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Critical exponent of ternary words with few distinct palindromes

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We study infinite ternary words that contain few distinct palindromes. In particular, we classify such words according to their critical exponents.

Keywords: palindromes, repetitions, critical exponent, repetition threshold

1 Introduction

Repetitions have been a central theme in combinatorics on words since the pioneering work of Thue [34, 35]. Here we study fractional repetitions of the form introduced by Dejean [14]. Roughly speaking, for a rational number $r > 1$, a word w is said to be an *r-power* if it consists of a nonempty word x repeated r times. For example, the English word `alfalfa` is a $\frac{7}{3}$ power, and the French word `chercher` is a 2-power (also called a square). For a real number $\beta \geq 1$, a word is said to be *β-free* (resp. *β^+-free*) if it contains no r-powers with $r \geq \beta$ (resp. $r > \beta$). The *critical exponent* of a word w is the infimum of all β such that w is β^+-free. For example, the word `banana` has critical exponent $\frac{5}{2}$, since it is $\frac{5}{2}^+$-free, but contains the $\frac{5}{2}$-power `anana`.

Thue [34] demonstrated the existence of an infinite square-free (i.e., 2-free) ternary word, and Dejean [14] strengthened this result by showing that there is an infinite ternary word with critical exponent $\frac{7}{4}$, and that this is the minimum critical exponent among all infinite ternary words. Roughly speaking, this says that the number $\frac{7}{4}$ represents the boundary between avoidable and unavoidable repetitions in infinite ternary words.

In general, the *repetition threshold* for k letters, denoted $\mathrm{RT}(k)$, is defined by

$$\mathrm{RT}(k) = \inf\{\beta \in \mathbb{R} \colon \text{ there is an infinite word over } k \text{ letters with critical exponent } \beta\}.$$

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In addition to establishing that $\mathrm{RT}(3) = \frac{7}{4}$, Dejean [14] conjectured that

$$\mathrm{RT}(k) = \begin{cases} \frac{7}{4}, & \text{if } k = 3; \\ \frac{7}{5}, & \text{if } k = 4; \\ \frac{k}{k-1}, & \text{if } k = 2 \text{ or } k \geq 5. \end{cases}$$

This conjecture has since been proven through the work of many authors, and is now known as Dejean's theorem. In particular, Carpi [4] proved all but finitely many cases, and the last cases were proven independently by Currie and Rampersad [13] and Rao [31].

Especially since the completion of the proof of Dejean's theorem, much work has been done on determining the minimum critical exponent (i.e., the repetition threshold) among all infinite words in various restricted classes of words, including episturmian words [17], balanced words [15, 18, 30], rich words [10, 11], and words with given factor complexity [8, 25, 32].

Here we are concerned with determining the minimum critical exponent among all infinite words containing at most a fixed number of distinct palindromes.[1] In other words, we study the trade-off between the number of palindromes and the critical exponent in infinite words. This trade-off has recently been investigated for binary words [16]. Removing the constraint on the alphabet size does not give anything new:

- The only infinite word containing at most 4 palindromes is $(\texttt{012})^\omega$, which has infinite critical exponent.
- For $k \geq 5$, it follows from Dejean's theorem that there is an infinite word (over $k-1$ letters) containing only k palindromes with critical exponent $\mathrm{RT}(k-1)$, and this is best possible.

So in this paper, we consider this trade-off in infinite ternary words. Our results completely answer questions of the following form: Do infinite β^+-free ternary words with at most p palindromes exist? In every case that such words exist, we also determine whether there are exponentially or polynomially many such words. The results are summarized in Table 1. A white cell means that no such infinite word exists. Extremal white cells are labelled with the corresponding item of Theorem 1. A green (resp. red) cell means that there are exponentially (resp. polynomially) many such words. We have labelled the cells that correspond to an item of Theorem 3, 5, or 8.

The remainder of the paper is organized as follows. In Section 2, we give the necessary background and definitions. In Section 3, we prove Theorem 1, which establishes the negative results corresponding to the labelled white cells in Table 1. In Section 4, we prove Theorem 3, which establishes the exponential growth in the cases corresponding to the labelled green cells in Table 1. In Section 5, we prove Theorem 5, which shows that there are only polynomially many overlap-free words (i.e., 2^+-free words) with at most 6 palindromes. The remaining polynomial cases are covered in Section 6 and Section 7. In Section 6, we relate the structure of all infinite $\frac{52}{27}$-free words with at most 16 palindromes and all $\frac{25}{13}$-free words with at most 17 palindromes to one particular infinite word with 16 palindromes. The proof that this word has critical exponent $\frac{41}{22}$ is long and technical, and is given in Section 7. Finally, in Section 8, we observe that for ternary square-free words, having at most most 16 palindromes is equivalent to avoiding overpals and also to avoiding the letter pattern $abcacba$. We use this observation to prove a conjecture of

[1] For brevity, going forward every mention of a number of palindromes will refer to a number of distinct palindromes, including the empty word.

18	3(f)							
17	1(d)		3(e)					
16		8		3(d)				
15				1(c)				
14								
13								
12								
11								
10								
9								
8								
7					3(c)			
6					5	3(b)		
5						1(b)	3(a)	
4							1(a)	$(\texttt{012})^\omega$
p / β^+	$\frac{7}{4}^+$	$\frac{41}{22}^+$	$\frac{25}{13}^+$	$\frac{52}{27}^+$	2^+	$\frac{9}{4}^+$	$\frac{10}{3}^+$	∞

Tab. 1: Infinite β^+-free ternary words with at most p palindromes.

Rajasekaran, Rampersad, and Shallit [29] on words avoiding overpals. We also simplify the proofs (and improve certain results) of Petrova [27] on the minimum critical exponent of square-free ternary words avoiding other letter patterns.

Several proofs throughout the paper make use of computer checks. The programs used can be found at the link below.

`http://www.lirmm.fr/ ochem/morphisms/palin3.htm`

2 Preliminaries

An *alphabet* $\mathcal{A}$ is a finite set and its elements are called *letters*. A *word* u over $\mathcal{A}$ of *length* n is a finite string $u = u_0u_1 \cdots u_{n-1}$, where $u_j \in \mathcal{A}$ for all $j \in \{0, 1, \ldots, n-1\}$. The length of u is denoted $|u|$ and $|u|_f$ denotes the number of occurrences of the factor $f \in \mathcal{A}^+$ in the word u. If $\mathcal{A} = \{\texttt{0}, \texttt{1}, \ldots, \texttt{d}-\texttt{1}\}$, the *Parikh vector* $\vec{u} \in \mathbb{N}^d$ is the vector defined as $\vec{u} = (|u|_\texttt{0}, |u|_\texttt{1}, \ldots, |u|_{\texttt{d}-\texttt{1}})^T$. The set of all finite words over $\mathcal{A}$ is denoted $\mathcal{A}^*$. The set $\mathcal{A}^*$ equipped with concatenation as the operation forms a monoid with the *empty word* ε as the neutral element. If $w = ux$ for some $u, x \in \mathcal{A}^*$, then $wx^{-1} = u$. We will also consider the set $\mathcal{A}^\omega$ of infinite words (that is, right-infinite words) and the set ${}^\omega\mathcal{A}^\omega$ of bi-infinite words. A word v is an *e-power* of a word u if v is a prefix of the infinite periodic word $uuu\cdots = u^\omega$ and $e = |v|/|u|$. We write $v = u^e$. We also call u^e a repetition with period u and exponent e. For instance, the Czech word $\texttt{kavka}$ (a kind of bird – jackdaw) can be written in this formalism as $(\texttt{kav})^{5/3}$. A word (finite or infinite) is *α^+-free* (resp. *α-free*) if it contains no repetition with exponent β such that $\beta > \alpha$ (resp. $\beta \geqslant \alpha$). For example, the word $\texttt{kavka}$ is $\frac{5}{3}^+$-free and square-free (i.e., 2-free), but not $\frac{5}{3}$-free.

The *critical exponent* $E(\mathbf{u})$ of an infinite word $\mathbf{u}$ is defined as

$$E(\mathbf{u}) = \inf\left\{\beta \in \mathbb{R} : \mathbf{u} \text{ is } \beta^+\text{-free}\right\}$$

or equivalently as

$$E(\mathbf{u}) = \sup\{e \in \mathbb{Q} : \ v^e \text{ is a factor of } \mathbf{u} \text{ for a non-empty word } v\}.$$

If each factor of $\mathbf{u}$ has infinitely many occurrences in $\mathbf{u}$, then $\mathbf{u}$ is *recurrent*. Moreover, if for each factor the distances between its consecutive occurrences are bounded, then $\mathbf{u}$ is *uniformly recurrent*. The *reversal* of the word $w = w_0w_1\cdots w_{n-1}$, where the w_i are letters, is the word $w^R = w_{n-1}\cdots w_1w_0$. A word w is a *palindrome* if $w = w^R$.

Consider a factor w of a recurrent infinite word $\mathbf{u} = u_0u_1u_2\cdots$. Let $j < \ell$ be two consecutive occurrences of w in $\mathbf{u}$, i.e., $u_ju_{j+1}\cdots u_{j+|w|-1} = w = u_\ell u_{\ell+1}\cdots u_{\ell+|w|-1}$. Then the word $u_ju_{j+1}\cdots u_{\ell-1}$ is a *return word* to w in $\mathbf{u}$.

Let $\mathcal{L}(\mathbf{u})$ denote the set of factors of $\mathbf{u}$, sometimes referred to as the *language* of $\mathbf{u}$. Given a word $w \in \mathcal{L}(\mathbf{u})$, we define the sets of left extensions, right extensions, and bi-extensions of w in $\mathbf{u}$ over an alphabet $\mathcal{A}$ respectively as

$$\mathrm{Lext}_{\mathbf{u}}(w) = \{a \in \mathcal{A} : aw \in \mathcal{L}(\mathbf{u})\}, \qquad \mathrm{Rext}_{\mathbf{u}}(w) = \{b \in \mathcal{A} : wb \in \mathcal{L}(\mathbf{u})\}$$

and

$$\mathrm{Bext}_{\mathbf{u}}(w) = \{(a,b) \in \mathcal{A} \times \mathcal{A} : awb \in \mathcal{L}(\mathbf{u})\}.$$

If $\#\mathrm{Lext}_{\mathbf{u}}(w) > 1$, then w is called *left special (LS)*. If $\#\mathrm{Rext}_{\mathbf{u}}(w) > 1$, then w is called *right special (RS)*. If w is both LS and RS, then it is called *bispecial (BS)*. The *bilateral order* [2] of a factor w is defined as $b(w) = \#\mathrm{Bext}_{\mathbf{u}}(w) - \#\mathrm{Lext}_{\mathbf{u}}(w) - \#\mathrm{Rext}_{\mathbf{u}}(w) + 1$ and we distinguish *ordinary BS factors* with $b(w) = 0$, *weak BS factors* with $b(w) < 0$ and *strong BS factors* with $b(w) > 0$.

A *morphism* is a map $\psi : \mathcal{A}^* \to \mathcal{B}^*$ such that $\psi(uv) = \psi(u)\psi(v)$ for all words $u, v \in \mathcal{A}^*$. The morphism ψ is *non-erasing* if $\psi(a) \neq \varepsilon$ for each $a \in \mathcal{A}$. Morphisms can be naturally extended to infinite words by setting $\psi(u_0u_1u_2\cdots) = \psi(u_0)\psi(u_1)\psi(u_2)\cdots$. A *fixed point* of a morphism $\psi : \mathcal{A}^* \to \mathcal{A}^*$ is an infinite word $\mathbf{u}$ such that $\psi(\mathbf{u}) = \mathbf{u}$.

If $\mathcal{A} = \{\mathtt{0}, \mathtt{1}, \ldots, \mathtt{d}-\mathtt{1}\}$, then for each morphism $\psi : \mathcal{A}^* \to \mathcal{A}^*$ there is an associated *(incidence) matrix* M_ψ defined for all $i, j \in \{0, 1, \ldots, d-1\}$ by $[M_\psi]_{ij} = |\psi(\mathtt{j})|_{\mathtt{i}}$. Note that multiplying the Parikh vector of a word $u \in \mathcal{A}^*$ by the incidence matrix M_ψ gives the Parikh vector of $\psi(u)$, i.e., we have $\overrightarrow{\psi(u)} = M_\psi \vec{u}$ for all $u \in \mathcal{A}^*$. If there exists $N \in \mathbb{N}$ such that M_ψ^N has all positive entries, then ψ is called a *primitive* morphism.

Let $\mathbf{u}$ be an infinite word over an alphabet $\mathcal{A}$ and let $\psi : \mathcal{A}^* \to \mathcal{B}^*$ be a morphism. Consider a factor w of $\psi(\mathbf{u})$. We say that (w_1, w_2) is a *synchronization point* of w if $w = w_1w_2$ and for all $p, s \in \mathcal{L}(\psi(\mathbf{u}))$ and $v \in \mathcal{L}(\mathbf{u})$ such that $\psi(v) = pws$ there exists a factorization $v = v_1v_2$ of v with $\psi(v_1) = pw_1$ and $\psi(v_2) = w_2s$. We denote the synchronization point by $w_1 \bullet w_2$. For instance, consider the morphism $\psi : \mathtt{0} \to \mathtt{012}, \quad \mathtt{1} \to \mathtt{0012}$. Then the factor $\mathtt{12001}$ of $\psi((\mathtt{01})^\omega)$ has the synchronization point $\mathtt{12} \bullet \mathtt{001}$ since 2 occurs only as the last letter of $\psi(\mathtt{0})$ and $\psi(\mathtt{1})$.

Given a factorial language L (a language where every factor of a word in the language is also part of that language) and an integer ℓ, let L^ℓ denote the words of length ℓ in L. The *Rauzy graph* of L of order ℓ is the directed graph whose vertices are the words of $L^{\ell-1}$, the arcs are the words of L^ℓ, and the arc corresponding to the word w goes from the vertex corresponding to the prefix of w of length $\ell - 1$ to the vertex corresponding to the suffix of w of length $\ell - 1$.

[2] The bilateral order was introduced by Casssaigne [6] as a tool for the computation of factor complexity.

An *overpal* is a word of the form $axax^Ra$, where x^R is the reversal of the (possibly empty) word x and a is a single letter. Overpals were introduced by Rajasekaran, Rampersad, and Shallit [29].

A *pattern* p is a non-empty finite word over the alphabet $\Delta = \{A, B, C, \ldots\}$ of capital letters called *variables*. An *occurrence* of p in a word w is a non-erasing morphism $h : \Delta^* \to \Sigma^*$ such that $h(p)$ is a factor of w. A *letter pattern* is a non-empty finite word over the alphabet $\{a, b, c, \ldots\}$ of lowercase variables such that every variable stands for one letter from the alphabet $\{\texttt{0}, \texttt{1}, \texttt{2}, \ldots\}$ and different variables denote different letters. For example, the occurrences of the letter pattern $abaca$ over the ternary alphabet are `01020`, `02010`, `10121`, `12101`, `20212`, `21202`. Letter patterns were introduced by Elena Petrova [27].

3 Negative results

Theorem 1. *There exists no infinite ternary β-free word containing only p palindromes for the following pairs (p, β).*

(a) $(4, \beta)$ *for any fixed* β.

(b) $(5, \frac{10}{3})$

(c) $(15, 2)$

(d) $(17, \frac{41}{22})$

Proof. Item (a) follows from the fact that, up to renaming of the letters, the word $(\texttt{012})^\omega$ is the only infinite word with at most 4 palindromes. The other cases are obtained by backtracking searches, the results of which are described below.

- The longest $10/3$-free ternary word with at most 5 palindromes has length 70.
- The longest 2-free ternary word with at most 15 palindromes has length 42.
- The longest $41/22$-free ternary word with at most 17 palindromes has length 449. □

4 Exponential cases

We need some terminology and a lemma from [23]. A morphism $f : \Sigma^* \to \Delta^*$ is *q-uniform* if $|f(a)| = q$ for every $a \in \Sigma$, and is called *synchronizing* if for all $a, b, c \in \Sigma$ and $u, v \in \Delta^*$, if $f(ab) = uf(c)v$, then either $u = \varepsilon$ and $a = c$, or $v = \varepsilon$ and $b = c$.

Lemma 2. *[23, Lemma 23] Let $r, s \in \mathbb{R}$ satisfy $1 < r < s$. Let $\rho \in \{r, r^+\}$ and $\sigma \in \{s, s^+\}$. Let $h\colon \Sigma^* \to \Delta^*$ be a synchronizing q-uniform morphism. Set*

$$t = \max\left(\frac{2s}{s-r}, \frac{2(q-1)(2s-1)}{q(s-1)}\right).$$

If $h(w)$ is σ-free for every ρ-free word $w \in \Sigma^*$ with $|w| \leqslant t$, then $h(z)$ is σ-free for every ρ-free word $z \in \Sigma^*$.

Most results in this subsection use the following steps. We find an appropriate uniform synchronizing morphism h by exhaustive search. We use Lemma 2 to show that h maps every ternary $\frac{7}{4}^+$-free word (resp. 4-ary $\frac{7}{5}^+$-free word) to a suitable ternary β^+-free word. Since there are exponentially many ternary $\frac{7}{4}^+$-free words (resp. 4-ary $\frac{7}{5}^+$-free words) [24], there are also exponentially many ternary β^+-free words.

Theorem 3. *There exist exponentially many infinite ternary β^+-free words containing at most p palindromes for the following pairs (p, β).*

(a) $(5, \frac{10}{3})$

(b) $(6, \frac{9}{4})$

(c) $(7, 2)$

(d) $(16, \frac{52}{27})$

(e) $(17, \frac{25}{13})$

(f) $(18, \frac{7}{4})$

Proof.

(a) $(5, \frac{10}{3})$: We claim that applying the following morphism f to any binary cube-free word gives a ternary $\frac{10}{3}^+$-free word containing only 5 palindromes.

$$\begin{aligned} \mathtt{0} &\to \mathtt{012} \\ \mathtt{1} &\to \mathtt{0012} \end{aligned}$$

Since there are exponentially many cube-free binary words [3], it follows that there are exponentially many $\frac{10}{3}^+$-free ternary words containing at most 5 palindromes.

To prove the claim, let w be a binary cube-free word. Suppose towards a contradiction that $f(w)$ contains a $\frac{10}{3}^+$-power, say $X = U^Q$, where $Q > \frac{10}{3}$. First suppose that $|U| \leqslant 17$. Then X contains a $\frac{10}{3}^+$-power of length at most $4|U| \leqslant 68$. But every factor of $f(w)$ of length at most 68 appears in the f-image of some cube-free word y of length at most $\frac{68-2}{3} + 2 = 24$. Using a computer, we enumerate every binary cube-free word y of length at most 24 and check that $f(y)$ is $\frac{10}{3}^+$-free. So we may assume that $|U| \geqslant 18$. Then we can write $X = \alpha f(x)\beta$, where α is a proper suffix of $f(\mathtt{0})$ or $f(\mathtt{1})$, β is a proper prefix of $f(\mathtt{0})$ or $f(\mathtt{1})$, and x is a factor of w. Note that $|\alpha| \leqslant 3$ and $|\beta| \leqslant 3$. Since $|U| \geqslant 18$ and $X = U^Q$ for some $Q > \frac{10}{3}$, we have

$$|f(x)| = |X| - |\alpha| - |\beta| > \frac{10}{3}|U| - 6 \geqslant 3|U|.$$

So $f(x)$ has exponent at least 3. It follows that x has exponent at least 3, which contradicts the assumption that w is cube-free.

(b) $(6, \frac{9}{4})$: Applying the following 25-uniform morphism to any ternary $\frac{7}{4}^+$-free word gives a ternary $\frac{9}{4}^+$-free word containing only 6 palindromes.

$$0 \to \texttt{0112001200112011200112012}$$
$$1 \to \texttt{0012011200112012001120012}$$
$$2 \to \texttt{0011201120012011201200112}$$

(c) $(7, 2)$: Applying the following 4-uniform morphism to any ternary $\frac{7}{4}^+$-free word gives a ternary 2^+-free word containing only 7 palindromes.

$$0 \to \texttt{0012}$$
$$1 \to \texttt{0112}$$
$$2 \to \texttt{0122}$$

(d) $(16, \frac{52}{27})$: Applying the following 609-uniform morphism to any ternary $\frac{7}{4}^+$-free word gives a ternary $\frac{52}{27}^+$-free word containing only 16 palindromes.

```
0 → p21012010201210120212010201202120121012010201202120102012101202120121012010201
    21012021201020120212012101202120102012101201020120212010201210120212012101201 0
    20121012021201020120212012101201020120212010201210120212012101201020120212012 1
    01202120102012101201020120212012101201020121012021201020120212012101202120102 0
    12101201020120212010201210120212012101201020121012021201020120212012101202120 1
    0201210120102
1 → p02012101202120121012010201202120121012021201020121012010201202120121012010201
    21012021201020120212012101202120102012101201020120212010201210120212012101201 0
    20121012021201020120212012101202120102012101201020120212012101201020121012021 2
    01020120212012101201020120212010201210120212012101201020120212012101202120102 0
    12101201020120212012101201020121012021201020120212012101202120102012101201020 1
    2021201020121
2 → p02012101202120121012010201202120121012021201020121012010201202120121012010201
    21012021201020120212012101202120102012101201020120212010201210120212012101201 0
    20121012021201020120212012101201020120212010201210120212012101201020120212012 1
    01202120102012101201020120212012101201020121012021201020120212012101201020120 2
    12010201210120212012101201020121012021201020120212012101202120102012101201020 1
    2021201020121
```

where $p =$ `0120212012101201020121012021201020120212012101201020120212010201210120 2`
`1201210120102012021201210120212010201210120102012021201020121012021201210120102 01`
`2101202120102012021201210120212010201210120102012021201.`

(e) $(17, \frac{25}{13})$: Applying the following 121-uniform morphism to any ternary $\frac{7}{4}^+$-free word gives a ternary $\frac{25}{13}^+$-free word containing only 17 palindromes.

$$0 \to p\texttt{1202101201021202102010210120210201021202101201020120210120102120210}$$
$$1 \to p\texttt{1202101201020120210120102120210201021012021020102120210120102120210}$$
$$2 \to p\texttt{1012021201020120210120102120210201021012021020102120210120102012021}$$

where $p = \texttt{201021012021201020120210120102012021201021012021020102}$.

(f) $(18, \frac{7}{4})$: Applying the following 87-uniform morphism to any 4-ary $\frac{7}{5}^+$-free word gives a ternary $\frac{7}{4}^+$-free word containing only 18 palindromes.

$$0 \to p\texttt{2101202102010212021012010201210120210201021012021201210}$$
$$1 \to p\texttt{2101202102010210121021202101201020121012021020102120210}$$
$$2 \to p\texttt{2101202102010210120212010201210120210201021012102120210}$$
$$3 \to p\texttt{0201202101210212021012010201202120121012021020102101202}$$

where $p = \texttt{12010201202101210201021012021201}$. □

Trying to estimate the growth rate in the cases above is certainly a difficult task. However, we note that Fleischer and Shallit [20] have shown that the number of ternary words of length n with at most 5 palindromes (sequence A329023 in the OEIS) is $\Theta(\kappa^n)$, where $\kappa = 1.2207440846\cdots$ is the root of $x^4 = x + 1$.

5 First polynomial case

We consider the morphic word $t^\omega(\texttt{0})$, where t is the morphism defined as follows.

$$t(\texttt{0}) = \texttt{01120}$$
$$t(\texttt{1}) = \texttt{12001}$$
$$t(\texttt{2}) = \texttt{2}$$

Lemma 4. *The word $t^\omega(\texttt{0})$ is obtained from the Thue-Morse word (i.e., the fixed point $\mu^\omega(\texttt{0})$ of the morphism μ defined by $\texttt{0} \mapsto \texttt{01}$ and $\texttt{1} \mapsto \texttt{10}$) by inserting the letter $\texttt{2}$ in the middle of every factor $\texttt{10}$.*

Proof. If we erase every occurrence of $\texttt{2}$ from $t^\omega(\texttt{0})$, we get the fixed point of $\texttt{0} \mapsto \texttt{0110}$; $\texttt{1} \mapsto \texttt{1001}$, that is, the Thue-Morse word. Now let us check that the letter $\texttt{2}$ is inserted correctly. The word $t^\omega(\texttt{0})$ avoids $\texttt{10}$, so that $\texttt{2}$ must have been inserted in the middle of every factor $\texttt{10}$. The fixed point of t also avoids $\texttt{02}$, $\texttt{21}$, and $\texttt{22}$, so that $\texttt{2}$ is not inserted anywhere else. □

Theorem 5. *The word $t^\omega(\texttt{0})$ contains 6 palindromes and is 2^+-free. Every bi-infinite ternary $\frac{9}{4}$-free word containing at most 6 palindromes has the same factor set as one of the six words obtained from $t^\omega(\texttt{0})$ by letter permutation.*

Proof. Matthieu Rosenfeld has written a program that implements the algorithm of Julien Cassaigne [5] to test whether a morphic word w avoids a pattern P, with w and P as input. We use it to show that $t^\omega(\texttt{0})$ avoids the pattern $ABABA$. Moreover, $t^\omega(\texttt{0})$ avoids $\texttt{000}$, $\texttt{111}$, and $\texttt{22}$. So, $t^\omega(\texttt{0})$ is 2^+-free.

Alternatively, one could argue that the insertion of the letter $\mathtt{2}$ in the middle of every factor $\mathtt{10}$ in the Thue-Morse word does not create an overlap.

Suppose now that $\mathbf{w}$ is a ternary $\frac{9}{4}$-free word containing at most 6 palindromes. Note that $\mathbf{w}$ must contain the four palindromes ε, $\mathtt{0}$, $\mathtt{1}$, and $\mathtt{2}$. Further, we see that $\mathbf{w}$ must contain two palindromes from the set $\{\mathtt{00},\mathtt{11},\mathtt{22}\}$, because a backtracking search shows that there are only finitely many ternary $\frac{9}{4}$-free words containing at most six palindromes and containing at most one palindrome from $\{\mathtt{00},\mathtt{11},\mathtt{22}\}$. By permuting the letters if necessary, we assume that $\mathbf{w}$ contains the palindromes $\mathtt{00}$ and $\mathtt{11}$.

So $\mathbf{w}$ contains the six palindromes ε, $\mathtt{0}$, $\mathtt{1}$, $\mathtt{2}$, $\mathtt{00}$, and $\mathtt{11}$, and hence no $\mathtt{22}$ and no palindrome of the form aba or $abba$, where a and b are letters. Thus, it is not hard to see that every return word to $\mathtt{2}$ in $\mathbf{w}$ belongs to $R_1 \cup R_2$, where

$$R_1 = \{\mathtt{201},\mathtt{2001},\mathtt{2011},\mathtt{20011}\}$$

and

$$R_2 = \{\mathtt{210},\mathtt{2110},\mathtt{2100},\mathtt{21100}\}.$$

Further, we see that $\mathbf{w}$ does not contain factors from both R_1 and R_2 (otherwise it would contain either $\mathtt{020}$ or $\mathtt{121}$). By permuting $\mathtt{0}$ and $\mathtt{1}$ if necessary, we may assume that $\mathbf{w}$ contains no factors from R_2. It follows that $\mathbf{w}$ is obtained from a binary word $\mathbf{v}$ by inserting the letter $\mathtt{2}$ in the middle of every factor $\mathtt{10}$. (Note that $\mathbf{v}$ is obtained from $\mathbf{w}$ by deleting every $\mathtt{2}$.)

We claim that $\mathbf{v}$ is $\frac{7}{3}$-free. Suppose otherwise that $\mathbf{v}$ contains a factor, say u, of exponent greater than $\frac{7}{3}$. It follows that u has a prefix of the form $xyxyx$, where $|xyxyx|/|xy| > \frac{7}{3}$, or equivalently

$$|y| < 2|x|. \tag{1}$$

Let U be the word obtained from $xyxyx$ by inserting the letter $\mathtt{2}$ in the middle of every factor $\mathtt{10}$, so that U is a factor of $\mathbf{w}$. Let X be the word obtained from x by inserting the letter $\mathtt{2}$ in the middle of every factor $\mathtt{10}$, and let Y be the word such that $V = XYXYX$. (So Y is obtained from y by inserting the letter $\mathtt{2}$ in the middle of every factor $\mathtt{10}$, and possibly adding a $\mathtt{2}$ at the beginning and/or the end.)

If $|x| \leqslant 6$, then we have $|y| < 2|x| \leqslant 12$, hence there are only finitely many possibilities for the word $xyxyx$. For each possible word $xyxyx$, we check by computer that the corresponding word $U = XYXYX$ has exponent greater than $\frac{9}{4}$, which contradicts the assumption that $\mathbf{w}$ is $\frac{9}{4}$-free. So we may assume that $|x| \geqslant 7$.

We now claim that

$$|X| \geqslant \tfrac{5}{4}|x| - 1 \tag{2}$$

and

$$|Y| \leqslant \tfrac{3}{2}|y| + 2. \tag{3}$$

For (2), note that $|X| = |x| + |x|_{\mathtt{10}}$. Since $\mathbf{w}$ (and $\mathbf{v}$, in turn) contains neither $\mathtt{000}$ nor $\mathtt{111}$, the factor $\mathtt{10}$ must occur at least $\frac{|x|}{4} - 1$ times in x if $|x|$ is a multiple of 4, and at least $\left\lfloor \frac{|x|}{4} \right\rfloor$ times otherwise. Thus we have

$$|X| = |x| + |x|_{\mathtt{10}} \geqslant |x| + \tfrac{|x|}{4} - 1 = \tfrac{5}{4}|x| - 1,$$

as desired. For (3), we have

$$|Y| \leqslant |y| + |y|_{\mathtt{10}} + 2 \leqslant |y| + \tfrac{|y|}{2} + 2 = \tfrac{3}{2}|y| + 2$$

by similar reasoning. Thus, we have

$$\begin{aligned} |Y| &\leqslant \tfrac{3}{2}|y| + 2 && \text{by (3)} \\ &< 3|x| + 2 && \text{by (1)} \\ &\leqslant \tfrac{12}{5}|X| + \tfrac{22}{5} && \text{by (2)} \\ &< 3|X| && \text{since } |x| \geqslant 7 \text{, hence } |X| \geqslant 8. \end{aligned}$$

Thus, we have shown that $|Y| < 3|X|$, which is equivalent to $|XYXYX|/|XY| > \frac{9}{4}$. In other words, we have shown that $U = XYXYX$ has exponent greater than $\frac{9}{4}$, which contradicts the assumption that $\mathbf{w}$ is $\frac{9}{4}$-free.

Therefore, we conclude that $\mathbf{v}$ is $\frac{7}{3}$-free. Now it follows from a well-known theorem of Karhumäki and Shallit [21] that $\mathbf{v}$ has the same factor set as the Thue-Morse word. Thus, by Lemma 4, we conclude that $\mathbf{w}$ has the same factor set as $t^{\omega}(\texttt{0})$. □

6 Second polynomial case

We consider the morphic word $\gamma(\eta^{\omega}(\texttt{0}))$ defined by the following morphisms.

$$\begin{aligned} \eta(\texttt{0}) &= \texttt{010203} & \gamma(\texttt{0}) &= \texttt{012021201210120102} \\ \eta(\texttt{1}) &= \texttt{2013} & \gamma(\texttt{1}) &= \texttt{012101202120102} \\ \eta(\texttt{2}) &= \texttt{0132} & \gamma(\texttt{2}) &= \texttt{012021201020121} \\ \eta(\texttt{3}) &= \texttt{0102013203} & \gamma(\texttt{3}) &= \texttt{012021201210120212010201210120102} \end{aligned}$$

We say that a word (finite or infinite) is η-good if it avoids AA and

$$\begin{aligned} F_{\eta} = \{&\texttt{12}, \texttt{21}, \texttt{23}, \texttt{31}, \texttt{103}, \texttt{302}, \texttt{303}, \texttt{132013}, \texttt{320132}, \\ &\texttt{2010201}, \texttt{2013201}, \texttt{3013203}, \texttt{030102030}\}. \end{aligned}$$

We say that a word (finite or infinite) is γ-good if it avoids AA and

$$\begin{aligned} F_{\gamma} = \{&\texttt{0210}, \texttt{1021}, \texttt{2102}, \texttt{01201020121012021201 2}, \texttt{020121012021201020121}, \\ &\texttt{101202120102012101202}, \texttt{120121012021201020120}, \\ &\texttt{201202120102012101201}, \texttt{212010201210120212010}\}. \end{aligned}$$

Lemma 6. *A bi-infinite word is η-good if and only if it has the same factor set as $\eta^{\omega}(\texttt{0})$.*

Proof. It is easy to check that $\eta^{\omega}(\texttt{0})$ avoids F_{η} and we use Rosenfeld's program to show that $\eta^{\omega}(\texttt{0})$ avoids AA. So $\eta^{\omega}(\texttt{0})$ is η-good.

We construct the set S_{η}^{58} defined as follows: a word v is in S_{η}^{58} if and only if there exists an η-good word pvs such that $|p| = |v| = |s| = 58$. We check that S_{η}^{58} is exactly the set of factors of length 58 of $\eta^{\omega}(\texttt{0})$. Thus, if w is any bi-infinite η-good word, then its factors of length 58 are also factors of $\eta^{\omega}(\texttt{0})$. Now, every element of S_{η}^{58} contains the factor $\eta(\texttt{0}) = \texttt{010203}$ and every element of S_{η}^{58} with prefix $\eta(\texttt{0})$ has a prefix in

$$\{\eta(\texttt{010}), \eta(\texttt{020}), \eta(\texttt{030}), \eta(\texttt{0130}), \eta(\texttt{0320}), \eta(\texttt{01320})\}.$$

So $w \in \{\eta(\texttt{01}),\eta(\texttt{02}),\eta(\texttt{03}),\eta(\texttt{013}),\eta(\texttt{032}),\eta(\texttt{0132})\}^\omega$, hence $w \in \{\eta(\texttt{0}),\eta(\texttt{1}),\eta(\texttt{2}),\eta(\texttt{3})\}^\omega$. Thus we have $w = \eta(u)$ for some bi-infinite 4-ary word u. Notice that u must be square-free since $w = \eta(u)$ is square-free. We also check that for every factor $f \in F_\eta$, S_η^{58} does not contain $\eta(f)$. Since $\max_{f\in F_\eta} |\eta(f)| = 58$, the pre-image u of w does not contain any factor in F_η. So u is also η-good. By induction, u and w have the same factor set as $\eta^\omega(\texttt{0})$. □

Lemma 7. *A bi-infinite word is γ-good if and only if it has the same factor set as $\gamma(\eta^\omega(\texttt{0}))$.*

Proof. It is easy to check that $\gamma(\eta^\omega(\texttt{0}))$ avoids F_γ and we use Rosenfeld's program to show that $\gamma(\eta^\omega(\texttt{0}))$ avoids AA. So $\gamma(\eta^\omega(\texttt{0}))$ is γ-good.

We construct the set S_γ^{186} defined as follows: a word v is in S_γ^{186} if and only if there exists a γ-good word pvs such that $|p| = |v| = |s| = 186$. We check that S_γ^{186} is exactly the set of factors of length 186 of $\gamma(\eta^\omega(\texttt{0}))$. Thus, if w is any bi-infinite γ-good word, then its factors of length 186 are also factors of $\gamma(\eta^\omega(\texttt{0}))$. Now, every element of S_γ^{186} contains the factor $\gamma(\texttt{0}) = \texttt{012021201210120102}$ and every element of S_γ^{186} with prefix $\gamma(\texttt{0})$ has a prefix in

$$\{\gamma(\texttt{010}),\gamma(\texttt{020}),\gamma(\texttt{030}),\gamma(\texttt{0130}),\gamma(\texttt{0320}),\gamma(\texttt{01320})\}.$$

So $w \in \{\gamma(\texttt{01}),\gamma(\texttt{02}),\gamma(\texttt{03}),\gamma(\texttt{013}),\gamma(\texttt{032}),\gamma(\texttt{0132})\}^\omega$, hence $w \in \{\gamma(\texttt{0}),\gamma(\texttt{1}),\gamma(\texttt{2}),\gamma(\texttt{3})\}^\omega$. Thus we have $w = \gamma(u)$ for some bi-infinite 4-ary word u. Notice that u must be square-free since $w = \gamma(u)$ is square-free. We also check that for every factor $f \in F_\eta$, S_γ^{186} does not contain $\gamma(f)$. Since $\max_{f\in F_\eta} |\gamma(f)| = 186$, the pre-image u of w does not contain any factor in F_η. So u is η-good. By Theorem 6, u has the same factor set as $\eta^\omega(\texttt{0})$. So w has the same factor set as $\gamma(\eta^\omega(\texttt{0}))$. □

Theorem 8.

(a) $\gamma(\eta^\omega(\texttt{0}))$ *contains exactly 16 palindromes and is* $\frac{41}{22}^+$*-free.*

(b) *Every bi-infinite ternary* $\frac{52}{27}$*-free word containing at most* 16 *palindromes has the same factor set as either* $\gamma(\eta^\omega(\texttt{0}))$ *or* $\gamma(\eta^\omega(\texttt{0}))^R$.

(c) *Every recurrent ternary* $\frac{25}{13}$*-free word containing at most* 17 *palindromes has the same factor set as either* $\gamma(\eta^\omega(\texttt{0}))$ *or* $\gamma(\eta^\omega(\texttt{0}))^R$.

Proof. Let us prove Item (a). $\gamma(\eta^\omega(\texttt{0}))$ contains the palindromes ε, 0, 1, 2, the six palindromes of the form bcb and the six palindromes of the form $abcba$. It is easy to check that it contains no other palindrome, i.e., no $cabcbac$. To show that $\gamma(\eta^\omega(\texttt{0}))$ is $\frac{41}{22}^+$-free, we consider the morphic word $g(h^\omega(\texttt{0}))$ defined by the following morphisms h and g.

$$\begin{aligned} h(\texttt{0}) &= \texttt{01213012} & g(\texttt{0}) &= \texttt{0102012}\\ h(\texttt{1}) &= \texttt{31} & g(\texttt{1}) &= \texttt{0212}\\ h(\texttt{2}) &= \texttt{01201312} & g(\texttt{2}) &= \texttt{0121012}\\ h(\texttt{3}) &= \texttt{0121301312} & g(\texttt{3}) &= \texttt{01020121012} \end{aligned}$$

We check that $g(h^\omega(\texttt{0}))$ avoids F_γ. We will show in Section 7 that $g(h^\omega(\texttt{0}))$ is $\frac{41}{22}^+$-free. Since $g(h^\omega(\texttt{0}))$ is square-free and avoids F_γ, we see that $g(h^\omega(\texttt{0}))$ is γ-good. By Lemma 7, $g(h^\omega(\texttt{0}))$ and $\gamma(\eta^\omega(\texttt{0}))$ have the same factor set.[3] Thus, $\gamma(\eta^\omega(\texttt{0}))$ is also $\frac{41}{22}^+$-free.

Let us prove Item (b). We say that a ternary word (finite or infinite) is 16-good if it is $\frac{52}{27}$-free and contains at most 16 palindromes. We construct the set S_{16}^{36} defined as follows: a word v is in S_{16}^{36} if and only if there exists a 16-good word pvs such that $|p| = |v| = |s| = 36$. Then we construct the Rauzy graph G_{16}^{36} based on S_{16}^{36}, that is, such that vertices correspond to factors of length 20 and arcs correspond to factors of length 36. We notice that G_{16}^{36} consists of two connected components that are images of each other with respect to reversal. Moreover, the connected component containing the factor $\texttt{0120}$ is equal to the Rauzy graph of the factors of length 36 of $\gamma(\eta^\omega(\texttt{0}))$. Let w be a bi-infinite 16-good word belonging to this connected component, that is, w contains $\texttt{0120}$. Since $\max_{f\in F_\gamma}|f| = 36$, w is γ-good. Then by Lemma 7, w has the same factor set as $\gamma(\eta^\omega(\texttt{0}))$. By symmetry, this proves Item (b).

Let us prove Item (c). We say that a ternary word (finite or infinite) is 17-good if it is $\frac{25}{13}$-free and contains at most 17 palindromes. Let w be a recurrent 17-good word. We check by backtracking that no infinite 17-good word avoids $\texttt{010}$. By symmetry, w must contain the six palindromes of the form bcb and thus $abcba$. So w contains the 16 palindromes of $\gamma(\eta^\omega(\texttt{0}))$. Notice that w cannot contain a palindrome of the form $abacaba$ since $abacaba$ cannot be extended into a ternary square-free word. So, without loss of generality, we assume that the potential 17th palindrome of w is $\texttt{0120210}$. We say that a ternary word (finite or infinite) is 17-great if it is 17-good and every palindrome it contains is a factor of $\texttt{0120210}$, $\texttt{01210}$, $\texttt{02120}$, $\texttt{10201}$, $\texttt{20102}$, or $\texttt{21012}$. Thus w is 17-great. We construct the set S_{17}^{36} defined as follows: a word v is in S_{17}^{36} if and only if there exists a 17-great word pvs such that $|p| = |v| = |s| = 36$. Then we construct the Rauzy graph G_{17}^{36} based on S_{17}^{36}. Since w is recurrent, w is in a strongly connected component of G_{17}^{36}. We have checked that the graph induced by the strongly connected components of G_{17}^{36} is isomorphic to G_{16}^{36}. Similarly, this shows that w has the same factor set as either $\gamma(\eta^\omega(\texttt{0}))$ or $\gamma(\eta^\omega(\texttt{0}))^R$. □

7 Critical exponent of $g(h^\omega(\texttt{0}))$

Let us recall the definition of morphisms h and g:

$$
\begin{aligned}
h(\texttt{0}) &= \texttt{01213012} & g(\texttt{0}) &= \texttt{0102012}\\
h(\texttt{1}) &= \texttt{31} & g(\texttt{1}) &= \texttt{0212}\\
h(\texttt{2}) &= \texttt{01201312} & g(\texttt{2}) &= \texttt{0121012}\\
h(\texttt{3}) &= \texttt{0121301312} & g(\texttt{3}) &= \texttt{01020121012}
\end{aligned}
$$

In order to show that the critical exponent of $g(h^\omega(\texttt{0}))$ equals $\frac{41}{22}$, we will describe bispecial factors of $g(h^\omega(\texttt{0}))$ and their shortest return words. We will then apply the following result.

Theorem 9 ([15], Theorem 3)**.** *Let* $\mathbf{u}$ *be a uniformly recurrent aperiodic infinite word. Let* $(w_n)_{n\in\mathbb{N}}$ *be the sequence of all bispecial factors in* $\mathbf{u}$ *ordered by length. For every* $n \in \mathbb{N}$*, let* r_n *be the shortest return word to the bispecial factor* w_n *in* $\mathbf{u}$*. Then*

$$E(\mathbf{u}) = 1 + \sup\left\{\frac{|w_n|}{|r_n|} \;:\; n \in \mathbb{N}\right\}.$$

[3] Notice that $g(h^\omega(\texttt{0}))$ is not bi-infinite, as required by Lemma 7. However, $g(h^\omega(\texttt{0}))$ is uniformly recurrent, which is somehow a stronger condition for our purpose.

Since h is a primitive morphism, its fixed point is uniformly recurrent. Hence, $g(h^\omega(\mathtt{0}))$ is uniformly recurrent, too. We will see in the sequel that $g(h^\omega(\mathtt{0}))$ contains infinitely many bispecial factors, thus it is aperiodic.

We will proceed in two steps, first we determine bispecial factors and their shortest return words in the fixed point $h^\omega(\mathtt{0})$, second we will do the same thing in the word $g(h^\omega(\mathtt{0}))$.

7.1 Bispecial factors in $h^\omega(\mathtt{0})$

To describe the structure of all bispecial factors, we use a method by Klouda [22, Theorem 36]. This method does not work directly with bispecial factors, but with bispecial triplets $((a,b),w,(c,d))$, where (a,b) and (c,d) are unordered pairs of left and right extensions (respectively) of w with $a \neq b, c \neq d$, and both awc and bwd or both awd and bwc are factors.[4] Obviously, each bispecial factor is a middle element of a bispecial triplet. We say that the bispecial factor is *associated* with the bispecial triplet. A bispecial factor may be associated with more than one bispecial triplet if it has more than two left or two right extensions.

The method by Klouda says that there is a mapping that maps one bispecial triplet to another and that all bispecial triplets are obtained by repetitive application of the mapping to a finite set of *initial* bispecial triplets.

The mapping, in the paper called *f-image*, is defined as follows: The f-image of a bispecial triplet $((a,b),w,(c,d))$ is the bispecial triplet $((a',b'),u_1h(w)u_2,(c',d'))$, where u_1 is the largest common suffix of $h(a)$ and $h(b)$, $a'u_1$ is a suffix of $h(a)$ and $b'u_1$ is a suffix of $h(b)$, similarly, u_2 is the largest common prefix of $h(c)$ and $h(d)$, u_2c' is a prefix of $h(c)$ and u_2d' is a prefix of $h(d)$.

The set of initial bispecial triplets corresponds to, as proved in [22], the set of bispecial factors without a synchronization point. In our case, the initial bispecial factors are:
$\varepsilon, \mathtt{1}, \mathtt{3}, \mathtt{01}, \mathtt{12}, \mathtt{13}, \mathtt{31}, \mathtt{012}, \mathtt{1201}$. In order to get all bispecial factors we do not have to consider all initial bispecial triplets since some of them have the same f-image.

Example 10. $((\mathtt{0},\mathtt{1}),\varepsilon,(\mathtt{1},\mathtt{2}))$ is a bispecial triplet as $\mathtt{01}$ and $\mathtt{12}$ are factors of $h^\omega(\mathtt{0})$. Its f-image equals the initial triplet $((\mathtt{2},\mathtt{1}),\varepsilon,(\mathtt{3},\mathtt{0}))$, hence to find all bispecial factors we do not have to take $((\mathtt{0},\mathtt{1}),\varepsilon,(\mathtt{1},\mathtt{2}))$ into account.

Let us provide a list of initial bispecial triplets (giving rise to all bispecial factors) together with their first four f-images.

1. $((\mathtt{0},\mathtt{2}),\varepsilon,(\mathtt{1},\mathtt{3}))$

$$f\text{-image: } ((\mathtt{0},\mathtt{3}),\mathtt{12},(\mathtt{0},\mathtt{3}))$$
$$f^2\text{-image: } ((\mathtt{0},\mathtt{3}),\mathtt{12}h(\mathtt{12})\mathtt{0121301},(\mathtt{2},\mathtt{3}))$$
$$f^3\text{-image: } ((\mathtt{0},\mathtt{3}),\mathtt{12}h(\mathtt{12})h^2(\mathtt{12})h(\mathtt{0121301})\mathtt{012},(\mathtt{0},\mathtt{1}))$$
$$f^4\text{-image: } ((\mathtt{0},\mathtt{3}),\mathtt{12}h(\mathtt{12})h^2(\mathtt{12})h^3(\mathtt{12})h^2(\mathtt{0121301})h(\mathtt{012}),(\mathtt{0},\mathtt{3}))$$

[4] In general, a,b,c,d can be words longer than one, but in the case of the morphism h it suffices to consider letters.

2. $((1,2),\varepsilon,(2,3))$

$$\begin{aligned} f\text{-image: } & ((1,2), 012, (0,1)) \\ f^2\text{-image: } & ((1,2), h(012), (0,3)) \\ f^3\text{-image: } & ((1,2), h^2(012)0121301, (2,3)) \\ f^4\text{-image: } & ((1,2), h^3(012)h(0121301)012, (0,1)) \end{aligned}$$

3. $((1,2),\varepsilon,(0,3))$

$$\begin{aligned} f\text{-image: } & ((1,2), 0121301, (2,3)) \\ f^2\text{-image: } & ((1,2), h(0121301)012, (0,1)) \\ f^3\text{-image: } & ((1,2), h^2(0121301)h(012), (0,3)) \\ f^4\text{-image: } & ((1,2), h^3(0121301)h^2(012)0121301, (2,3)) \end{aligned}$$

4. $((2,3),\varepsilon,(0,3))$

$$\begin{aligned} f\text{-image: } & ((2,3), 013120121301, (2,3)) \\ f^2\text{-image: } & ((2,3), 01312h(013120121301)012, (0,1)) \\ f^3\text{-image: } & ((2,3), 01312h(01312)h^2(013120121301)h(012), (0,3)) \\ f^4\text{-image: } & ((2,3), 01312h(01312)h^2(01312)h^3(013120121301)h^2(012)0121301, (2,3)) \end{aligned}$$

5. $((2,3),\varepsilon,(1,3))$

$$\begin{aligned} f\text{-image: } & ((2,3), 01312, (0,3)) \\ f^2\text{-image: } & ((2,3), 01312h(01312)0121301, (2,3)) \\ f^3\text{-image: } & ((2,3), 01312h(01312)h^2(01312)h(0121301)012, (0,1)) \\ f^4\text{-image: } & ((2,3), 01312h(01312)h^2(01312)h^3(01312)h^2(0121301)h(012), (0,3)) \end{aligned}$$

6. $((0,3),1,(0,2))$

$$\begin{aligned} f\text{-image: } & ((0,3), 12h(1)012, (0,1)) \\ f^2\text{-image: } & ((0,3), 12h(12)h^2(1)h(012), (0,3)) \\ f^3\text{-image: } & ((0,3), 12h(12)h^2(12)h^3(1)h^2(012)0121301, (2,3)) \\ f^4\text{-image: } & ((0,3), 12h(12)h^2(12)h^3(12)h^4(1)h^3(012)h(0121301)012, (0,1)) \end{aligned}$$

7. $((2,3),1,(0,3))$

$$\begin{aligned} f\text{-image: } & ((2,3), 01312h(1)0121301, (2,3)) \\ f^2\text{-image: } & ((2,3), 01312h(01312)h^2(1)h(0121301)012, (0,1)) \\ f^3\text{-image: } & ((2,3), 01312h(01312)h^2(01312)h^3(1)h^2(0121301)h(012), (0,3)) \\ f^4\text{-image: } & ((2,3), 01312h(01312)h^2(01312)h^3(01312)h^4(1)h^3(0121301)h^2(012)0121301, (2,3)) \end{aligned}$$

8. $((0,3), 1, (0,3))$

f-image: $((0,3), 12h(1)0121301, (2,3))$
f^2-image: $((0,3), 12h(12)h^2(1)h(0121301)012, (0,1))$
f^3-image: $((0,3), 12h(12)h^2(12)h^3(1)h^2(0121301)h(012), (0,3))$
f^4-image: $((0,3), 12h(12)h^2(12)h^3(12)h^4(1)h^3(0121301)h^2(012)0121301, (2,3))$

9. $((2,3), 1, (2,3))$

f-image: $((2,3), 01312h(1)012, (0,1))$
f^2-image: $((2,3), 01312h(01312)h^2(1)h(012), (0,3))$
f^3-image: $((2,3), 01312h(01312)h^2(01312)h^3(1)h^2(012)0121301, (2,3))$
f^4-image: $((2,3), 01312h(01312)h^2(01312)h^3(01312)h^4(1)h^3(012)h(0121301)012, (0,1))$

10. $((1,2), 3, (0,1))$

f-image: $((1,2), h(3), (0,3))$
f^2-image: $((1,2), h^2(3)0121301, (2,3))$
f^3-image: $((1,2), h^3(3)h(0121301)012, (0,1))$
f^4-image: $((1,2), h^4(3)h^2(0121301)h(012), (0,3))$

11. $((2,3), 01, (2,3))$

f-image: $((2,3), 01312h(01)012, (0,1))$
f^2-image: $((2,3), 01312h(01312)h^2(01)h(012), (0,3))$
f^3-image: $((2,3), 01312h(01312)h^2(01312)h^3(01)h^2(012)0121301, (2,3))$
f^4-image: $((2,3), 01312h(01312)h^2(01312)h^3(01312)h^4(01)h^3(012)h(0121301)012, (0,1))$

12. $((1,2), 01, (2,3))$

f-image: $((1,2), h(01)012, (0,1))$
f^2-image: $((1,2), h^2(01)h(012), (0,3))$
f^3-image: $((1,2), h^3(01)h^2(012)0121301, (2,3))$
f^4-image: $((1,2), h^4(01)h^3(012)h(0121301)012, (0,1))$

13. $((0,3), 12, (0,1))$

f-image: $((0,3), 12h(12), (0,3))$
f^2-image: $((0,3), 12h(12)h^2(12)0121301, (2,3))$
f^3-image: $((0,3), 12h(12)h^2(12)h^3(12)h(0121301)012, (0,1))$
f^4-image: $((0,3), 12h(12)h^2(12)h^3(12)h^4(12)h^2(0121301)h(012), (0,3))$

14. $((0,2),13,(0,1))$

$$f\text{-image: } ((0,3),12h(13),(0,3))$$
$$f^2\text{-image: } ((0,3),12h(12)h^2(13)0121301,(2,3))$$
$$f^3\text{-image: } ((0,3),12h(12)h^2(12)h^3(13)h(0121301)012,(0,1))$$
$$f^4\text{-image: } ((0,3),12h(12)h^2(12)h^3(12)h^4(13)h^2(0121301)h(012),(0,3))$$

15. $((1,2),31,(0,2))$

$$f\text{-image: } ((1,2),h(31)012,(0,1))$$
$$f^2\text{-image: } ((1,2),h^2(31)h(012),(0,3))$$
$$f^3\text{-image: } ((1,2),h^3(31)h^2(012)0121301,(2,3))$$
$$f^4\text{-image: } ((1,2),h^4(31)h^3(012)h(0121301)012,(0,1))$$

16. $((2,3),012,(1,3))$

$$f\text{-image: } ((2,3),01312h(012),(0,3))$$
$$f^2\text{-image: } ((2,3),01312h(01312)h^2(012)0121301,(2,3))$$
$$f^3\text{-image: } ((2,3),01312h(01312)h^2(01312)h^3(012)h(0121301)012,(0,1))$$
$$f^4\text{-image: } ((2,3),01312h(01312)h^2(01312)h^3(01312)h^4(012)h^2(0121301)h(012),(0,3))$$

17. $((1,3),012,(0,3))$

$$f\text{-image: } ((1,2),h(012)0121301,(2,3))$$
$$f^2\text{-image: } ((1,2),h^2(012)h(0121301)012,(0,1))$$
$$f^3\text{-image: } ((1,2),h^3(012)h^2(0121301)h(012),(0,3))$$
$$f^4\text{-image: } ((1,2),h^4(012)h^3(0121301)h^2(012)0121301,(2,3))$$

18. $((0,3),1201,(2,3))$

$$f\text{-image: } ((0,3),12h(1201)012,(0,1))$$
$$f^2\text{-image: } ((0,3),12h(12)h^2(1201)h(012),(0,3))$$
$$f^3\text{-image: } ((0,3),12h(12)h^2(12)h^3(1201)h^2(012)0121301,(2,3))$$
$$f^4\text{-image: } ((0,3),12h(12)h^2(12)h^3(12)h^4(1201)h^3(012)h(0121301)012,(0,1))$$

Observing Items 1 to 18, we can see that the f^n-image and f^{n+3}-image of the same initial bispecial triplet have the same left and right extensions. Therefore, we may write down recurrence relations and their explicit solutions for the Parikh vectors of bispecial factors $w_{\ell+3n}$ associated with the $f^{\ell+3n}$-image of each initial bispecial triplet, where $\ell \in \{1,2,3\}$. For this purpose, we will denote M the matrix of the morphism h, i.e.,

$$M = \begin{pmatrix} 2 & 0 & 2 & 2 \\ 3 & 1 & 3 & 4 \\ 2 & 0 & 2 & 2 \\ 1 & 1 & 1 & 2 \end{pmatrix}.$$

We will limit our considerations to the cases 1 and 15; the other ones are analogous and are left to the reader.

Case 1 We have $\vec{w}_1 = \left(\begin{smallmatrix} 0\\1\\1\\0 \end{smallmatrix}\right)$ and the recurrence relations read, for each $n \in \mathbb{N}$,

$$\begin{aligned} \vec{w}_{4+3n} &= \left(\begin{smallmatrix} 0\\1\\1\\0 \end{smallmatrix}\right) + M\left(\begin{smallmatrix} 0\\1\\1\\0 \end{smallmatrix}\right) + M^2\left(\begin{smallmatrix} 0\\1\\1\\0 \end{smallmatrix}\right) + M^3\vec{w}_{1+3n} + M^2\left(\begin{smallmatrix} 2\\3\\1\\1 \end{smallmatrix}\right) + M\left(\begin{smallmatrix} 1\\1\\1\\0 \end{smallmatrix}\right), \\ \vec{w}_{2+3n} &= \left(\begin{smallmatrix} 0\\1\\1\\0 \end{smallmatrix}\right) + M\vec{w}_{1+3n} + \left(\begin{smallmatrix} 2\\3\\1\\1 \end{smallmatrix}\right), \\ \vec{w}_{3+3n} &= \left(\begin{smallmatrix} 0\\1\\1\\0 \end{smallmatrix}\right) + M\vec{w}_{2+3n} + \left(\begin{smallmatrix} 1\\1\\1\\0 \end{smallmatrix}\right). \end{aligned}$$

The explicit solution, for each $n \in \mathbb{N}$,

$$\vec{w}_{1+3n} = \sum_{j=0}^{3n} M^j \left(\begin{smallmatrix} 0\\1\\1\\0 \end{smallmatrix}\right) + \sum_{j=0}^{n-1} M^{3j}\left(M^2\left(\begin{smallmatrix} 2\\3\\1\\1 \end{smallmatrix}\right) + M\left(\begin{smallmatrix} 1\\1\\1\\0 \end{smallmatrix}\right)\right).$$

Case 15 We have $\vec{w}_1 = M\left(\begin{smallmatrix} 0\\1\\0\\1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1\\1\\1\\0 \end{smallmatrix}\right)$ and the recurrence relations read, $n \in \mathbb{N}$,

$$\begin{aligned} \vec{w}_{4+3n} &= M^3\vec{w}_{1+3n} + M\left(\begin{smallmatrix} 2\\3\\1\\1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1\\1\\1\\0 \end{smallmatrix}\right), \\ \vec{w}_{2+3n} &= M\vec{w}_{1+3n}, \\ \vec{w}_{3+3n} &= M\vec{w}_{2+3n} + \left(\begin{smallmatrix} 2\\3\\1\\1 \end{smallmatrix}\right). \end{aligned}$$

The explicit solution, $n \in \mathbb{N}$,

$$\vec{w}_{1+3n} = M^{3n+1}\left(\begin{smallmatrix} 0\\1\\0\\1 \end{smallmatrix}\right) + \sum_{j=0}^{n} M^{3j}\left(\begin{smallmatrix} 1\\1\\1\\0 \end{smallmatrix}\right) + \sum_{j=0}^{n-1} M^{3j+1}\left(\begin{smallmatrix} 2\\3\\1\\1 \end{smallmatrix}\right).$$

7.2 Shortest return words to bispecial factors in $h^\omega(0)$

To find the shortest return words to bispecial factors, we will use the following statement. For words r and s, we write $\vec{r} \leqslant \vec{s}$ if the i-th component of $\vec{r}$ is less than or equal to the i-th component of $\vec{s}$ for all i.

Proposition 11. *If a bispecial factor w in $h^\omega(\mathtt{0})$ is associated to a bispecial triplet*

$$((a,b), u_1 h(v) u_2, (c,d)),$$

where u_1 is the longest common suffix of $h(a)$ and $h(b)$ and u_2 is the longest common prefix of $h(c)$ and $h(d)$, and $h(v)$ has the synchronization points $\bullet h(v)\bullet$ and v has only two left and two right extensions, then the Parikh vectors of return words to w are the same as the Parikh vectors of images by h of return words to v.

In particular, if r is a return word to v such that $\vec{r} \leqslant \vec{s}$ for any return word s to v, then $\overrightarrow{h(r)}$ is the Parikh vector of the shortest return word to w.

Proof. Let z be a return word to $w = u_1 h(v) u_2$, so that we have

$$zw = \underbrace{u_1 h(v) u_2 \cdots}_{z} u_1 h(v) u_2.$$

Since v has only two left and two right extensions, using the definition of bispecial triplet, $h(v)$ is always preceded by u_1 and followed by u_2, hence $h(v)$ occurs only twice in $u_1 h(v) u_2 \cdots u_1 h(v) u_2$. Knowing the synchronization points $\bullet h(v) \bullet u_2 \cdots u_1 \bullet h(v) \bullet$ and by the injectivity of h, there exists a unique return word r to v such that $h(v)u_2 \cdots u_1 h(v) = h(r)h(v)$. Again, since $h(v)$ is always preceded by u_1 and followed by u_2, we get $u_1 h(r) h(v) u_2 = u_1 h(r) u_1^{-1} u_1 h(v) u_2$, consequently, $z = u_1 h(r) u_1^{-1}$ and $\vec{z} = \overrightarrow{h(r)}$.

The second statement is then obvious. □

Observation 12. *The words $h(\mathtt{0}), h(\mathtt{2}), h(\mathtt{3}), h(i\mathtt{1}), h(\mathtt{1}i)$, where $i \in \{\mathtt{0},\mathtt{2},\mathtt{3}\}$, have synchronization points $\bullet h(\mathtt{0})\bullet, \bullet h(\mathtt{2})\bullet, \bullet h(\mathtt{3})\bullet, \bullet h(i\mathtt{1})\bullet, \bullet h(\mathtt{1}i)\bullet$.*

By Proposition 11, it suffices to find the shortest return words to the shortest bispecial factors having only two left and two right extensions, in each class 1 to 18. By the shortest return word we mean a return word r satisfying $\vec{r} \leqslant \vec{s}$ for any other return word. Here is the complete list:

1. 12h(12)h(01)01213013 is the shortest return word to 12h(12)0121301
2. h(01231) is the shortest return word to h(012)
3. 0121301231 is the shortest return word to 0121301
4. 013120121301231012 is the shortest return word to 013120121301
5. 0131201213 is the shortest return word to 01312
6. 12h(1)012013 is the shortest return word to 12h(1)012
7. 01312h(1)0121301231012 is the shortest return word to 01312h(1)0121301
8. 12h(1)0121301231012013 is the shortest return word to 12h(1)0121301
9. 01312h(1)0120131201213 is the shortest return word to 01312h(1)012

10. 301 and 312 are the shortest return words to 3
11. $01312h(01)012130131231012$ is the shortest return word to $01312h(01)012$
12. $h(01)01201312$ is the shortest return word to $h(01)012$
13. $12h(12)012130$ is the shortest return word to $12h(12)$
14. 130 is the shortest return word to 13
15. 312 is the shortest return word to 31
16. $01312h(012)3101213$ is the shortest return word to $01312h(012)$
17. $h(01231)$ is the shortest return word to $h(012)0121301$
18. 12013 is the shortest return word to 1201

Applying Proposition 11, we get the Parikh vectors of the shortest return words to all bispecial factors, besides eventually a few shortest ones. Let us list the shortest return words to all bispecial factors in cases 1 and 15.

Case 1: $\vec{r}_2 = M \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$ is the Parikh vector of the shortest return word to $w_2 = 12h(12)0121301$. By Proposition 11, $\vec{r}_n = M^{n-2}\vec{r}_2$ is the Parikh vector of the shortest return word to w_n for $n \geqslant 3$.

Case 15: $r_0 = 312$ is the shortest return word to $w_0 = 31$. By Proposition 11, $\vec{r}_n = M^n \vec{r}_0$ is the Parikh vector of the shortest return word to w_n for $n \geqslant 1$.

7.3 Bispecial factors in $g(h^\omega(0))$

Let us describe how to get all sufficiently long bispecial factors in $g(h^\omega(0))$ using our knowledge of bispecial triplets in $h^\omega(0)$.

Proposition 13. *Let w be a bispecial factor in $g(h^\omega(0))$ such that it has at least two synchronization points. Then there exists a unique bispecial factor v in $h^\omega(0)$ such that $w = xg(v)y$, where x is the longest common suffix of $g(a)$ and $g(b)$ for some $a, b \in \{0, 1, 12, 3\}$, $a \neq b$, and y is the longest common prefix of $g(c)$ and $g(d)$ for some $c, d \in \{01, 1, 2, 3\}$, $c \neq d$, and either avc and bvd or avd and bvc are factors of $h^\omega(0)$.*

Moreover, if v has only two left and two right extensions, then the Parikh vectors of return words to w are the same as the Parikh vectors of images by g of return words to v.

In particular, if r is a return word to v such that $\vec{r} \leqslant \vec{s}$ for any return word s to v, then $\overrightarrow{g(r)}$ is the Parikh vector of the shortest return word to w.

Proof. Let us highlight the first and the last synchronization point of $w = x \bullet \cdots \bullet y$, then by injectivity of g, there exists a unique factor v in $h^\omega(0)$ such that $w = xg(v)y$. Now, v is bispecial, otherwise we have a contradiction with the choice of synchronization points. Since $g(0)$ is a prefix of $g(3)$ and $g(2)$ is a suffix of $g(3)$, in order to determine x and y we need to consider 12 instead of 2 among the left extensions of v (this is without loss of generality since 2 is always preceded by 1 in $h^\omega(0)$). Similarly, we have to consider 01 instead of 0 among the right extensions of v. The statement on how to get x and y is then straightforward.

The remaining part is analogous to the proof of Proposition 11. □

Using Proposition 13, to get the Parikh vectors of all bispecial factors and their shortest return words in $g(h^\omega(0))$, we have to treat three cases:

a) Bispecial factors in $g(h^\omega(0))$ having at most one synchronization point.

b) Bispecial factors in $g(h^\omega(0))$ having at least two synchronization points arising from bispecial factors in $h^\omega(0)$ with more extensions.

c) Bispecial factors in $g(h^\omega(0))$ having at least two synchronization points arising from bispecial factors in $h^\omega(0)$ with exactly two left and two right extensions.

Case a). Table 2 contains the list of all bispecial factors having at most one synchronization point in $g(h^\omega(0))$ together with their shortest return words (there are more of them in some cases, but we always list only one).

bispecial factor	shortest return word
0	01
1	12
2	20
10	10120
01	012
02	02012
20	201
12	120
21	21201
201	2010
012	0121
120	1202
2012	2012021
0120	0120102
1201	1201210
201210120	20121012010
120102012	12010201210

Tab. 2: List of bispecial factors and their shortest return words in case a).

Case b) consists of bispecial factors in $g(h^\omega(0))$ arising from bispecial factors in $h^\omega(0)$ having more left or right extensions than two, i.e., from 1, 01, 12, 012. Table 3 contains their list together with the shortest return words (there are more of them in some cases, but we always list only one).

Case c). We will list the Parikh vectors of bispecial factors and their shortest return words coming only from cases 1 and 15 in $h^\omega(0)$. Let us denote by W_n the bispecial factor in $g(h^\omega(0))$ arising from w_n in

bispecial factor	shortest return word
$\mathtt{012}g(\mathtt{1})\mathtt{01}$	$\mathtt{012}g(\mathtt{1})\mathtt{0102}$
$\mathtt{012}g(\mathtt{1})\mathtt{0102012}$	$\mathtt{012}g(\mathtt{1})\mathtt{010201210120102}$
$\mathtt{20121012}g(\mathtt{1})\mathtt{01}$	$\mathtt{20121012}g(\mathtt{1})\mathtt{0102012021}$
$\mathtt{20121012}g(\mathtt{1})\mathtt{0102012}$	$\mathtt{20121012}g(\mathtt{1})\mathtt{0102012021}$
$\mathtt{12}g(\mathtt{01})\mathtt{01}$	$\mathtt{12}g(\mathtt{01})\mathtt{01210}$
$\mathtt{20121012}g(\mathtt{01})\mathtt{01}$	$\mathtt{20121012}g(\mathtt{01})\mathtt{01210120212010}$
$\mathtt{012}g(\mathtt{12})\mathtt{0}$	$\mathtt{012}g(\mathtt{12})\mathtt{0102}$
$\mathtt{012}g(\mathtt{12})\mathtt{0102012}$	$\mathtt{012}g(\mathtt{12})\mathtt{0102012021201020121}$
$\mathtt{12}g(\mathtt{012})\mathtt{0}$	$\mathtt{12}g(\mathtt{012})\mathtt{0102012101202}$
$\mathtt{20121012}g(\mathtt{012})\mathtt{0}$	$\mathtt{20121012}g(\mathtt{012})\mathtt{0212010}$
$\mathtt{12}g(\mathtt{012})\mathtt{0102012}$	$\mathtt{12}g(\mathtt{012})\mathtt{0102012101202}$

Tab. 3: List of bispecial factors and their shortest return words in case b).

$h^\omega(\mathtt{0})$. Moreover, we will use the matrix N of the morphism g, i.e.,

$$N = \begin{pmatrix} 3 & 1 & 2 & 4 \\ 2 & 1 & 3 & 4 \\ 2 & 2 & 2 & 3 \end{pmatrix}.$$

By Proposition 13, in all cases, the shortest return word to W_n has the Parikh vector equal to $N\vec{r}_n$, where r_n is the shortest return word to w_n in $h^\omega(\mathtt{0})$.

Case 1 For all $n \in \mathbb{N}$

$$\begin{aligned} W_{1+3n} &= \mathtt{012}g(w_{1+3n})\mathtt{0102012}\,, \quad n \geqslant 1\,, \\ W_{2+3n} &= \mathtt{012}g(w_{2+3n})\mathtt{01}\,, \\ W_{3+3n} &= \mathtt{012}g(w_{3+3n})\mathtt{0}\,. \end{aligned}$$

Case 15 For all $n \in \mathbb{N}$

$$\begin{aligned} W_0 &= \mathtt{12}g(\mathtt{31})\mathtt{01}\,, \\ W_{1+3n} &= \mathtt{12}g(w_{1+3n})\mathtt{0}\,, \\ W_{2+3n} &= \mathtt{12}g(w_{2+3n})\mathtt{0102012}\,, \\ W_{3+3n} &= \mathtt{12}g(w_{3+3n})\mathtt{01}\,. \end{aligned}$$

We have prepared everything to be able to compute the lengths of all bispecial factors in $g(h^\omega(\mathtt{0}))$ together with the lengths of their shortest return words. Thus, using Theorem 9 we will be able to determine the critical exponent of $g(h^\omega(\mathtt{0}))$.

Theorem 14. *The critical exponent of the word $g(h^\omega(\mathtt{0}))$ equals $\frac{41}{22}$.*

Proof. According to Theorem 9, we have to show that for each bispecial factor W and the shortest return word R to W in $g(h^\omega(\mathtt{0}))$, the ratio $\frac{|W|}{|R|}$ is smaller than or equal to $\frac{19}{22}$. We have divided bispecial factors into three classes.

a) For all bispecial factors in the first class, the ratio $\frac{|W|}{|R|}$ is at most $\frac{9}{11} < \frac{19}{22}$, see Table 2.

b) For all bispecial factors in the second class, the ratio $\frac{|W|}{|R|}$ is at most $\frac{19}{22}$, see Table 3. Moreover, the exponent $\frac{41}{22}$ is reached by the factor

$$\texttt{20121012}g(1)\texttt{0102012021201210 12}g(1)\texttt{0102012} = (\texttt{2012101202120102012021})^{\frac{41}{22}}\,.$$

c) In this last class, we will treat only the bispecial factors described in Cases 1 and 15. To compute the lengths of bispecial factors and their shortest return words, it is necessary to diagonalize the matrix M. Using standard linear algebra, we obtain $M = XDX^{-1}$, where

$$D = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; X = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 0 \\ 1 & 2 & 0 & -1 \\ 1 & -3 & -1 & 0 \end{pmatrix}, \; X^{-1} = \frac{1}{15}\begin{pmatrix} 4 & 1 & 4 & 5 \\ 3 & -3 & 3 & 0 \\ -5 & 10 & -5 & -10 \\ 10 & -5 & -5 & 5 \end{pmatrix}.$$

We will repeatedly use the formula

$$(1,1,1)NM^j = (1,1,1)NXD^jX^{-1} = \frac{1}{5}(44\cdot 6^j - 9, 11\cdot 6^j + 9, 44\cdot 6^j - 9, 55\cdot 6^j)\,.$$

Case 1: For $n \in \mathbb{N}, n \geqslant 1$,

$$\begin{aligned}
\frac{|W_{1+3n}|}{|R_{1+3n}|} &= \frac{(1,1,1)\left(\left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right) + N\vec{w}_{1+3n} + \left(\begin{smallmatrix}3\\2\\2\end{smallmatrix}\right)\right)}{(1,1,1)NM^{3n-1}\vec{r}_2} \\
&= \frac{(1,1,1)\left(\left(\begin{smallmatrix}4\\3\\3\end{smallmatrix}\right) + N\left(\sum_{j=0}^{3n} M^j \left(\begin{smallmatrix}0\\1\\1\\0\end{smallmatrix}\right) + \sum_{j=0}^{n-1} M^{3j+2}\left(\begin{smallmatrix}2\\3\\1\\1\end{smallmatrix}\right) + \sum_{j=0}^{n-1} M^{3j+1}\left(\begin{smallmatrix}1\\1\\1\\0\end{smallmatrix}\right)\right)\right)}{(1,1,1)NM^{3n}\left(\begin{smallmatrix}1\\2\\1\\1\end{smallmatrix}\right)} \\
&= \frac{10 + 11\sum_{j=0}^{3n} 6^j + 44\sum_{j=0}^{n-1} 6^{3j+2} + \frac{1}{5}\sum_{j=0}^{n-1}(99\cdot 6^{3j+1} - 9)}{33\cdot 6^{3n}} \\
&= \frac{10 + \frac{11}{5}(6^{3n+1}-1) + \frac{44\cdot 36}{215}(6^{3n}-1) + \frac{99\cdot 6}{5\cdot 215}(6^{3n}-1) - \frac{9}{5}n}{33\cdot 6^{3n}} \\
&< \frac{688}{1075} + \frac{10}{33\cdot 6^{3n}} < \frac{19}{22}\,.
\end{aligned}$$

For $n \in \mathbb{N}$

$$\begin{aligned}
\frac{|W_{2+3n}|}{|R_{2+3n}|} &= \frac{(1,1,1)\left(\left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right) + N\vec{w}_{2+3n} + \left(\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right)\right)}{(1,1,1)NM^{3n}\vec{r}_2} \\
&= \frac{(1,1,1)\left(\left(\begin{smallmatrix}2\\2\\1\end{smallmatrix}\right) + N\left(\sum_{j=0}^{3n+1} M^j \left(\begin{smallmatrix}0\\1\\1\\0\end{smallmatrix}\right) + \sum_{j=0}^{n} M^{3j}\left(\begin{smallmatrix}2\\3\\1\\1\end{smallmatrix}\right) + \sum_{j=0}^{n-1} M^{3j+2}\left(\begin{smallmatrix}1\\1\\1\\0\end{smallmatrix}\right)\right)\right)}{(1,1,1)NM^{3n+1}\left(\begin{smallmatrix}1\\2\\1\\1\end{smallmatrix}\right)} \\
&= \frac{5 + 11\sum_{j=0}^{3n+1} 6^j + 44\sum_{j=0}^{n} 6^{3j} + \frac{1}{5}\sum_{j=0}^{n-1}(99\cdot 6^{3j+2} - 9)}{33\cdot 6^{3n+1}} \\
&= \frac{5 + \frac{11}{5}(6^{3n+2} - 1) + \frac{44}{215}(6^{3n+3} - 1) + \frac{99\cdot 36}{5\cdot 215}(6^{3n} - 1) - \frac{9}{5}n}{33\cdot 6^{3n+1}} \\
&< \frac{688}{1075} + \frac{5}{33\cdot 6^{3n+1}} < \frac{19}{22}\,.
\end{aligned}$$

For $n \in \mathbb{N}$

$$\begin{aligned}
\frac{|W_{3+3n}|}{|R_{3+3n}|} &= \frac{(1,1,1)\left(\left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right) + N\vec{w}_{3+3n} + \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)\right)}{(1,1,1)NM^{3n+1}\vec{r}_2} \\
&= \frac{(1,1,1)\left(\left(\begin{smallmatrix}2\\1\\1\end{smallmatrix}\right) + N\left(\sum_{j=0}^{3n+2} M^j \left(\begin{smallmatrix}0\\1\\1\\0\end{smallmatrix}\right) + \sum_{j=0}^{n} M^{3j+1}\left(\begin{smallmatrix}2\\3\\1\\1\end{smallmatrix}\right) + \sum_{j=0}^{n} M^{3j}\left(\begin{smallmatrix}1\\1\\1\\0\end{smallmatrix}\right)\right)\right)}{(1,1,1)NM^{3n+2}\left(\begin{smallmatrix}1\\2\\1\\1\end{smallmatrix}\right)} \\
&= \frac{4 + 11\sum_{j=0}^{3n+2} 6^j + 44\sum_{j=0}^{n} 6^{3j+1} + \frac{1}{5}\sum_{j=0}^{n}(99\cdot 6^{3j} - 9)}{33\cdot 6^{3n+2}} \\
&= \frac{4 + \frac{11}{5}(6^{3n+3} - 1) + \frac{44\cdot 6}{215}(6^{3n+3} - 1) + \frac{99}{5\cdot 215}(6^{3n+3} - 1) - \frac{9}{5}(n+1)}{33\cdot 6^{3n+2}} \\
&< \frac{688}{1075} + \frac{4}{33\cdot 6^{3n+2}} < \frac{19}{22}\,.
\end{aligned}$$

Case 15: Let us start with $\frac{|W_0|}{|R_0|} = \frac{(1,1,1)\left(\left(\begin{smallmatrix}0\\1\\1\end{smallmatrix}\right) + N\left(\begin{smallmatrix}0\\1\\0\\1\end{smallmatrix}\right) + \left(\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right)\right)}{(1,1,1)N\left(\begin{smallmatrix}0\\1\\1\\1\end{smallmatrix}\right)} = \frac{19}{22}$. Thus we have found another factor having the exponent equal to $\frac{41}{22}$:

$$\mathtt{12}g(\mathtt{31})\mathtt{0121012}g(\mathtt{31})\mathtt{01} = (\mathtt{1201020121012021201210})^{\frac{41}{22}}\,.$$

For $n \in \mathbb{N}$

$$\begin{aligned}
\frac{|W_{1+3n}|}{|R_{1+3n}|} &= \frac{(1,1,1)\left(\left(\begin{smallmatrix}0\\1\\1\end{smallmatrix}\right) + N\vec{w}_{1+3n} + \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)\right)}{(1,1,1)NM^{3n+1}\vec{r}_0} \\
&= \frac{(1,1,1)\left(\left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right) + N\left(M^{3n+1}\left(\begin{smallmatrix}0\\1\\0\\1\end{smallmatrix}\right) + \sum_{j=0}^{n} M^{3j}\left(\begin{smallmatrix}1\\1\\1\\0\end{smallmatrix}\right) + \sum_{j=0}^{n-1} M^{3j+1}\left(\begin{smallmatrix}2\\3\\1\\1\end{smallmatrix}\right)\right)\right)}{(1,1,1)NM^{3n+1}\left(\begin{smallmatrix}0\\1\\1\\1\end{smallmatrix}\right)} \\
&= \frac{3 + \frac{66\cdot 6^{3n+1}+9}{5} + \frac{1}{5}\sum_{j=0}^{n}(99\cdot 6^{3j} - 9) + 44\sum_{j=0}^{n-1} 6^{3j+1}}{22\cdot 6^{3n+1}} \\
&< \frac{817}{1075} + \frac{2}{55\cdot 6^{3n}} < \frac{19}{22}\,.
\end{aligned}$$

For $n \in \mathbb{N}$

$$\begin{aligned}
\frac{|W_{2+3n}|}{|R_{2+3n}|} &= \frac{(1,1,1)\left(\left(\begin{smallmatrix}0\\1\\1\end{smallmatrix}\right) + N\vec{w}_{2+3n} + \left(\begin{smallmatrix}3\\2\\2\end{smallmatrix}\right)\right)}{(1,1,1)NM^{3n+2}\vec{r}_0} \\
&= \frac{(1,1,1)\left(\left(\begin{smallmatrix}3\\3\\3\end{smallmatrix}\right) + N\left(M^{3n+2}\left(\begin{smallmatrix}0\\1\\0\\1\end{smallmatrix}\right) + \sum_{j=0}^{n} M^{3j+1}\left(\begin{smallmatrix}1\\1\\1\\0\end{smallmatrix}\right) + \sum_{j=0}^{n-1} M^{3j+2}\left(\begin{smallmatrix}2\\3\\1\\1\end{smallmatrix}\right)\right)\right)}{(1,1,1)NM^{3n+2}\left(\begin{smallmatrix}0\\1\\1\\1\end{smallmatrix}\right)} \\
&= \frac{9 + \frac{66\cdot 6^{3n+2}+9}{5} + \frac{1}{5}\sum_{j=0}^{n}(99\cdot 6^{3j+1} - 9) + 44\sum_{j=0}^{n-1} 6^{3j+2}}{22\cdot 6^{3n+2}} \\
&< \frac{817}{1075} + \frac{9}{110\cdot 6^{3n+1}} < \frac{19}{22}\,.
\end{aligned}$$

For $n \in \mathbb{N}$

$$\begin{aligned}
\frac{|W_{3+3n}|}{|R_{3+3n}|} &= \frac{(1,1,1)\left(\left(\begin{smallmatrix}0\\1\\1\end{smallmatrix}\right) + N\vec{w}_{3+3n} + \left(\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right)\right)}{(1,1,1)NM^{3n+3}\vec{r}_0} \\
&= \frac{(1,1,1)\left(\left(\begin{smallmatrix}1\\2\\1\end{smallmatrix}\right) + N\left(M^{3n+3}\left(\begin{smallmatrix}0\\1\\0\\1\end{smallmatrix}\right) + \sum_{j=0}^{n} M^{3j+2}\left(\begin{smallmatrix}1\\1\\1\\0\end{smallmatrix}\right) + \sum_{j=0}^{n} M^{3j}\left(\begin{smallmatrix}2\\3\\1\\1\end{smallmatrix}\right)\right)\right)}{(1,1,1)NM^{3n+3}\left(\begin{smallmatrix}0\\1\\1\\1\end{smallmatrix}\right)} \\
&= \frac{4 + \frac{66\cdot 6^{3n+3}+9}{5} + \frac{1}{5}\sum_{j=0}^{n}(99\cdot 6^{3j+2} - 9) + 44\sum_{j=0}^{n} 6^{3j}}{22\cdot 6^{3n+3}} \\
&< \frac{817}{1075} + \frac{29}{110\cdot 6^{3n+3}} < \frac{19}{22}\,.
\end{aligned}$$

□

8 Overpals and letter patterns

The notions of overpals and letter patterns defined in the preliminaries happen to be related to square-free words with few palindromes.

Lemma 15. *Let w be a bi-infinite ternary square-free word. Then the following properties are equivalent:*

(a) *w contains at most 16 palindromes.*

(b) *w avoids overpals.*

(c) *w avoids the letter pattern $abcacba$.*

Proof. First, notice that since w is square-free, the length of every non-empty palindrome in w is odd. Also, w contains the palindromic letter pattern aba if and only if w contains $cabac$.

Item (a) $\Longrightarrow$ Item (b): A computer check shows that there are only finitely many ternary square-free words avoiding `010` and containing at most 16 palindromes. So, if w contains at most 16 palindromes, then w must contain all the 6 factors aba. This implies that w contains the following 16 palindromes: ε, `0`, `1`, `2`, six palindromes aba, and six palindromes $abcba$. Thus w contains no other palindrome. In particular, w avoids overpals, since an overpal is a palindrome such that the first letter occurs at least three times.

Item (b) $\Longrightarrow$ Item (c): The letter pattern $abcacba$ is an overpal $axax^Ra$ with $x = bc$. So if w avoids overpals, then w avoids $abcacba$.

Item (c) $\Longrightarrow$ Item (a): Notice that w avoids $cbcacbc$, since $cbcacbc$ has no square-free extension. If w also avoids $abcacba$, then the palindrome $bcacb$ cannot be extended to a larger palindrome. Thus, w contains no palindrome other than the 16 palindromes mentioned above. □

Using Lemma 15, we obtain the analogs of Theorems 3(d), 8(a), and 8(b), where "containing (at most) 16 palindromes" is replaced by "avoiding overpals" or "avoiding $abcacba$". This proves Conjecture 17 in [29] that there exists an infinite ternary $\frac{41}{22}^+$-free word that avoids overpals. This also complements Currie's results [7] about words avoiding $abcacba$ with respect to the critical exponent and the factor complexity.

Petrova [27] considered the other letter patterns that are minimally avoidable by infinite ternary square-free words, namely $abaca$, $abcab$, and $abacbc$. We give simpler proofs of her results as well as the minimal critical exponent $\frac{7}{4}^+$ for words avoiding $abacbc$. Here, contrary to the case of $abcacba$, there is no critical exponent such that the factor complexity is polynomial.

Theorem 16. *There exist exponentially many ternary words that are:*

(a) *$\frac{15}{8}^+$-free and avoid $abaca$.*

(b) *$\frac{11}{6}^+$-free and avoid $abcab$.*

(c) *$\frac{7}{4}^+$-free and avoid $abacbc$.*

Proof. Notice that checking if the proposed word contains the letter pattern p only requires to check the factors up to length $|p|$.

(a) Applying the following 72-uniform morphism to any ternary $\frac{7}{4}^+$-free word gives a ternary $\frac{15}{8}^+$-free word avoiding $abaca$.

0 $\to$ 010210120102120121020120210120102101202102012102120102101201021201210212

1 $\to$ 010210120210201202101201021012021020121021201210201202101201021201210212

2 $\to$ 010210120210201210212012102012021012010210120210201202101201021201210212

(b) Applying the following 73-uniform morphism to any ternary $\frac{7}{4}^+$-free word gives a ternary $\frac{11}{6}^+$-free word avoiding $abcab$.

0 $\to$ 0121020102120210120212010212021012102012101202120102120210120212010201210

1 $\to$ 1202101210201021201020121020102120210120212010201210201021201020121012021

2 $\to$ 2010212021012102012101202101210201021201020121012021012102012101202120102

(c) Applying the following 128-uniform morphism to any 4-ary $\frac{7}{5}^+$-free word gives a ternary $\frac{7}{4}^+$-free word avoiding $abacbc$.

0 $\to$ p120102101210212012101201020120210201021012102120210201202120121021202102
010210121

1 $\to$ p120102101210212021020102101201020120210201021012102120121012010210121021
202102012

2 $\to$ p120102101210212021020120212012102120210201021012010201202102010210121021
202102012

3 $\to$ p212021020102101210212012101201021012102120210201021012010201202120121021
202102012

where $p =$ 02120121012010201202102010210120102012021201210. □

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