x \subset I_y \cup I_z$. At this case the vertex x can serve as the pivot vertex for an even split, A is the component containing the whole triangle T_G, B is the one containing only x. Now A is a rich component, since the deletion of the edges of T_G leaves only two components. Let us consider the intervals representing B. Since I_x cannot intersect any of those intervals, and $d_B(x) = d_G(x) - 2$, there must be three intervals among those intervals such that their intersection is not empty, and a triangle T_B of B corresponds to this. Now we can repeat the argument above, proving either B is a rich component with T_B, or we can proceed by induction. ■

Remark. The stronger form of Theorem 1.3 holds for general simple graphs; one just have to write $\lceil d(v)/2 \rceil$ instead of $d(v)/2$ in the statement. The way of proof is standard: join a new vertex x to G, and connect it to all vertices of odd degree.

C. An Application

It was shown in [15] that $i(G) \leq 3$ for any planar graph G. By the help of Theorem 3 we can point out an interesting special case.

Theorem 3.2. *If G is a planar graph with maximum degree at most 4, then $i(G) \leq 2$.*

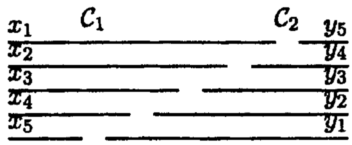
Proof. One can assume that G is connected. If G is not 4-regular, then applying Theorem 1.3 the statement follows. Thus, in the rest of the proof we can assume that G is 4-regular.

Since G is planar, the Euler formula $n(G) + l(G) = e(G) + 2$ holds, where $n(G)$, $l(G)$ and $e(G)$ are the number of vertices, faces, and edges, respectively. G must contain a triangle, since otherwise the Euler formula would give that $e(G) \leq 2n(G) - 4$, while the 4-regularity implies that $e(G) = 2n(G)$. Our aim is to show that G has a rich decomposition, and finish the proof by the use of Theorem 1.3. If there is a triangle such that its deletion leaves at most two non-trivial components, then G is a rich component itself. Let us assume that the deletion of the edges of any triangle leaves exactly three non-trivial components. The numbers of vertices in the non-trivial components vary; let T be a triangle such that one of these components, say H, is the smallest possible.

Of course $e(H) \leq 2n(H) - 4$, since H is planar and H cannot contain a triangle. On the other hand one vertex of H has degree 2 and all the others have degree 4, giving us the equality $2n(H) = e(G) + 1$, which is a contradiction. It means that there must be a triangle such that the deletion of its edges results in less than three non-trivial components, that is G has a rich decomposition. ■

4. PROOF OF THEOREM 1.4

Proof of Theorem 1.4. The idea of the proof is to partition the vertex set of G into $\chi(\bar{G})$ cliques $\mathcal{C}_1, \ldots, \mathcal{C}_{\chi(\bar{G})}$, then represent the edges in the cliques and between the cliques. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two arbitrary cliques of that partition. Since G is C_4-free, the vertices of $\mathcal{C}_1 = \{x_1, \ldots, x_l\}$ can be ordered in such a way that $N_2(x_1) \supset N_2(x_2) \supset \cdots \supset N_2(x_l)$, where $N_2(x)$ denotes the vertices of $\mathcal{C}_2$ neighboring to x. To represent the edges in and between $\mathcal{C}_1$ and $\mathcal{C}_2$ it suffices to draw two piles of intervals containing only one interval per vertex. (The bigger the neighborhood of a vertex, the longer the interval that is assigned to it. The picture illustrates the representation the edges between two five-element cliques when $|N_2(x_i)| = |N_2(x_{i+1})| + 1$.)



Certainly, we can use both ends of these intervals (the left ends of those assigned to $\mathcal{C}_1$ and the right ends of those assigned to $\mathcal{C}_2$). One can think of this as a generalization of the proof of Theorem 1, where the "degree" of each clique is not more than $\chi(\bar{G}) - 1$. More exactly, we consider an Eulerian trail (or circuit) containing all edges of the complete graph on $\chi(\bar{G})$ vertices, and draw the consecutive piles according to their appearance in that Eulerian trail. The number of piles we need to draw for any clique $\mathcal{C}$ is at most $\lceil \chi(\bar{G})/2 \rceil$. Since $\{\mathcal{C}_i\}_{i=1}^{\chi(\bar{G})}$ is a partition of the vertex set of G, $\lceil \chi(\bar{G})/2 \rceil$ is also an upper bound on the number of intervals that are used for any vertex in the representation. ■

Proof of Corollary 1.1. First we need recall some basic facts about perfect graphs, see the details in [4]. Chordal and comparability graphs are perfect. A graph G is perfect iff for every induced subgraph $H \subset G$ it holds that $\omega(H) = \chi(H)$. The complement of a perfect graph is perfect (this is the celebrated Perfect Graph Theorem).

A chordal graph G is induced C_4-free, and since it is perfect we have $\chi(\bar{G}) = \alpha(G)$ and Theorem 4 applies, giving Corollary 1.1.

5. PROOF OF THEOREM 1.5

For any k positive integer, we define the *universal split graph* G_k.

The vertex set of G_k is indexed by the numbers $\{1, \ldots, k\} = A$, and the non-empty subsets of $\{1, \ldots, k\}$. There are no edges inside A, while $V(G_k)\backslash A$ is a clique. Furthermore a vertex $i \in A$ is connected to a vertex indexed by $S \subset A$ if and only if $i \in S$. A graph H is *essential* if $N(x) \neq N(y)$ for $x \neq y \in V(H)$. G_k contains every connected essential split graph H of independence number k as an induced subgraph, and also $\alpha(G_k) = k$. To conclude the proof it is enough to show that $(1/2 - o(1))k/\log_2 k \leq i(G_k) \leq (1 + o(1))k/\log_2 k$. (If a split graph H is not essential, then we may represent the biggest induced essential subgraph H^* of it, and take a copy of the intervals of $x \in V(H^*)$ for a $y \notin V(H^*)$ if $N(x) = N(y)$.)

We start with the proof of the upper bound. Let us divide A into blocks of size $t = \log_2 k - \log_2 \log_2 k$. For each subset of a block we secure an unused part of the line, and place small disjoint intervals for each element of the subset. This way we use up not more than $2^t \leq k/(2 \log_2 k)$ intervals per vertex. To represent the edges between A and a vertex x of the clique on $V(G_k) \backslash A$, we use one interval per block, putting a long interval under the intervals corresponding to the subset $A \cap N(x)$. Finally we represent the edges of the clique by one interval per vertex. This procedure uses only $\max(2^{t-1}, k/t) + 1$ intervals per vertex. Since any essential split graph H is an induced subgraph of G_k, if $k = \alpha(H)$, we have completed the first part of proof.

Next we show that $i(G_k) \geq (1/2 - o(1))k/\log_2 k$. Because $V(G_k) - A$ is a clique and A is an independent set we may assume that in a t-representation of G_k there are exactly t intervals corresponding to each vertex of A, and all of those are single points of the line, which set is denoted by $\mathcal{A}$.

Clearly, $|\mathcal{A}| = kt$. Let I_x be the set of intervals representing vertex x. If $x \notin A$, then $\mathcal{A} \cap I_x$ consists of at most t consecutive segments of $\mathcal{A}$. This means, the number of such sets is not more than $\sum_{l=1}^{t} \binom{kt+1}{2l}$. (Indeed, a consecutive sequence of points is determined by its endpoints that we can choose among the points of $\mathcal{A}$ without repetition. However, degenerated segments may occur, that is why we have to make the summation from $l = 1$ to $l = t$.)

For distinct vertices x and y, we have $\mathcal{A} \cap I_x \neq \mathcal{A} \cap I_y$, which implies that

$$2^k - 1 \leq \sum_{l=1}^{t} \binom{kt+1}{2l} \leq t \binom{kt+1}{2t}.$$

Applying the inequality $\binom{a}{b} \leq (ea/b)^b$ reduces this inequality to $t \geq (1/2 - o(1))k/\log_2 k$. ■

Note that a very similar argument is used in [13] in order to obtain lower bounds on the interval number for triangulated graphs.

6. PROOF OF THEOREM 1.6

The proof parallels that of Theorem 1.1, and uses the following lemma.

Lemma 6.1. *If the vertex set of a graph G can be partitioned into k cliques such that the subgraphs induced by any two of those cliques are comparability graphs, then $i(G) \leq k$.*

Proof. We imitate the proof of Theorem 1.4. However, the structure of the neighborhoods is more complicated here. Let $\mathcal{C}_1 = \{x_1, \ldots, x_l\}$ and $\mathcal{C}_2 = \{y_1, \ldots, y_t\}$ be two complete subgraphs of G, spanning a comparability graph. As $\mathcal{C}$ is a clique, its vertices can be linearly ordered, inheriting the order from a poset realizing the comparability graph. W.l.o.g. the order is $x_1 \geq x_2 \geq \cdots$. Now $N_2(x) = N_2^<(x) \cup N_2^>(x)$, where $N_2^<(x)$ is the set of those vertices in $\mathcal{C}_2$ that are smaller than x in the partial order of the induced subgraph, while $N_2^>(x)$ is the subset of $\mathcal{C}_2$ consisting of the vertices bigger than x. It is easy to see that

$$N_2^<(x_1) \supset N_2^<(x_2) \supset \cdots \supset N_2^<(x_l) \quad \text{and} \quad N_2^>(x_1) \subset N_2^>(x_2) \subset \cdots \subset N_2^>(x_l).$$

We may represent only one type of those neighborhoods by drawing two piles, although the free ends of the piles are re-usable again. The representation is similar to that of Theorem 1.4. We consider an directed Eulerian circuit in the directed complete graph (i.e., this graph has $\chi(\bar{G}) * (\chi(\bar{G}) - 1)$ edges) on $\chi(\bar{G})$ vertices, to which we associate the cliques of G, and we draw piles for a clique at each occurance of the vertex in the Eulerian circuit that corresponds to it. Now, for each pair of cliques $(\mathcal{C}_i, \mathcal{C}_j)$ we have two pairs of intersecting piles that can represent both types of neighborhoods, and we get Lemma 6.1. ■

Lemma 6.1 implies that the interval number of a comparability graph G is bounded by the number of cliques needed to cover $V(G)$, that is $i(G) \leq \chi(\bar{G})$. Since the comparability graphs are perfect, this bound is equals to $\alpha(G)$, and Theorem 6 is proved. ■

7. PROOF OF THEOREM 1.7

We need a result of Erdős and Hajnal, see [5]. A set $L \subset V(G)$ is *homogeneous* if the vertices of L are pairwise adjacent or pairwise non-adjacent. Note that $\log n = \log_2 n$ in these section.

If a graph G on n vertices does not contain a fixed graph H as an induced subgraph, then it contains a large homogeneous set, with size at least $\exp(c\sqrt{\log n}/2)$, where $c < 1/|V(H)|$. For any k, if n is big enough, then $\exp(c\sqrt{\log n}/2) \gg \log^{2k} n$.

We repeatedly use this theorem, always cutting off a big homogeneous set L from the remaining part of $V(G)$. If L is a clique, we take about $\log_2 n - \log_2 \log_2 n$ vertices of it, otherwise (i.e., if L is an independent set) we take a subset of size between $\log_2^{2k+1} n$ and $2\log_2^{2k+1} n$. This procedure can go on until the leftover graph has only $o(n/\log_2 n)$ vertices.

Summing up, the procedure gives a partition of the vertices of G:

$$V(G) = K_1 \cup \cdots \cup K_\ell \cup E_1 \cup \cdots \cup E_r \cup A,$$

where

- $|A| = o(n/\log n)$,
- each K_i is a clique of size $\log_2 n - 2\log\log n$,
- each E_j is an independent set such that $\log^{2k+1} n \le |E_j| \le 2\log^{2k+1}$.

Now we represent the edges of G step by step, starting by the ones that having an endpoint in A, then those having one endpoint in $\cup_i K_i$, finally the edges having both endpoints in $\cup_j E_j$.

First let us create pairwise disjoint displayed intervals for the vertices of A. For each vertex of G, place a small interval within the displayed intervals for each of its neighbors in A. This uses at most $o(n/\log n)$ intervals for each vertex of G, as $|A| = o(n/\log_2 n)$.

For the cliques, we construct a Scheinerman-type of displayed system, see [14] or [3]. For a clique Q of size q, one places 2^{q-1} intervals per vertex in such a way that for every subset P of Q there exists an interval I_P of the line where intervals representing P and no other vertices intersect. The use of such system is that one can represent all edges between a vertex x and Q by only one additional interval of x, placing that interval into $I_{N(x)\cap Q}$. Since in our case $q = \log_2 n - 2\log_2\log_2 n$, we have $2^{q-1} = n/2(\log_2 n)^2$.

For each pair (K_i, K_j) (with $i < j$) to represent all the edges between K_i and K_j, we can choose one of them and use only one additional interval for each vertex of it. We add an interval for the vertices in K_i if $j - i \le \ell/2$; otherwise we add an interval for the vertices in K_j. That is for each vertex of $\cup_i K_i$ the number of additional intervals will be at most $\ell/2$, where ℓ is the number of cliques.

This way for each vertex in a clique we use at most

$$n/(\log_2 n)^2 + \ell/2 + o(n/\log_2 n) \le (1/2 + o(1))n/(\log_2 n)$$

intervals (where the magnitude of the upper bound is coming from $\ell \le n/(\log_2 n - 2\log\log n)$).

Finally for a vertex in an independent set we use no more than $\ell \le n/(\log_2 n - \log_2\log_2 n) = (1 + o(1))n/\log_2 n)$ intervals to represent its edges to the cliques.

Because we have assumed that G is $K_{k,k}$-free, the result of Kővári, Sós, and Turán [10] implies that the number of the edges in G joining two independent sets E_i and E_j is at most $2m^{2-1/k}$, where $m = \max\{|E_i|, |E_j|\}$. With this, we can bound the number of edges among the independent sets of the partition. Since $\log_2^{2k+1} n \le |E_j| \le 2\log_2^{2k+1} n$, we have that $r \le n/(\log_2 n)^{2k+1}$ and $m \le 2(\log_2 n)^{2k+1}$. Thus, the number of edges is no more than

$$\binom{r}{2} 2m^{2-1/k} \le \frac{4n^2}{(\log_2 n)^{2+1/k}}.$$

In order to represent the edges among all E_j we use the edge-bound theorem from [2], which states that for a graph G, having e edges, $i(G) \le \lceil \sqrt{e}/2 \rceil + 1$.

Combining this bound and the bound on the number of edges in this subgraph we get a representation of that part of G using only $o(n/\log_2 n)$ intervals per vertex.

Putting together all these bounds on the representations of the subgraphs of G yields $i(G) \leq n/\log_2 n + o(n/\log_2 n)$, which proves Theorem 1.7. ■

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