ANOTHER REMARK ON THE RADICAL OF AN ODD PERFECT NUMBER

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Abstract. Ellia recently proved that if N is an odd perfect number such that $\mathrm{rad}(N) > \sqrt{N}$, then its special prime p satisfies $p > 148207$ if $3 \nmid N$ and $p > 223$ otherwise. He also suggested that these bounds can be improved with some computation. We obtain that if N is an odd perfect number such that $\mathrm{rad}(N) > \sqrt{N}$, then $p > 10^{60}$.

1. Introduction

A natural number N is said *perfect* if it is equal to the sum of its positive divisors (excluding N). It is well known that an even natural number N is perfect if and only if $N = 2^{k-1}(2^k - 1)$ for an integer k such that $2^k - 1$ is a Mersenne prime. On the other hand, it is a long-standing open question whether an odd perfect number exists.

In order to investigate this question, several authors gave necessary conditions for the existence of an odd perfect number N. Let $\Omega(n)$ and $\omega(n)$ denote respectively the total number of prime factors and the number of distinct prime factors of the integer n. Euler proved that $N = p^e m^2$ for a prime p, with $p \equiv e \equiv 1 \pmod 4$, p is prime, and $p \nmid m$. The prime p is said to be the *special prime*. Recent results show that $\omega(N) \geq 10$ [4] and $\Omega(N) \geq \max\{101, 2\omega(N) + 51, (18\omega(N) - 31)/7\}$ [5, 6]. Moreover, the bound $N > 10^{1500}$ [5] has been improved [1] to $N > 10^{2000}$.

Let $\mathrm{rad}(n)$ denote the *radical* or *square-free part* of the postive integer n, that is, $\mathrm{rad}(n) = \Pi_{p|n} p$ where p runs over primes. Luca and Pomerance [3] have investigated the radical of an odd perfect number and obtained that $\mathrm{rad}(N) < N^{17/26}$. Then Ellia [2] proved that if $\mathrm{rad}(N) > \sqrt{N}$, then $p > 148207$ and if $3 \nmid N$ and $p > 223$ otherwise.

In this paper, we explain the computations that we performed to obtain the following result:

Theorem 1.1. *If $N = p^e m^2$ is an odd perfect number, then at least one of the following holds:*

- *N has no prime factor less than 10^6,*
- *there exists a component $q^a \| N$ such that $q^{a-2} > 10^{60}$.*

Then we use Theorem 1.1 to improve Ellia's bounds:

Theorem 1.2. *If $N = p^e m^2$ is an odd perfect number such that $\mathrm{rad}(N) > \sqrt{N}$, then $p > 10^{60}$.*

2. Proof of Theorem 1.1

Let $\sigma(n)$ denote the sum of the positive divisors of the natural number n, and let $\sigma_{-1}(n) = \sigma(n)/n$ denote the *abundancy* of n. Clearly, n is perfect if and only if $\sigma_{-1}(n) = 2$. Recall that, if q is prime, then $\sigma(q^a) = \frac{q^{a+1}-1}{q-1}$ and $\sigma_{-1}(q^\infty) = \lim_{a\to\infty} \sigma_{-1}(q^a) = q/(q-1)$. Also, if $\gcd(i,j) = 1$, then $\sigma(ij) = \sigma(i)\sigma(j)$ and $\sigma_{-1}(ij) = \sigma_{-1}(i)\sigma_{-1}(j)$.

[1] see `www.lirmm.fr/~ochem/opn/`

To prove Theorem 1.1, we use the general method and the computer program discussed in [5, 6], which are based on the method of factor chains introduced in [1].

We suppose that N has no component q^a with q prime such that $q^{a-2} > 10^{60}$ and we use factor chains to rule out (in increasing order) all the odd primes less than 10^6 as a factor of N. These chains are constructed using *branchings*. To branch on a prime q means that we sequentially branch on all possible components q^a. To branch on a component q^a for q prime means that we suppose $q^a \times \sigma(q^a) \mid 2N$ and we either reach a contradiction or recursively branch on the overall largest available prime factor N that has not been already branched on.

We use the following contradictions to bound the tree of factor chains:

- The abundancy of the current number is strictly greater than 2.
- The current number has a non-special component q^a such that $q^{a-2} > 10^{60}$.

When branching on a prime q, we first branch on q^1 if $q \equiv 1 \pmod 4$ and no other prime is currently considered as the special prime. Then, we branch on the components q^a such that $a+1$ is an odd prime, as long as $q^{a-2} < 10^{60}$. This is because $\sigma\left(q^a\right) \mid \sigma\left(q^{(a+1)t-1}\right)$, so any contradiction obtained thanks to the factors of $\sigma\left(q^a\right)$ when supposing $q^a \| N$ also gives a contradiction in the case $q^{(a+1)t-1} \| N$. So q^a is a representative for all $q^{(a+1)t-1}$, and to compute lower bounds on the abundancy or to test against the bound 10^{60}, we can suppose that the multiplicity of p is exactly a. During the computation, we have encountered composite numbers with no known factors. However, there always existed an available prime that could be branched on.

For the reader familiar with the terminology of [5, 6], this explanation of the computation can be summarized as follows:

- we branch the overall largest available prime factor,
- we only do standard branchings,
- we did not encounter any roadblock.

3. Proof of Theorem 1.2

Suppose that $N = p^e m^2$ is an odd perfect number such that $\mathrm{rad}(N) > \sqrt{N}$. This implies obviously that $e = 1$. Let us write $m^2 = \Pi q_i^{a_i}$ where the q_i's are distinct primes. We have

$$(\mathrm{rad}(N))^2 = (\mathrm{rad}(p\Pi q_i^{a_i}))^2 = p^2 \Pi q_i^2 = \frac{p}{\Pi q_i^{a_i-2}} N.$$

Hence, $\mathrm{rad}(N) > \sqrt{N}$ implies that $p > \Pi q_i^{a_i-2}$.

By Theorem 1.1, we know that N has no prime factor less than 10^6, since otherwise there would exist a component q^a such that $p > \Pi q_i^{a_i-2} \geq q^{a-2} > 10^{60}$. Theorem 2 in [6] states that $\Omega(N) \geq 2\omega(N)+51$. Since $e=1$, we have $\Omega(N)-2\omega(N) = \Sigma(a_i-2)-1$, so that $\Sigma(a_i-2) \geq 52$. Now, every prime factor is greater than 10^6, so we have $p > \Pi q_i^{a_i-2} > \left(10^6\right)^{52} > 10^{60}$, which concludes the proof.

Notice that we can also obtain a better lower bound on $\Omega(N) - 2\omega(N)$ by using the (non-computational) inequality $\Omega(N) \geq (18\omega(N) - 31)/7$ in [6] and some lower bound on $\omega(N)$.

References

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