SIEVE METHODS FOR ODD PERFECT NUMBERS

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Abstract. Using a new factor chain argument, we show that 5 does not divide an odd perfect number indivisible by a sixth power. Applying sieve techniques, we also find an upper bound on the smallest prime divisor. Putting this together we prove that an odd perfect number must be divisible by the sixth power of a prime or its smallest prime factor lies in the range $10^8 < p < 10^{1000}$. These results are generalized to much broader situations.

1. Introduction

A positive integer of the form $2^n - 1$ is called a Mersenne number. These numbers can only be prime when n is prime. As it currently stands, there are forty-seven known Mersenne primes. The largest is $2^{43112609} - 1$, which is also currently the largest explicitly known prime, and the first such found with more than ten-million digits. As observed by Euclid, a number of the form $N = 2^{p-1}(2^p - 1)$, where $2^p - 1$ is a Mersenne prime, possesses the property that $\sigma(N) = 2N$, and Euler proved the converse in the case when N is even. A positive integer N satisfying $\sigma(N) = 2N$ is called a *perfect number*. There are two questions, open since antiquity, concerning perfect numbers.

Question 1. Are there infinitely many even perfect numbers?

Equivalently, one can ask if there are infinitely many Mersenne primes. Due to heuristic arguments, and the fact that prime-hunters keep finding more of them at about the right times, the widely held belief is that the answer to this question is yes. However, little concrete progress has been made in formally answering the question. A positive answer would give an answer to other difficult open questions, such as whether or not the quadratic polynomial $2x^2 - 1$ is prime infinitely often. Those wishing to look for Mersenne primes can participate in GIMPS (the Great Internet Mersenne Prime Search) by downloading a free program that runs on spare computer cycles from `http://www.mersenne.org`.

Question 2. Are there any odd perfect numbers?

The popular opinion is that the answer is no. Further, there are a number of known limiting conditions. We give an updated version of the list provided in [24]. Write $N = \prod_{i=1}^{k} p_i^{a_i}$ where each p_i is prime, $p_1 < p_2 < \ldots < p_k$, and $k = \omega(N)$ is the number of distinct prime factors. Then:

- *Eulerian Form:* We have $N = \pi^\alpha m^2$ for some integers $\pi, \alpha, m \in \mathbb{Z}_{>0}$, $\pi \nmid m$, with $\pi \equiv \alpha \equiv 1 \pmod 4$ and π prime. The prime π is called the *special* prime of N.
- *Lower Bound:* Brent, Cohen, and te Riele [2] using a computer search found that $N > 10^{300}$. As of this writing, the third named author has completed the calculations which improve this bound to $N > 10^{1500}$, and further progress is likely. William Lipp is helping coordinate this effort via his website `http://www.oddperfect.org`.
- *Upper Bound:* Dickson [6] proved that there are finitely many odd perfect numbers with a fixed number of distinct prime factors. Pomerance [26] gave an effective bound in terms of k. This was improved in succession by Heath-Brown [13], Cook [5], and finally by the second author to

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$N < 2^{4^k}$. Recently Pollack [25] used these bounds to give an effective bound on the number of odd perfect numbers in terms of k.

- *Large Factors:* Goto and Ohno [9] proved that $p_k > 10^8$, and Iannucci [14, 15] proved $p_{k-1} > 10^4$ and $p_{k-2} > 10^2$.
- *Small Factors:* The smallest prime factor satisfies $p_1 < \frac{2}{3}k + 2$ as proved by Grün [11]. For $2 \leqslant i \leqslant 6$, Kishore [20] showed that $p_i < 2^{2^{i-1}}(k - i + 1)$, and this has been slightly improved by Cohen and Sorli [3].
- *Number of Total Prime Factors:* Hare [12] proved that the total number of (not necessarily distinct) prime factors of N must be at least 75. The third author is performing computations which increase this bound to 101.
- *Number of Distinct Prime Factors:* The second author [24] demonstrated that $\omega(N) \geqslant 9$ and if $3 \nmid N$ then $\omega(N) \geqslant 12$. In a work in progress these bounds are increased further.
- *The Exponents:* Set $d = \gcd_i(a_i + 1)$, where the a_i run over the exponents on the non-special primes. McDaniel [23] proved that $d \not\equiv 0 \pmod 3$. If the exponents of the non-special primes are all equal (to $d-1$), Yamada [30] proved that there is an effective upper bound on N depending only on d. In this case, a number of authors have demonstrated that certain values of d are unacceptable.

The focus of this paper will be on the exponents of a purported odd perfect number, the last of the bulleted points above. As above, let $d = \gcd_i(a_i + 1)$, where the a_i run over the exponents on the non-special primes. With modern computing power it is a simple matter to show that $d \not\equiv 0 \pmod 3$; taking fewer than 30 cases to check. We even provide such a proof later in the paper. If we replace 3 by a larger prime, like 5, already the computation becomes much more difficult, and previously had never been accomplished. Nonetheless, there is still a straightforward algorithm to deal with this case; assuming you have a large amount of computational power at hand. However, that leaves open what happens in the case $d = \gcd_i(a_i + 1) = 1$. For example, it is known that the non-special primes cannot all have an exponent of 2, nor can they all have exponent 4, but it is unknown what happens in the mixed case when we allow both 2 and 4 to occur.

A new idea was developed by Yamada [29] in a preprint available at the mathematics preprint server `http://arxiv.org` since 2005. Using sieve methods, he proved the following:

Yamada's Theorem. *Let $n, d, b_1, b_2, \ldots, b_t$ be positive integers, where the b_j belong to a finite set $\mathcal{P}$. If $N = p_1^{a_1} \cdots p_s^{a_s} q_1^{b_1} \cdots q_t^{b_t}$ is the prime factorization of N and $\sigma(N)/N = n/d$, then N has a prime factor smaller than a constant C, effectively computable in terms of n, s, and $\mathcal{P}$.*

Yamada explicitly computed the constant C in the case that N is an odd perfect number (i.e. $n = 2$ and $d = 1$) where the exponents of the non-special prime factors are either 2 or 4 (so $s \leq 1$ and $\mathcal{P} = \{2, 4\}$). The constant he obtained was $\exp(4.97401 \cdot 10^{10}) > 10^{20,000,000,000}$.

In this paper we strengthen Yamada's theorem by weakening the condition "the b_j belong to a finite set" to "there is a finite set of primes so that $b_j + 1$ is divisible by one of them." Furthermore, we simplify the proof a great deal, and the computable constant obtained is much smaller.

Applying our theorem, we arrive at the following:

Main Theorem. *Let N be an odd perfect number such that if $p^a || N$ and p is not the special prime then either $3|(a+1)$ or $5|(a+1)$. The smallest prime factor of N belongs to the range $10^8 < p < 10^{1000}$.*

In Sections 2-4 we develop the machinery necessary to establish an improved version of Yamada's theorem. Emphasis is given to simplifying the computations, rather than optimizing constants. Section 5-6 describe how the lower bound of the main theorem is pushed to 10^8. The most difficult problem, and the real heart of the paper, is dealing with the case when $p = 5$. In Sections 7-10 we finish the proof of the upper bound in the main theorem. The last few sections focus on open questions and possible future improvements.

2. Break-down of the main ideas

We begin with some standard notation. We define the function $\sigma_{-1}(a) = \frac{\sigma(a)}{a} = \sum_{d|a} d^{-1}$ for positive integers a, and note that this function is multiplicative. Clearly, an integer N is perfect if and only if $\sigma_{-1}(N) = 2$. Given a prime p we have $\sigma_{-1}(p^a) = \frac{p^{a+1}-1}{p^a(p-1)}$ which is increasing in a and $\lim_{a\to\infty} \sigma_{-1}(p^a) = \frac{p}{p-1}$.

An integer $N > 1$ is said to be n/d-perfect if $\sigma_{-1}(N) = n/d$. Reducing if necessary, we will always assume n and d are relatively prime positive integers. In the case $n = 2$ and $d = 1$, we let π denote the special prime factor of N. We also let $\Phi_m(x)$ denote the mth cyclotomic polynomial.

Hereafter, we fix an integer $s \geq 0$ and a finite set of primes $\mathcal{P}$. We will think of the set $\mathcal{P}$ as limiting the exponents of all but s of the prime factors of N, but to make this more formal we need further notations. We set $P = \prod_{p\in\mathcal{P}} p$ and let P_{max} denote the largest prime in $\mathcal{P}$. We let N be an n/d-perfect number with a prime factorization

$$N = \prod_{i=1}^{s} r_i^{a_i} \prod_{j=1}^{t} q_j^{b_j}$$

where for each $1 \leq j \leq t$ there is a prime $p_j \in \mathcal{P}$ with $p_j|(b_j + 1)$. In particular we have $\Phi_{p_j}(q_j)|\sigma(q_j^{b_j})$, for any such prime $p_j \in \mathcal{P}$. For each $p \in \mathcal{P}$ we let $Q_p = \{q \ : \ q^b||N \text{ and } p|(b+1)\}$. Notice that prime factors of N may occur in more than one of the sets Q_p, but (by design) there are at most s prime factors of N which belong to none of them. Without loss of generality we also assume $r_1 < r_2 < \ldots < r_s$. We speak of the primes q_j as the constrained primes, as their exponents are constrained to specific residue classes. The primes r_i are unconstrained, but limited in number.

We will construct a constant, which we call C, that is an upper bound on the smallest prime factor of N. The definition of C is done in three stages; we will assume that C is the smallest constant satisfying the conditions given in each stage. For the first stage we simply assume $C \geq \max\{P_{max}, n\} + 1$. This takes care of the case that one of n, d, or P shares a prime factor in common with N. Hereafter, we will assume that N does not share such a factor. Noting that $\sigma(N)$ is an integer, we derive from the equation $\sigma(N) = Nn/d$ that $d|N$, and hence we may now assume $d = 1$.

The second stage is also straightforward, and yields a large improvement over [29]. From the equation $\sigma_{-1}(N) = n$ we have

$$n = \prod_{i=1}^{s} \sigma_{-1}(r_i^{a_i}) \prod_{j} \sigma_{-1}(q_j^{b_j}) \leq \prod_{i=1}^{s} \frac{r_1 + i - 1}{r_1 + i - 2} \prod_{p\in\mathcal{P}} \prod_{q\in Q_p} \frac{q}{q-1} = \frac{r_1 + s - 1}{r_1 - 1} \prod_{p\in\mathcal{P}} \prod_{q\in Q_p} \frac{q}{q-1}.$$

We wish to turn this inequality into a non-trivial lower-bound on the quantity $\prod_{p\in\mathcal{P}} \prod_{q\in Q_p} \frac{q}{q-1}$. This occurs only when $n > \frac{r_1+s-1}{r_1-1}$. Equivalently we need $r_1 > \frac{s+n-1}{n-1}$. For ease later, we define the following constants: $C_0 = \frac{s+n-1}{n-1}$ and

$$C_0' = \frac{n(r_1 - 1)}{s + r_1 - 1} \leq \prod_{p\in\mathcal{P}} \prod_{q\in Q_p} \frac{q}{q-1}.$$

Our second assumption is $C \geq C_0 + 1$. The observant reader will note that the constant C_0' depends on more than just n, s, and $\mathcal{P}$; namely the prime r_1. However, note that with the assumption $C \geq C_0 + 1$ in place we lose no generality by assuming $r_1 > C_0$. Thus, we also have the inequality $C_0' > 1$, and C_0' is bounded away from 1 by a positive quantity only depending on n, s, and $\mathcal{P}$.

The third stage, which is explained fully in the next two sections, relies on the following idea: We wish to count the number of primes factors of N which are congruent to 1 (mod P). To this end, fix one of the constrained primes $q \in Q_p$. If $q \equiv 1 \pmod P$ then $p|\Phi_p(q)|\sigma(q^b)|\sigma(N) = Nn$. From our first assumption on C, we know $p \nmid N$, hence $p|n$. Thus there are at most $v_p(n)$ primes $q \in Q_p$ with $q \equiv 1 \pmod P$, where v_p denotes the p-adic valuation. In particular, there are at most $C_1 = s + \sum_{p\in\mathcal{P}} v_p(n)$ distinct

prime factors of N which are $\equiv 1 \pmod P$. We will use the large sieve to show that this constraint implies that $\prod_{p\in\mathcal{P}}\prod_{q\in Q_p} q/(q-1)$ converges quickly. If this double product involves only sufficiently large primes then we can guarantee it is smaller than C_0', which yields a contradiction.

3. The Large Sieve

In this section we fix a prime $p_0 \in \mathcal{P}$ and we let p represent an arbitrary prime. To use the large sieve we first need to construct appropriate sieving sets. For $p \equiv 1 \pmod P$ we set

$$\Omega_p = \{0 \pmod p\} \cup \{a \pmod p \ : \ a \not\equiv 1 \pmod p,\ a^{p_0} \equiv 1 \pmod p\},$$

and for all other primes we set $\Omega_p = \{0 \pmod p\}$. We wish to estimate the size of the set $\mathcal{S} = \{n \leq x \ : \ n \notin \Omega_p \text{ for all } p < u\}$, where $x, u \in \mathbb{R}_{>0}$.

To see why we want to bound the size of $\mathcal{S}$ consider the following argument. Assume $q \in Q_{p_0}$, $q \not\equiv 1 \pmod p$, and $q^{p_0} \equiv 1 \pmod p$. From the definition of Q_{p_0}, this implies that $p|N$. If we further assume that $C_1 = 0$ then there are no primes $p|N$ with $p \equiv 1 \pmod P$. Thus, under the assumptions above, $q \notin \Omega_p$ unless $q = p$. In particular we have the upper-bound $|\{q \in Q_{p_0} \ : \ q < x\}| \leq |\mathcal{S}| + u$. Since C_1 is not necessarily 0 a small corrective factor needs to be introduced, which we do at the end of this section.

If $p \equiv 1 \pmod P$ we set $\kappa(p) = p_0$. Otherwise we set $\kappa(p) = 1$. We extend κ multiplicatively, and define it to be zero on non-square-free arguments. Note that $\kappa(p) = |\Omega_p|$. We now describe how to give an effective bound on $|S|$.

Proposition 1 (The Large Sieve [16, Theorem 7.14])**.** *We have*

$$|\mathcal{S}| \leq \frac{x + u^2}{G(u)}$$

where $G(u) = \sum_{n\leq u} \mu^2(n) \prod_{p|n} \left(\frac{\kappa(p)}{p-\kappa(p)}\right)$.

Note that we are in the situation where "small sieves" would apply, and this is the avenue Yamada explored. However, to avoid complications with error-term estimates we prefer the large sieve. In the sieve we take $u = \sqrt{x}$, and thus we need to find a lower-bound for $G(\sqrt{x})$. Set $V(P(w)) = \prod_{p<w}\left(1 - \frac{\kappa(p)}{p}\right)$. One can approximate $G(u)$ from above and below by suitable multiples of $V(P(w))^{-1}$, choosing w appropriately in either case. We are concerned with the lower bound, and thus appeal to the following proposition, which follows from a trick of Rankin.

Proposition 2 ([10, Theorem 1, p. 52])**.** *Suppose there is a constant $B > 1$ so that*

$$\frac{1}{\log(z)} \sum_{p<z} \frac{\kappa(p)\log(p)}{p} < B$$

for some $z \geq 2$. Writing $z = x^{1/s}$, if $s > 2B$ we have

$$G(\sqrt{x}) \geq \frac{1 - e^{-C_2(s)}}{V(P(x^{1/s}))}$$

where $C_2(s) = \frac{s}{2}\log(s/2B) - \frac{s}{2} + B$.

So we first find an upper-bound of the form

$$\frac{1}{\log(z)} \sum_{p<z} \frac{\kappa(p)\log(p)}{p} < B.$$

An effective but simplistic upper-bound is easy to achieve. It is well known (e.g., see [28, Equation 3.24]) that $\frac{1}{\log(z)}\sum_{p<z}\frac{\log(p)}{p} < 1$. Since $\kappa(p) \leq p_0 \leq P$ we can take $B = P$, and then take $s = 2P + 1$.

Next, we need to find an upper bound for $V(P(w))$, where $w = x^{1/s}$. We begin by noting

$$\begin{aligned} V(P(w)) \leq \prod_{p<w}\left(1-\frac{1}{p}\right)^{\kappa(p)} &= \prod_{p<w}\left(1-\frac{1}{p}\right)\prod_{p<w,\, p\equiv 1 \pmod P}\left(1-\frac{1}{p}\right)^{p_0-1} \\ &= \prod_{p<w}\left(1-\frac{1}{p}\right)^{p_0}\prod_{p<w,\, p\not\equiv 1 \pmod P}\left(1-\frac{1}{p}\right)^{-p_0+1}. \end{aligned}$$

Thus, it suffices to find an effective upper-bound for the Mertens' product over the arithmetic progression $1 \pmod P$; or lower-bounds for the products over the other arithmetic progressions modulo P. The bounds given in [1] suffice. (To apply the results of that paper we may need to increase P so we have $P \geq 37$, which can be done by adding primes to $\mathcal{P}$ if necessary.) We arrive at an effective inequality of the form

$$V(P(w)) < \frac{C_3}{\log(w)^{1+\frac{p_0-1}{\varphi(P)}}}$$

where $C_3 > 0$ depends only on P, and φ is the totient function.

At this point we still need to deal with the possibility that there might be prime factors of N which are $1 \pmod P$. Let $\mathcal{T}$ denote the set of all such prime factors, if any. We redefine Ω_p and κ, so that $\Omega_p = \{0 \pmod P\}$ and $\kappa(p) = 1$ whenever $p \in \mathcal{T}$. Notice that we now truly do have the bound

$$|\{q \in Q_{p_0} \ : \ q < x\}| \leq |\mathcal{S}| + u.$$

One of the effects of this change in the definition of Ω_p and κ is that the needed constant B is no larger, so we use the same constant $B = P$ as in our previous computation. The upper-bound on $V(P(w))$ is changed by a constant which depends only on the set $\mathcal{T}$. Further, the worst case is when $\mathcal{T}$ consists of the first C_1 primes which are $1 \pmod P$. Letting $\mathcal{T}'$ consist of the first C_1 primes which are $1 \pmod P$, we have

$$V(P(w)) < \frac{C_3'}{\log(w)^{1+\frac{p_0}{\varphi(P)}}}$$

where

$$C_3' = C_3 \prod_{p\in\mathcal{T}'} \frac{1-\frac{1}{p}}{1-\frac{p_0}{p}} > 0$$

is still effectively computable. Note that at the cost of weaker bounds, we then have

$$|\{q \in Q_{p_0} \ : \ q < x\}| < C_4 x/\log(x)^{1+1/\varphi(P)}$$

where $C_4 > 0$ depends on $\mathcal{P}$, but is independent of p_0.

4. Partial summation bound

We are now ready to estimate $\prod_{q\in Q_{p_0}} \frac{q}{q-1}$. Suppose that all the prime factors of N are greater than some integer $y > 1$. By partial summation we have

$$\begin{aligned} \log\left(\prod_{q\in Q_{p_0}} \frac{q}{q-1}\right) &= -\sum_{q\in Q_{p_0}} \log(1-1/q) = \sum_{q\in Q_{p_0}}\left(\frac{1}{q}+\frac{1}{2q^2}+\frac{1}{3q^3}+\cdots\right) \\ &< \int_y^\infty \frac{C_4}{x\log(x)^{1+1/\varphi(P)}}\,dx + \sum_{q>y}\frac{1}{q(q-1)} < \frac{\varphi(P)C_4}{\log(y)^{1/\varphi(P)}} + \sum_{n>y}\frac{1}{n(n-1)} = \frac{\varphi(P)C_4}{\log(y)^{1/\varphi(P)}} + \frac{1}{y}. \end{aligned}$$

We can incorporate $1/y$ into the main term (as the main term dominates), by changing C_4 to a new constant $C_5 > 0$. Thus

$$0 < \log(C_0') \leq \log\left(\prod_{p_0 \in \mathcal{P}} \prod_{q \in Q_{p_0}} \frac{q}{q-1}\right) < \frac{|\mathcal{P}|\varphi(P)C_5}{\log(y)^{1/\varphi(P)}}.$$

Solving for y we have

$$y < \exp\left(\left(\frac{|\mathcal{P}|\varphi(P)C_5}{\log(C_0')}\right)^{\varphi(P)}\right)$$

and our third assumption on C is that it is larger than the quantity on the right.

We have thus proven:

Theorem 3. *Let $N > 1$ be an odd integer with $\sigma_{-1}(N) = n/d$. Let $s \geq 1$ be a fixed integer, and let $\mathcal{P}$ be a finite set of primes. Suppose the prime factorization of N has the form $N = \prod_{i=1}^{s} r_i^{a_i} \prod_{j=1}^{t} q_j^{b_j}$, where for each j there is a prime $p_j \in \mathcal{P}$ with $p_j | (b_j + 1)$. There exists a computable constant C, depending only on $\mathcal{P}$, n, and s, which gives an upper bound on the smallest prime factor of N.*

5. Our special case, dealing with small primes $p \neq 5$

In the remainder of the paper we specialize to the case when $\sigma_{-1}(N) = 2$, $\mathcal{P} = \{3, 5\}$, and $s = 1$. We further assume that the one prime factor of N not necessarily limited by $\mathcal{P}$ is the special prime, π. Note that while the exponent of π is not limited by $\mathcal{P}$ we do know that the exponent is $\equiv 1 \pmod 4$. In particular, $\Phi_2(\pi) = \pi + 1$ divides $2N$. To abbreviate these assumptions, we refer to an integer N satisfying the above conditions by the acronym R-OPN (a restricted odd perfect number).

Let N be an R-OPN. Suppose for a moment that Q_5 is empty, and so all of the exponents of the non-special primes are $\equiv 2 \pmod 3$. This implies that $\Phi_3(q)|N$ for all $q|N$, $q \neq \pi$. We wish to show that $3 \nmid N$ when N is an R-OPN, working by contradiction. The next paragraph demonstrates how this is accomplished in one special case.

If $3|N$ then since $\pi \neq 3$ we have $13 = \Phi_3(3)|N$. In other words, starting with the prime 3 we are forced to have 13 as another prime divisor of N. We say that 3 contributes the prime factor 13. Since $13 \equiv 1 \pmod 4$ it might be the special prime, and so we will first consider that case. With $\pi = 13$ we have $2 \cdot 7 = \pi + 1 = \Phi_2(\pi)|2N$. This implies that $7 \in Q_3$ (since 13 is already the special prime, and we are assuming $Q_5 = \emptyset$) and so 7 contributes the factors $3 \cdot 19 = \Phi_3(7)$ to N. Repeating this argument we have that 19 contributes $3 \cdot 127 = \Phi_3(19)$, which in turn contributes the factors $3 \cdot 5419 = \Phi_3(127)|N$, and finally we have $3 \cdot 31 \cdot 313 \cdot 1009 = \Phi_3(5419)$. We could continue this process of obtaining more and more factors of N, but pausing for a moment to collect our data we currently have $M = 3^4 \cdot 7^2 \cdot 13^1 \cdot 19^2 \cdot 31^2 \cdot 127^2 \cdot 313^2 \cdot 1009^2 \cdot 5419^2 | N$. On the other hand $\sigma_{-1}(M) > 2$. Adding more prime factors to M, or increasing the exponents on the factors we already have, only increases the size of $\sigma_{-1}(M)$, which contradicts $\sigma_{-1}(N) = 2$.

One calls an integer $M > 1$ for which $\sigma_{-1}(M) > 2$ an *abundant* number. If $M|N$ and M is abundant, then N is abundant. Using this fact we were able to show that the situation in the previous paragraph led to a contradiction. However, there are more cases to consider; such as when 13 is not the special prime. The print-out below covers all cases. Some explanation of the notation in the print-out may be necessary. Every line is of the form "$p^a \Rightarrow \Phi_{a+1}(p)$", where $\Phi_{a+1}(p)$ is completely factored to tell us which new primes are contributed. The exponent a is determined by the case under consideration (we have $a = 1$ if p is the special prime, and $a = 2$ otherwise). Also note that the letter `A` tells us that the case under consideration gives us an abundant number, which allows us to back-track to the next case. The indentation tells us how far into a chain of factors we are, and also aids us in finding the next case to consider.

```
3^2 => 13^1
 13^1 => 2^1 7^1
  7^2 => 3^1 19^1
   19^2 => 3^1 127^1
    127^2 => 3^1 5419^1
     5419^2 => 3^1 31^1 313^1 1009^1  A
 13^2 => 3^1 61^1
  61^1 => 2^1 31^1
   31^2 => 3^1 331^1
    331^2 => 3^1 7^1 5233^1
     7^2 => 3^1 19^1  A
  61^2 => 3^1 13^1 97^1
   97^1 => 3^1 7^2
    7^2 => 3^1 19^1  A
   97^2 => 3^1 3169^1
    3169^1 => 2^1 5^1 317^1  A
    3169^2 => 3^1 3348577^1
     3348577^1 => 2^1 1674289^1
      1674289^2 => 3^1 934415109937^1
       934415109937^2 => 3^1 61^1 79^1 127^1 226903^1 2095837299571^1
        79^2 => 3^1 7^2 43^1  A
     3348577^2 => 3^1 3737657091169^1
      3737657091169^1 => 2^1 5^1 443^1 843714919^1  A
      3737657091169^2 => 3^1 181^1 26042690887^1 987900542491^1
       181^1 => 2^1 7^1 13^1
        7^2 => 3^1 19^1  A
       181^2 => 3^1 79^1 139^1
        79^2 => 3^1 7^2 43^1  A
```

Note that the line `331^2 => 3^1 7^1 5233^1` contributes the two primes 7 and 5233, either of which we could use to start the next line. We chose 7 simply because it was the smallest prime we hadn't used, and we make similar choices throughout the chain. This algorithm above is often referred to as "creating a factor chain." Each possible chain of factors leads to a contradiction, i.e. the number must always become abundant. We thus can conclude $3 \nmid N$.

At this point we might ask what happens if we disregard our assumption that Q_5 is empty. This allows one more branch for each prime factor since we do not know whether we have $\Phi_3(q)|N$ or $\Phi_5(q)|N$. For example, instead of just dealing with 13 and 13^2, we also have chains involving 13^4. Surprisingly, the algorithm still successfully finishes in just under 1000 steps. In particular if N is an R-OPN then $3 \nmid N$.

We can try to repeat this process for the prime 5, but the number of factors needed to achieve abundance increases dramatically. Even if we assume Q_3 is empty the number of cases to check is large; the chain must go to depths of about $80,000$ prime factors which is entirely unfeasible. We describe how to overcome these difficulties in the next section. On the other hand, for any prime larger than 5 we quickly get a smaller prime (at least in practice) and thus reduce to a previous case. For example, the following print-out shows that if $7|N$ then a smaller odd prime must divide N. When a smaller odd prime appears, we write `S`:

```
7^2 => 3^1 19^1  S
7^4 => 2801^1
```

```
2801^1 => 2^1 3^1 467^1  S
2801^2 => 37^1 43^1 4933^1
 37^1 => 2^1 19^1
  19^2 => 3^1 127^1  S
  19^4 => 151^1 911^1
   151^2 => 3^1 7^1 1093^1  S
   151^4 => 5^1 104670301^1  S
 37^2 => 3^1 7^1 67^1  S
 37^4 => 11^1 41^1 4271^1
  11^2 => 7^1 19^1
   19^2 => 3^1 127^1  S
   19^4 => 151^1 911^1
    151^2 => 3^1 7^1 1093^1  S
    151^4 => 5^1 104670301^1  S
  11^4 => 5^1 3221^1  S
2801^4 => 5^1 1956611^1 6294091^1  S
```

Notice that we do not even need to involve abundance computations. Continuing in this manner, one can prove that if $7 \leq q < 10^8$ is prime and $q|N$ then a smaller odd prime divides N. Full print-outs (of these results, and the results of the next section) are available on the second author's website. With more effort, the upper bound on q can be improved slightly; but a large improvement is not possible without improved factorization techniques (which is, of course, a very difficult problem!). Eventually the numbers we consider are too large to factor in a reasonable amount of time. It is interesting to note that $\Phi_3(\Phi_5(x)) = (x^2 - x + 1)(x^6 + 3x^5 + 5x^4 + 6x^3 + 7x^2 + 6x + 3)$, which helps factor some of the larger integers.

Computations suggest that any factor chain as above starting with a large prime quickly yields a smaller prime. It is an open question whether one can create a chain (using odd primes) which eventually yields no new prime factors.

6. Dealing with the prime 5

We still need to deal with the case $q = 5$, and then we can conclude that an R-OPN N has no prime factors $< 10^8$. It should be noted that even if we assume that N is not divisible by a sixth power the computation is difficult because chains do not necessarily repeat prime factors. So we will continue to work with the assumption that N is an R-OPN, even though our main interest lies in that special case. Keep in mind that we do know that $3 \nmid N$ from our work in the previous section.

The main difficulty in producing factor chains starting with the prime 5 is that abundance is more difficult to achieve. When we started with the prime 3 it fortunately contributed more to the abundance computations which helped in ending the chains quickly. To reach abundance when starting with 5 some cases will require hundreds of thousands of different primes. It should also be mentioned that moving from the case $\mathcal{P} = \{3\}$ to the case $\mathcal{P} = \{3, 5\}$ increases the number of chains exponentially.

The key insight is to notice that certain primes occur quickly on any chain one considers. For example, we can always eventually get the prime 11 because of the following factor chain:

```
5^1 => 3 S
5^2 => 31
 31^2 => 3 S
 31^4 => 11 X
5^4 => 11 X
```

Note: To save room we do not print the full factorization of $\Phi_a(p)$ to the right of the arrow, which in some cases is difficult (or even impossible) to provide since full factorization is time intensive. Instead, we merely print the single prime factor we will use to continue the factor chain. So, for example, the last line tells us that $11|\Phi_5(5)$ and that is all we need to know. Since we know 3 cannot divide N, hereafter we will drop all lines involving a 3.

Now that we know that the prime 11 must occur on any chain we can use it to get other primes. A factor chain starting with the prime 11 which is seventy-two lines long tells us that 31 must occur in any chain. The prime 31 quickly gives us the prime 17351, and in turn we get 41 using the chain:

```
17351^2 => 21787
 21787^4 => 41 X
17351^4 => 1648012040336791
 1648012040336791^4 => 42751
  42751^4 => 2223796173481
   2223796173481^1 => 95239
    95239^4 => 41 X
   2223796173481^4 => 8431
    8431^4 => 631
     631^4 => 41 X
```

Note that we do not use the contradiction `S` when we have a prime smaller than 17351 (except on the dropped lines involving 3) since this chain is really a subchain of a factor chain starting with 5.

In general, the only primes that we can capture in this way are all $\equiv 1 \pmod 5$. This has to do with the fact that if $q|\Phi_5(p)$ then $q = 5$ or $q \equiv 1 \pmod 5$. This makes abundance computations more difficult (since we don't always have small primes like 7 and 13). However, it is possible to build a large list of primes that must occur in all chains. The ultimate goal is to collect enough primes, that each must occur in the chain, so that we reach abundance

Since we can ignore lines giving us 3, and since $3|\Phi_3(q)$ whenever $q \equiv 1 \pmod 3$, one strategy that is highly effective is to branch along primes which are $1 \pmod 3$ as much as possible. This allows us to avoid having to start lines with q^2. For example, once we have the prime 104670301 we get the prime 191 using the following factor chain:

```
104670301^1 => 109
 109^4 => 191 X
104670301^4 => 17440542156505477796383741
 17440542156505477796383741^1 => 9446713932985381771
  9446713932985381771^4 => 14397718401641737347815951307647326 21
   1439771840164173734781595130764732621^4 => 2179291
    2179291^4 => 43411
     43411^4 => 6571
      6571^4 => 152211901
       152211901^4 => 4025521
        4025521^4 => 2341
         2341^4 => 191 X
 17440542156505477796383741^4 => 28225...251 (big prime)
  28225...251^4 => 191 X
```

This strategy is extremely useful in the "endgame" when there are a lot of primes with which we can start chains.

The possibility that a prime can be the special prime has a quadratic effect on the number of lines in a given factor chain. But the special prime is special precisely because there is only one prime factor of N with its properties. We can exploit this fact by creating two separate factor chains which involve distinct sets of possible special primes. If the special prime occurs in the first chain then it cannot appear in the second chain and vice versa. Thus, we can write both chains without any contributions from a special prime, and at least one of them gives us the prime we need (since the special prime occurs in at most one of the two chains). To give an explicit example, suppose we want to get the prime 1051 and we already have the primes 241 and 104670301. Consider the two chains:

```
241^4 => 11106421
 11106421^4 => 3263466811411
  3263466811411^4 => 8821
   8821^4 => 1051 X
```

and

```
104670301^4 => 1744054215650547779638374 1
 1744054215650547779638374 1^4 => 1274458273681
  1274458273681^4 => 111805142252581
   111805142252581^4 => 49080432523124928361
    49080432523124928361^4 => 662698623515864248531
     662698623515864248531^4 => 43801
      43801^4 => 1051 X
```

The only possible special primes which can appear in the first chain are 241, 8821 or 11106421, since they are the only primes $p \equiv 1 \pmod 4$. Thus, if the special prime is not one of these three primes then the first chain shows us that we obtain 1051. On the other hand, if one of those three primes is the special prime, then the first chain is insufficient but the second chain suffices since it does not contain any of those possible special primes. This process reduces the quadratic effect of the special prime to a doubling effect.

There is one final simplifying technique that we put into use. Although it takes many primes to reach abundance, if a chain does ever involve the prime 7 (or other combinations of small primes) it is much easier to reach abundance. Thus, once we know that all chains eventually contain each of the first hundred primes congruent to 1 (mod 5), we also know that if 7 occurs in a chain we have abundance. Hence, after that point we may ignore any line containing 7. Similarly, with enough primes we can start ignoring lines with 13, 17, and so forth. Even before we know we can ignore 7, we know that we can ignore chains which give us both 7 and 13, or both 7 and 19. We put this into use when trying to obtain the prime 2531. We found two chains (without duplicated possible special primes) that either give us 2531 or 7. We then found two new chains (again without duplicated possible special primes) that either give us 2531, 13, or 19. Since 7 and 13 cannot happen together, and neither can 7 and 19, we see that we must get 2531.

After six months of running programs we wrote for Mathematica, we gathered enough chains and primes to achieve abundance. This proves that R-OPNs are not divisible by 5.

7. Minimizing the upper bound

Let N be an R-OPN, and let C be the upper bound we constructed for the smallest prime factor of N. Recall that our first assumption on C was made to prevent N from having prime factors in common with n, d, or $\mathcal{P}$. Due to the calculations above, we lose nothing in assuming $C > 10^8$, and we achieve the same goal. It turns out this simple fact will greatly reduce a number of constants

we deal with. For example, in the case we are considering we have $C_0 = 2$. Merely assuming that $C \geq \max\{P_{max}, n, C_0\} + 1 = 6$ we have $p_1 \geq 6$ and so $C_0' \geq 5/3$. But as we have proven that $C \geq 10^8$ we also have $C_0' \geq 99999999/50000000 = 1.99999998$ which is a much better (indeed, nearly optimal) constant.

In all computations hereafter we will implicitly use the fact that N has no prime factors less than 10^8, and we will take $C_0' = 2(1 - 10^{-8})$.

8. Working with Q_3

8.1. **Defining Ω_p and κ.** Our next job is to estimate $\prod_{q \in Q_3} \frac{q}{q-1}$. As before, we will use the large sieve. However, it turns out that we are able to improve our sieving sets Ω_p if we take into account the computations of the previous section. Notice that with $n = 2$ we have $v_3(n) = v_5(n) = 0$, and hence $C_1 = 1$. Thus, the only possible prime factor of N congruent to 1 (mod 15) is π, and we know $\pi > 10^8$.

Let's recall some standard notation. For a prime p, and an integer $a \in \mathbb{Z}$ with $p \nmid a$, we let $o_p(a)$ denote the order of a modulo p. In other words, this is the smallest positive integer for which $a^{o_p(a)} \equiv 1 \pmod{p}$. If r is a prime number and $r|(p-1)$ then there are exactly $r-1$ congruence classes mod p which have order r.

The abundance computations from the previous section were only useful for small primes. The reasons are two-fold. Primarily, when working with large primes we can end a chain much sooner by finding a smaller prime (which we will call the *smaller prime contradiction*). Secondarily, since we are assuming the prime factors of N are larger than 10^8 we would need an enormous number of primes to reach abundance. Because of these two reasons we wish to encode into the definition of the sets Ω_p as much information as we can glean from the smaller prime contradiction, and we can ignore abundance computations.

Fix $q \in Q_3$. We then know $\Phi_3(q)|N$. The simplest possible chain would occur if $\Phi_3(q)$ is divisible by a prime $p < 10^8$. Such a prime is necessarily congruent to 1 (mod 3), or equal to 3, due to congruence restrictions on the possible prime factors of $\Phi_3(x)$; see [24, Lemma 1]. Thus, for all primes $p < 10^8$ with $p \equiv 1 \pmod{3}$, we can sieve the two classes $a \pmod{p}$ with $o_p(a) = 3$; and we can also sieve by the class 1 (mod 3). This is a finite number of congruence classes, and we will see that it only changes the bound given in the large sieve by a constant factor.

The next simplest chain occurs as follows: Suppose we have a prime $p|\Phi_3(q)$. We may as well assume $q \equiv 2 \pmod{3}$ and so $p \equiv 1 \pmod{3}$. We then have $3|\Phi_3(p)$, and hence $p \notin Q_3$. Therefore, either $p \in Q_5$ or $p = \pi$, and we can deal with the latter contingency as we did in previous sections. In the case $p \in Q_5$ the chain stops as long as $\Phi_5(p)$ is divisible by a prime $r < 10^8$. By congruence conditions, we either have $r = 5$ or $r \equiv 1 \pmod{5}$. In that case either $p \equiv 1 \pmod{15}$, or $o_r(p) = 5$ and p belongs to one of the four congruence classes modulo r of order 5. Unlike the previous case, this gives us an infinite set of congruence classes to to sieve away. If one has good estimates for Mertens' products over arithmetic progressions, it would be valuable to not only sieve by primes in the class 1 (mod 15) but in classes such as $1 \pmod{3} \cap a \pmod{11}$ where $a = 3, 4, 5, 9$. The best explicit bounds we are aware of, that work for moduli that get increasingly large, are found in [1]. Unfortunately, even in the case of reasonable sized moduli, such as $k = 165$, the bounds are too large to be of value in the computations we will perform.

We could continue this process, defining Ω_p to deal with chains of deeper depths. Complications arise in this situation. For example, we still have to deal with the possibility that one of the factors in the chain is the special prime. In practice we will restrict ourselves to chains of depth one, or of depth two if they yield 3 or 5. Further note that since the the upper bound we reach for the smallest prime is 10^{1000} we may as well replace all instances of 10^8 by the upper bound we wish to obtain. Due to technical conditions, we will work up to 10^{100}.

We are now ready to define Ω_p. Let $\mathcal{U}$ be the set of primes $p \equiv 1 \pmod 3$ which are either $< 10^{100}$ or $\equiv 1 \pmod{15}$. As just mentioned, depending on what tools are available, another choice of $\mathcal{U}$ may be appropriate. For an arbitrary prime p, set

$$\Omega_p = \{0 \pmod p\} \cup \begin{cases} \{1 \pmod 3\} & \text{if } p = 3, \\ \{a \pmod p \ : \ o_p(a) = 3\} & \text{if } p \in \mathcal{U}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Define

$$\kappa(p) = \begin{cases} 2 & \text{if } p = 3, \\ 3 & \text{if } p \in \mathcal{U}, \\ 1 & \text{otherwise.} \end{cases}$$

We have $\kappa(p) = |\Omega_p|$ just as before.

8.2. **Finding bounds on $|\theta(x; 15, 1) - x/8|$.** In the following we let $\theta(t; k, \ell) = \sum_{p \le t,\ p \equiv \ell \pmod k} \log(p)$. We first need to find explicit bounds on the error term for $\theta(x; 15, 1) - x/8$. We begin with the following result found in [27], for values of x that are small:

Lemma 4. *For $1 \le x \le 10^{10}$ we have $|\theta(x; 15, 1) - x/8| \le 1.097307\sqrt{x}$.*

For slight larger x we follow [27, Section 4]. In the notation of that paper, taking $R = 6.41$ and $C_1(\chi) = 1$ due to work in [17] (which is also found in [19], and slightly improved in the preprint [18]), we get bounds of the form

$$|\theta(x; 15, 1) - x/8| < \epsilon x \text{ for all } x > x_0,$$

where ϵ depends on x_0. We note that, without verifying the Riemann Hypothesis to greater heights for Dirichlet L-function modulo 15, these bounds are only valid when $x_0 \le 10^{470}$. Here is a table of some of these values:

x_0	ϵ
10^{10}	0.00903538
10^{11}	0.00811594
10^{12}	0.00780339
10^{20}	0.00741842
10^{50}	0.00646773
10^{100}	0.00496541
10^{200}	0.00241077
10^{400}	0.00030846

For large values of x, we turn to the methods employed in [8]. In the notation of that paper, we take $R = 9.645908801$, $H = 2500$, and $C_1(\chi) = 9.14$. In [8, Theorem 5], we have $X_4 = 10$ and so the bounds we obtain are accurate only when $\sqrt{\log(x)/R} \ge X_4$, or in other words $x \ge 8.251 \cdot 10^{418}$. Putting it all together, we have

$$|\theta(x; 15, 1) - x/8| < 0.77101532 \frac{x \log(x)^{1/4}}{e^{0.321979\sqrt{\log(x)}}}$$

when $x > 10^{420}$.

8.3. **Bounding a Mertens type product.** We seek an effective bound on $\sum_{p \le x,\, p \equiv 1 \,(\mathrm{mod}\, 15)} \frac{1}{p}$. By methods employed in [28], specifically in the derivation of equation (4.15) using partial summation (see also [4, Lemma 1.3.1]), we have

$$\sum_{\substack{p \le x \\ p \equiv 1 \,(\mathrm{mod}\, 15)}} \frac{1}{p} = \frac{1}{8} \log(\log(x)) + M(15, 1) + \frac{\theta(x; 15, 1) - x/8}{x \log(x)} - \int_x^\infty (\theta(t; 15, 1) - t/8) \frac{1 + \log(t)}{t^2 \log(t)^2}\, dt$$

where $M(15,1) = -0.1506\ldots$ is an explicit constant (see [21]). The work in the previous section yields $|\theta(x;15,1) - x/8| < 1.32812x/\log(x)$ for all $x \geq 2$. Plugging this into the formula above, we then have

$$\left| \sum_{\substack{p \leq x \\ p \equiv 1 \pmod{15}}} \frac{1}{p} - \frac{1}{8}\log(\log(x)) - M(15,1) \right| < \frac{2}{\log(x)^2} + \frac{1.32812}{\log(x)}.$$

As it will become useful shortly, we develop a series of inequalities. We have

$$\frac{2}{\log(x)^2} + \frac{1.32812}{\log(x)} < \frac{1.48}{\log(x)} \text{ when } x \geq 10^6.$$

We also have

$$\exp(-1.48/\log(x)) > 1 - \frac{1.48}{\log(x)}$$

for $x \geq 2$, by a simple application of Taylor series remainders. Similarly,

$$\exp(1.48/\log(x)) < 1 + \frac{1.65}{\log(x)}$$

for $x \geq 10^6$. Next, we compute that

$$\sum_{\substack{p>x,\, n\geq 2 \\ p\equiv 1 \pmod{15}}} \frac{1}{np^n} < \frac{1}{2} \sum_{\substack{p>x,\, n\geq 2 \\ p\equiv 1 \pmod{15}}} \frac{1}{p^n} = \frac{1}{2} \sum_{\substack{p>x \\ p\equiv 1 \pmod{15}}} \frac{1}{p(p-1)} < \frac{1}{2}\sum_{n>x} \frac{1}{n(n-1)} < \frac{1}{2(x-1)}$$

and so

$$\exp\left(\sum_{\substack{p>x,\, n\geq 2 \\ p\equiv 1 \pmod{15}}} \frac{1}{np^n} \right) < \exp(1/2(x-1)) < 1 + \frac{1.1}{x} = \alpha(x)$$

for $x \geq 10^6$. Finally,

$$\alpha(x)\left(1 + \frac{1.65}{\log(x)}\right) < 1 + \frac{1.66}{\log(x)}$$

when $x \geq 10^6$.

Taking the logarithm of a product, expanding the Taylor series, and taking exponentials we obtain

$$\prod_{\substack{p\leq x \\ p\equiv 1 \pmod{15}}} \left(1 - \frac{1}{p}\right) = \exp\left[- \sum_{\substack{p\leq x \\ p\equiv 1 \pmod{15}}} \frac{1}{p} - \sum_{\substack{n\geq 2 \\ p\equiv 1 \pmod{15}}} \frac{1}{np^n} + \sum_{\substack{p>x,\, n\geq 2 \\ p\equiv 1 \pmod{15}}} \frac{1}{np^n} \right].$$

Using the bounds given above, we then have, for $x \geq 10^6$,

$$\frac{C(15,1)}{\log(x)^{1/8}}\left(1 - \frac{1.48}{\log(x)}\right) < \prod_{\substack{p\leq x \\ p\equiv 1 \pmod{15}}} \left(1 - \frac{1}{p}\right) < \frac{C(15,1)}{\log(x)^{1/8}}\left(1 + \frac{1.66}{\log(x)}\right)$$

where $C(15,1) = 1.1617\ldots$ according to [22].

8.4. **Bounding another Mertens type product.** Again, using the methods employed in [28], we have

$$\sum_{\substack{p\le x\\ p\equiv 1 \pmod 3}} \frac{1}{p} = \frac{1}{2}\log(\log(x)) + M(3,1) + \frac{\theta(x;3,1)-x/2}{x\log(x)} - \int_x^\infty (\theta(t;3,1)-t/2)\frac{1+\log(t)}{t^2\log(t)^2}\,dt$$

where $M(3,1) = -0.35689\ldots$ is an explicit constant (again see [21]). By [8], we have the bound $|\theta(t;3,1)-t/2| < 0.262\frac{t}{\log(t)}$ for $t \ge 1531$. Repeat the process in the last subsection to obtain

$$\left|\sum_{\substack{p\le x\\ p\equiv 1 \pmod 3}} \frac{1}{p} - \frac{1}{2}\log(\log(x)) - M(3,1)\right| < \frac{0.393}{\log(x)^2} + \frac{0.262}{\log(x)}.$$

For $x \ge 10^6$ we have

$$\frac{0.393}{\log(x)^2} + \frac{0.262}{\log(x)} < \frac{0.3}{\log(x)}.$$

Taylor series remainders prove that

$$\exp(-0.3/\log(x)) > 1 - \frac{0.3}{\log(x)}$$

and we also have

$$\alpha(x)\exp(0.3/\log(x)) < 1 + \frac{0.31}{\log(x)}$$

Thus, for $x \ge 10^6$ we have

$$\frac{C(3,1)}{\log(x)^{1/2}}\left(1-\frac{0.3}{\log(x)}\right) < \prod_{\substack{p\le x\\ p\equiv 1 \pmod 3}}\left(1-\frac{1}{p}\right) < \frac{C(3,1)}{\log(x)^{1/2}}\left(1+\frac{0.31}{\log(x)}\right)$$

where $C(3,1) = 1.403477\ldots$ according to [22].

8.5. **Estimating B.** Now, consider the quantity

$$\frac{1}{\log(z)}\sum_{p<z}\frac{\kappa(p)\log(p)}{p} \le \frac{1}{\log(z)}\left(\sum_{p<z}\frac{\log(p)}{p} + \frac{\log(3)}{3} + \sum_{\substack{p<z\\ p\equiv 1 \pmod 3}}\frac{2\log(p)}{p}\right).$$

A simple computation shows that the quantity on the right is smaller than 2 for $z \le 10^6$. In the calculations below, we implicitly assume $z > 10^6$ unless otherwise stated.

We first estimate the piece involving the arithmetic progression. Recall that by [8], we have the bound $|\theta(t;3,1)-t/2| < 0.262\frac{t}{\log(t)}$ for $t \ge 1531$. By partial summation, we compute for integer values of z that

$$\begin{aligned}\sum_{p\le z,\ p\equiv 1 \pmod 3}\frac{\log(p)}{p} &= \theta(z;3,1)\frac{1}{z} + \int_{t=7}^{z}\theta(t;3,1)\frac{1}{t^2}\,dt\\ &< \frac{1}{2} + 0.262\frac{1}{\log(z)} + \frac{1}{2}\log(z) + 0.262\log(\log(z)) - 1.14\end{aligned}$$

which is increasing in z. In particular,

$$\frac{1}{\log(z)}\sum_{p\le z,\ p\equiv 1 \pmod 3}\frac{2\log(p)}{p} \le 1 + 0.524\frac{\log(\log(z))}{\log(z)} - \frac{1.2}{\log(z)}$$

for all real $z > 10^6$. The function $0.524\frac{\log(\log(z))}{\log(z)} - \frac{1.2}{\log(z)}$ is bounded above by 0.02.

By Equation (3.22) in [28], we have

$$\sum_{p \le z} \frac{\log(p)}{p} < \log(z) + E + \frac{1}{2\log(z)}$$

for $z \ge 319$, where $E = -1.33258\ldots$ is a constant. Putting all of this together, we have

$$\frac{1}{\log(z)} \sum_{p<z} \frac{\kappa(p)\log(p)}{p} < 2.02$$

for all $z \ge 2$. Hence, we take $B = 2.02$. Notice that this choice of B continues to work with any of the other choices for $\mathcal{U}$ discussed above. Asymptotically, $B = 2$ would be the optimal constant when $\mathcal{U}$ consists of all primes $p \equiv 1 \pmod 3$.

8.6. **Estimating $V(P(w))$.** Another result of Dusart [7] tells us

$$\frac{e^{-\gamma}}{\log(w)}\left(1 - \frac{0.2}{\log(w)^2}\right) < \prod_{p<w}\left(1 - \frac{1}{p}\right) < \frac{e^{-\gamma}}{\log(w)}\left(1 + \frac{0.2}{\log(w)^2}\right), \qquad \text{for } w > 2973.$$

Rewriting $V(P(w))$, for $w \ge 10^{100}$ we have

$$\begin{aligned} V(P(w)) &= \frac{1}{2}\prod_{p\le w}\left(1-\frac{1}{p}\right) \prod_{\substack{p\le 10^{100}\\ p\equiv 1 \pmod 3}}\left(1-\frac{1}{p}\right)^2 \prod_{\substack{p\le w\\ p\equiv 1 \pmod{15}}}\left(1-\frac{1}{p}\right)^2 \prod_{\substack{p\le 10^{100}\\ p\equiv 1 \pmod{15}}}\left(1-\frac{1}{p}\right)^{-2} \\ &\quad * \prod_{\substack{p\le 10^{100}\\ p\equiv 1 \pmod 3}} \frac{\left(1-\frac{3}{p}\right)}{\left(1-\frac{1}{p}\right)^3} \prod_{\substack{p\le w\\ p\equiv 1 \pmod{15}}} \frac{\left(1-\frac{3}{p}\right)}{\left(1-\frac{1}{p}\right)^3} \prod_{\substack{p\le 10^{100}\\ p\equiv 1 \pmod{15}}} \frac{\left(1-\frac{1}{p}\right)^3}{\left(1-\frac{3}{p}\right)}. \end{aligned}$$

The first two products in the second line are both bounded above by 1. The third product on the second line can be bounded by

$$\begin{aligned} \prod_{\substack{p\le 10^{100}\\ p\equiv 1 \pmod{15}}} \frac{\left(1-\frac{1}{p}\right)^3}{\left(1-\frac{3}{p}\right)} &= \prod_{\substack{p\le 10^{6}\\ p\equiv 1 \pmod{15}}} \frac{\left(1-\frac{1}{p}\right)^3}{\left(1-\frac{3}{p}\right)} \prod_{\substack{10^6<p\le 10^{100}\\ p\equiv 1 \pmod{15}}} \frac{\left(1-\frac{1}{p}\right)^3}{\left(1-\frac{3}{p}\right)} \\ &< \prod_{\substack{p\le 10^{6}\\ p\equiv 1 \pmod{15}}} \frac{\left(1-\frac{1}{p}\right)^3}{\left(1-\frac{3}{p}\right)} \prod_{10^6<n\le 10^{100}} \frac{\left(1-\frac{1}{n}\right)^3}{\left(1-\frac{3}{n}\right)} < 1.00482. \end{aligned}$$

The other products were bounded previously, so we have for $w \ge 10^{100}$ that

$$\begin{aligned} V(P(w)) &< \frac{1}{2}(1.00482)\frac{e^{-\gamma}}{\log(w)}\left(1+\frac{0.2}{\log(10^{100})^2}\right)\frac{C(3,1)^2}{\log(10^{100})}\left(1+\frac{0.31}{\log(10^{100})}\right)^2 \\ &\quad * \frac{C(15,1)^2}{\log(w)^{1/4}}\left(1+\frac{1.66}{\log(10^{100})}\right)^2 \frac{\log(10^{100})^{1/4}}{C(15,1)^2}\left(1-\frac{1.48}{\log(10^{100})}\right)^{-2} \\ &< \frac{0.00969}{\log(w)^{5/4}}. \end{aligned}$$

Recall that we must deal with the possibility of a single element $\pi \in \mathcal{T}$; which is accomplished by multiplying our answer above by $\left(1-\frac{1}{10^8}\right)/\left(1-\frac{3}{10^8}\right)$, as $\pi > 10^8$. This quantity changes our constants so little that with some forethought it can be accounted for in previous computations.

8.7. **Picking a value for $s > 2B$.** We set $w = x^{1/s}$, and assume $w > 10^{100}$. By Proposition 2, and the work in the previous section, we have

$$\frac{1}{G(\sqrt{x})} \leq \left(\left(1 - e^{-C_2(s)}\right)^{-1} V(P(x^{1/s}))\right) < 0.00969 \left(1 - e^{-C_2(s)}\right)^{-1} \log(x)^{-5/4} s^{5/4}$$

with $C_2(s) = \frac{s}{2}\log(s/2B) - \frac{s}{2} + B$ and $B = 2.02$. This quantity is minimized when s is a little smaller than 10. We take $s = 10$ and obtain

$$\frac{1}{G(\sqrt{x})} \leq \frac{0.219}{\log(x)^{5/4}}.$$

Notice that since $w > 10^{100}$ we are also assuming $x > 10^{1000}$.

8.8. **Measuring the size of Q_3.** In the large sieve, taking $u = \sqrt{x}$ where $x > 10^{1000}$, we have

$$|\{q \in Q_3 \ : \ q < x\}| \leq \frac{2x}{G(\sqrt{x})} + \sqrt{x} < \frac{0.438x}{\log(x)^{5/4}}.$$

Repeating the computation done in Section 4, for $y \geq 10^{1000}$,

$$\begin{aligned}\log\left(\prod_{q\in Q_3} \frac{q}{q-1}\right) &= -\sum_{q\in Q_3} \log(1-1/q) = \sum_{q\in Q_3}\left(\frac{1}{q} + \frac{1}{2q^2} + \frac{1}{3q^3} + \cdots\right) \\ &< \int_y^\infty \frac{0.438}{x\log(x)^{5/4}}\,dx + \sum_{q>y}\frac{1}{q(q-1)} < \frac{1.752}{\log(y)^{1/4}} + \sum_{n>y}\frac{1}{n(n-1)} < \frac{1.76}{\log(y)^{1/4}}.\end{aligned}$$

9. Working with Q_5

We now turn our attention to the set Q_5. As many of the same computations and remarks that were made with regards to Q_3 hold for the new set, we will simply work quickly through the appropriate bounds. We will often continue our assumption that $w > 10^{100}$ (unless otherwise posited).

9.1. **Defining Ω_p and κ.** We take $\mathcal{U}$ to be the set of primes $q \equiv 1 \pmod 5$ which also either satisfy $q < 10^{100}$ or $q \equiv 1 \pmod{15}$. Set

$$\Omega_p = \{0 \pmod p\} \cup \begin{cases} \{1 \pmod 5\} & \text{if } p = 5, \\ \{a \pmod p \ : \ o_p(a) = 5\} & \text{if } p \in \mathcal{U}, \\ \emptyset & \text{otherwise} \end{cases}$$

and define

$$\kappa(p) = \begin{cases} 2 & \text{if } p = 5, \\ 5 & \text{if } p \in \mathcal{U}, \\ 1 & \text{otherwise.} \end{cases}$$

9.2. **Estimating B.** Computations suggest (just as before) that $B = 2$ is the optimal value, asymptotically. We will get close to this value. We begin with the following bounds found in [27].

Proposition 5. *For $t > 10^{10}$,*

$$\left|\theta(t;5,1) - \frac{t}{4}\right| < 0.002785\frac{t}{4}.$$

For all $0 < t \leq 10^{10}$,

$$\left|\theta(t;5,1) - \frac{t}{4}\right| < 1.412480\sqrt{t}.$$

For integer values of $z > 10^{10}$, we then have

$$\begin{aligned}\sum_{p\le z,\ p\equiv 1 \pmod 5} \frac{\log(p)}{p} &= \theta(z;5,1)\frac{1}{z} + \int_{t=11}^{z} \theta(t;5,1)\frac{1}{t^2}\,dt \\ &< 0.2507\log(z) + 0.5\end{aligned}$$

Combining this with work done in previous sections, we have

$$\frac{1}{\log(z)}\sum_{p<z}\frac{\kappa(p)\log(p)}{p} \le \frac{1}{\log(z)}\left(\sum_{p<z}\frac{\log(p)}{p} + \frac{\log(5)}{5} + \sum_{\substack{p<z\\ p\equiv 1 \pmod 5}}\frac{4\log(p)}{p}\right) < 2.05$$

for all real $z > 10^{10}$.

For integer values in the range $10^5 < z < 10^{10}$, we have

$$\begin{aligned}\sum_{p\le z,\ p\equiv 1 \pmod 5} \frac{\log(p)}{p} &= \theta(z;5,1)\frac{1}{z} + \int_{t=11}^{z} \theta(t;5,1)\frac{1}{t^2}\,dt \\ &< 0.25\log(z) + 0.503 - 1.41/\sqrt{z}.\end{aligned}$$

In particular, $\frac{1}{\log(z)}\sum_{p<z}\frac{\kappa(p)\log(p)}{p} \le 2.09$ for all real $z > 10^5$. Finally, for $z < 10^5$ one checks directly that 2 works as an upper bound. We may then take $B = 2.09$.

9.3. **Finding bounds on $|\theta(x;5,1) - x/4|$.** We bound $|\theta(x;5,1) - x/4|$ using the techniques of Section 8.2. By Proposition 5, for $1 \le x \le 10^{10}$ we have $|\theta(x;15,1) - x/8| \le 1.097307\sqrt{x}$. For slight larger x we again follow [27, Section 4]. For very large x we use [8]. Putting these computations together we have the bound

$$|\theta(x;15,1) - x/8| < 0.6\frac{x}{\log(x)}$$

for all $x \ge 10$.

9.4. **Bounding a third Mertens type product.** We have the equality

$$\sum_{\substack{p\le x\\ p\equiv 1 \pmod 5}} \frac{1}{p} = \frac{1}{4}\log(\log(x)) + M(5,1) + \frac{\theta(x;5,1) - x/4}{x\log(x)} - \int_x^{\infty}(\theta(t;5,1) - t/4)\frac{1+\log(t)}{t^2\log(t)^2}\,dt$$

where $M(5,1) = -0.2088344\ldots$ is a constant. Using the bound given in the last subsection, we have for $x \ge 10^{10}$ that

$$\left|\sum_{\substack{p\le x\\ p\equiv 1 \pmod 5}} \frac{1}{p} - \frac{1}{4}\log(\log(x)) - M(5,1)\right| < \frac{0.9}{\log(x)^2} + \frac{0.6}{\log(x)}.$$

For $x \ge 10^6$ we have

$$\frac{0.9}{\log(x)^2} + \frac{0.6}{\log(x)} < \frac{0.67}{\log(x)}.$$

Letting $x \ge 10^6$, Taylor remainders yield

$$\exp(0.67/\log(x)) < 1 + \frac{0.71}{\log(x)},$$

and since by previous computation we have

$$\exp\left(\sum_{\substack{p>x,\, n\geq 2\\ p\equiv 1 \,(\mathrm{mod}\, 5)}} \frac{1}{np^n}\right) < \exp(1/2(x-1)) < 1+\frac{1.1}{x} = \alpha(x)$$

we also have

$$\alpha(x)\left(1+\frac{0.71}{\log(x)}\right) < 1+\frac{0.72}{\log(x)}.$$

As

$$\prod_{\substack{p\leq x\\ p\equiv 1 \,(\mathrm{mod}\, 5)}}\left(1-\frac{1}{p}\right) = \exp\left[-\sum_{\substack{p\leq x\\ p\equiv 1 \,(\mathrm{mod}\, 5)}}\frac{1}{p} - \sum_{\substack{n\geq 2\\ p\equiv 1 \,(\mathrm{mod}\, 5)}}\frac{1}{np^n} + \sum_{\substack{p>x,\, n\geq 2\\ p\equiv 1 \,(\mathrm{mod}\, 5)}}\frac{1}{np^n}\right].$$

we obtain

$$\prod_{\substack{p\leq x\\ p\equiv 1 \,(\mathrm{mod}\, 5)}}\left(1-\frac{1}{p}\right) < \frac{C(5,1)}{\log(x)^{1/4}}\left(1+\frac{0.72}{\log(x)}\right)$$

where $C(5,1) = 1.2252\ldots$ according to [22].

9.5. **Estimating $V(P(w))$.** We have for $w > 10^{100}$,

$$\begin{aligned} V(P(w)) &< \frac{3}{4}(1.00482)\prod_{p\leq w}\left(1-\frac{1}{p}\right)\prod_{\substack{p\leq 10^{100}\\ p\equiv 1 \ (\mathrm{mod}\ 5)}}\left(1-\frac{1}{p}\right)^4 \\ &\quad * \prod_{\substack{p\leq w\\ p\equiv 1 \ (\mathrm{mod}\ 15)}}\left(1-\frac{1}{p}\right)^4\prod_{\substack{p\leq 10^{100}\\ p\equiv 1 \ (\mathrm{mod}\ 15)}}\left(1-\frac{1}{p}\right)^{-4} \\ &< \frac{0.0673}{\log(w)^{3/2}}. \end{aligned}$$

9.6. **Picking a value for $s > 2B$.** We take $s = 10 > 2B$, and so $x > 10^{1000}$ as before. We obtain

$$\frac{1}{G(\sqrt{x})} \leq \frac{2.78}{\log(x)^{3/2}}.$$

9.7. **Measuring the size of Q_5.** In the large sieve, taking $u = \sqrt{x}$ where $x > 10^{1000}$, we have

$$|\{q\in Q_5 \ : \ q<x\}| \leq \frac{2x}{G(\sqrt{x})} + \sqrt{x} < \frac{5.56x}{\log(x)^{3/2}}.$$

Repeating the computation done in Section 4, for $y \geq 10^{1000}$,

$$\begin{aligned} \log\left(\prod_{q\in Q_5}\frac{q}{q-1}\right) &= -\sum_{q\in Q_5}\log(1-1/q) = \sum_{q\in Q_5}\left(\frac{1}{q}+\frac{1}{2q^2}+\frac{1}{3q^3}+\cdots\right) \\ &< \int_y^\infty \frac{5.56}{x\log(x)^{3/2}}\,dx + \sum_{q>y}\frac{1}{q(q-1)} < \frac{11.12}{\log(y)^{1/2}} + \sum_{n>y}\frac{1}{n(n-1)} < \frac{11.2}{\log(y)^{1/2}}. \end{aligned}$$

10. Putting it all together

Recall that $C_0' = 2(1-10^{-8})$. Our computations tell us that

$$\log(C_0') > \frac{1.76}{\log(y)^{1/4}} + \frac{11.2}{\log(y)^{1/2}}.$$

for $y \geq 10^{1000}$. Thus y is larger than the smallest prime divisor of our purported number, and this finishes the proof of the Main Theorem. This improves Yamada's result by seven orders of magnitude.

11. Idealization

One might wonder what bounds are possible if we idealize the situation slightly. First, we may as well assume $C_0 = 2$. Second, we will suppose that π has no further effect on the computations. Third, let's take $\mathcal{U}$ to be the set of all primes $p \equiv 1 \pmod 3$ when working with Q_3, and take $\mathcal{U}$ to be the set of all primes $p \equiv 1 \pmod 5$ when working with Q_5. While Rankin's trick gives us effective bounds, it is not tight. By methods in [16, Chapter 1] and [10, Theorem 2.2.2], one can find an asymptotic and effective bound on $G(w)$. In fact, for Q_3 we have $G(w) \sim c_{3,g}\log(w)^2$ where

$$c_{3,g} = \frac{e^{-2\gamma}}{\Gamma(1+2)} \lim_{w\to\infty} \frac{1}{\log(w)^2 V(P(w))} = 0.164\ldots.$$

Similarly, for Q_5 we have $G(w) \sim c_{5,g}\log(w)^2$ where

$$c_{5,g} = \frac{e^{-2\gamma}}{\Gamma(1+2)} \lim_{w\to\infty} \frac{1}{\log(w)^2 V(P(w))} = 0.145\ldots.$$

Fourth, let's suppose we are in a sieving situation where $|\mathcal{S}| \leq x/G(\sqrt{x})$. Ignoring error terms we have

$$\log\left(\prod_{q\in Q_3} \frac{q}{q-1}\right) \approx \sum_{q\in Q_3} \frac{1}{q} \approx \int_y^\infty \frac{6.1}{x\log(x)^2}\,dx \approx \frac{6.1}{\log(y)}$$

and similarly

$$\log\left(\prod_{q\in Q_5} \frac{q}{q-1}\right) \approx \frac{6.9}{\log(y)}$$

Solving the inequality

$$\log(2) \leq \frac{13}{\log(y)}$$

yields $y \lessapprox 1.4\cdot 10^8$.

With a significant amount of work we could improve the lower bound in the Main Theorem to 10^{10}, or possibly to 10^{12} if the computation was distributed. This would easily surpass the idealized bound we obtained above. However, that bound is still not obtainable for two reasons. First, the choices for the sets $\mathcal{U}$ are not realistic. Second, and more importantly, obtaining effective asymptotic bounds for $G(w)$ with current techniques seems to require $w \gg 10^{10}$, and hence we are led to something close to $x \gg 10^{20}$, which is just too large.

12. Open questions and future directions

Let p be a prime and let $q|\Phi_3(p)$ be a prime divisor. Either $q = 3$ or $\Phi_3(q)$ is divisible by 3. Similarly, if $q|\Phi_5(p)$ is prime then either $q = 5$ or $\Phi_5(q)$ is divisible by 5. Thus, if we consider factor chains where new primes arise only from either factoring $\Phi_3(p)$ or $\Phi_5(p)$, and we can stop whenever 3 or 5 occurs, then the factor chain simply bounces back and forth between applying Φ_3 and Φ_5. Eventually we expect

one of the prime factors to be $q \equiv 1 \pmod{15}$, and then both $\Phi_3(q)$ and $\Phi_5(q)$ terminate our factor chain.

In practice, such chains reach primes $q \equiv 1 \pmod{15}$ very quickly. It would be interesting to prove that for all sufficiently large primes p, factor chains limited as above always reach 3 or 5. Moreover, it would be interesting to prove (or disprove) that there is a bound on the depth it takes to reach 3 or 5.

One can also ask: What happens if we expand our set of limited exponents to $\mathcal{P} = \{3, 5, 7\}$? In this case, each prime in a factor chain has three possible branches (and possibly a fourth, if it is the special prime). Our technique for dealing with the prime 5 is inadequate because no specific primes appears (at least quickly) in every chain, although in principle one should still be able to deal with 3 because abundance computations are less difficult in that case.

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