Lower bounds on odd perfect numbers

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Perfect numbers

- A number equal to the sum of its proper divisors.
- Examples : 6=1+2+3, 28=1+2+4+7+14, 496=1+2+4+8+16+31+62+124+248.
- Conjecture 1 : there are infinitely many perfect numbers.
- Conjecture 2 : there are no odd perfect numbers.

$\sigma_i(N)$

- $\sigma_i(N) = \Sigma_{d|N} d^i$.
- N is perfect : $\sigma_1(N) = 2N$.
- N is perfect : $\sigma_{-1}(N) = \sigma_1(N)/N = 2$. (abundancy)
- $GCD(a, b) = 1$ implies $\sigma_i(ab) = \sigma_i(a)\sigma_i(b)$.
- $\sigma_i(p_1^{e_1} p_2^{e_2} p_3^{e_3} \dots) = \sigma_i(p_1^{e_1})\sigma_i(p_2^{e_2})\sigma_i(p_3^{e_3}) \dots$.
- $\sigma_1(p^e) = 1 + p + p^2 + \dots + p^e = \frac{p^{e+1}-1}{p-1}$.
- $1 + \frac{1}{p} \leqslant \sigma_{-1}(p^e) < 1 + \frac{1}{p-1}$.
- $b > 1$ implies $\sigma_i(ab) > \sigma_i(a)$.

Even perfect numbers

- Suppose $2^k \parallel N$, $k \geqslant 1$, $\sigma_1(N) = 2N$.
- $\sigma_1(2^k) \mid \sigma_1(N)$, so $2^{k+1} - 1 \mid 2N$, so $2^{k+1} - 1 \mid N$.
- $N = 2^k(2^{k+1} - 1) \cdot i$, with i odd.

$$
\begin{aligned}
\sigma_{-1}(N) &= \sigma_{-1}(2^k(2^{k+1} - 1) \cdot i) \\
&\geqslant \sigma_{-1}(2^k(2^{k+1} - 1)) \\
&= \frac{\sigma_1(2^k)}{2^k} \times \frac{\sigma_1(2^{k+1} - 1)}{2^{k+1} - 1} \\
&= \frac{\sigma_1(2^{k+1} - 1)}{2^k} \\
&\geqslant \frac{1 + (2^{k+1} - 1)}{2^k} = 2
\end{aligned}
$$

So, N is an even perfect number iff $N = 2^k(2^{k+1} - 1)$ and $2^{k+1} - 1$ is prime.

Odd perfect numbers

- Suppose N is odd and $\sigma_1(N) = 2N$.
- [Euler] $N = p^e m^2$, p prime, $p \nmid m$, $p \equiv e \equiv 1 \pmod 4$.
- Proof :
  - $2N \equiv 2 \pmod 4$, so $\sigma_1(N) \equiv 2 \pmod 4$.
  - $\sigma_1(k) \equiv 1 \pmod 2$ iff $k = 2^t \cdot m^2$.
  - p prime and $p \equiv 3 \pmod 4$ implies $\sigma_1(p^{2i+1}) \equiv 0 \pmod 4$
  - p odd prime implies $\sigma_1(p^{4i+3}) \equiv 0 \pmod 4$
- p is the *special prime*. p^e is the *special component*.

Number of prime factors

Notation :

- $\omega(n)$: number of distinct prime factors of n.
- $\Omega(n)$: total number of prime factors of n.

Example :

- $360 = 2^3 \cdot 3^2 \cdot 5^1$.
- $\omega(360) = 3$.
- $\Omega(360) = 6$.

Odd perfect numbers

[O., Rao 2012] $N > 10^{1500}$ ($N > 10^{2000}$, unpublished).

previous bound : [Brent, Cohen, Riele 1991] $N > 10^{300}$.

[Nielsen 2014] $\omega(N) \geqslant 10$.

[O., Rao 2012] $\Omega(N) \geqslant \max(101,\ 2\omega(N) + 51,\ \frac{18\omega(N)-31}{7})$.

[Nielsen 2003] $N < 2^{4^{\omega(N)}}$.

[Goto, Ohno 2008] One prime factor $> 10^8$.

[Iannucci 1999] Two distinct prime factors $> 10^4$.

[Iannucci 2000] Three distinct prime factors $> 10^2$.

[O., Rao 2012] One component $> 10^{62}$.

Factor chains

Suppose $3^2 \parallel N$. Then $\sigma_1(3^2) \mid \sigma_1(N)$, i.e., $13 \mid N$.
Suppose $13^1 \parallel N$. Then $2 \cdot 7 \mid \sigma_1(N)$, i.e., $7 \mid N$.

$3^2 \Longrightarrow 13$	
$13^1 \Longrightarrow 2 \cdot 7$	
$7^2 \Longrightarrow 3 \cdot 19$	$[3^2 \cdot 7^2 \cdot 13 \cdot 19^2 > 10^6]$
$7^4 \Longrightarrow 2801$	$[3^2 \cdot 7^4 \cdot 13 \cdot 2801^2 > 10^6]$
$7^e, e \geqslant 6$	$[3^2 \cdot 7^6 \cdot 13 > 10^6]$
$13^2 \Longrightarrow 3 \cdot 61$	
$61^1 \Longrightarrow 2 \cdot 31$	$[3^2 \cdot 13^2 \cdot 61 \cdot 32^2 > 10^6]$
$61^e, e \geqslant 2$	$[3^2 \cdot 13^2 \cdot 61^2 > 10^6]$
$13^4 \Longrightarrow 30941$	$[3^2 \cdot 13^4 \cdot 30941 > 10^6]$
$13^e, e \geqslant 5$	$[3^2 \cdot 13^5 > 10^6]$

Factor chains

$3^4 \implies 11^2$	
$\quad 11^2 \implies 7 \cdot 19$	$[\sigma_{-1}(3^4 \cdot 7^2 \cdot 11^2 \cdot 19^2) = \frac{127}{63} > 2]$ $[> 10^6]$
$\quad 11^e, e \geqslant 4$	$[3^4 \cdot 11^4 > 10^6]$
$3^6 \implies 1093$	
$\quad 1093^1 \implies 2 \cdot 547$	$[3^6 \cdot 1093 \cdot 547^2 > 10^6]$
$\quad 1093^e, e \geqslant 2$	$[3^6 \cdot 1093^2 > 10^6]$
$3^8 \implies 13 \cdot 757$	$[3^8 \cdot 13^2 \cdot 757 > 10^6]$
$3^{10} \implies 23 \cdot 3851$	$[3^{10} \cdot 23^2 \cdot 3851^2 > 10^6]$
$3^{12} \implies 797161$	$[3^{12} \cdot 797161 > 10^6]$
$3^e, e \geqslant 14$	$[3^{14} > 10^6]$

Factor chains

$5^1 \implies 2 \cdot 3$	[3 is forbidden]
$5^2 \implies 31$	
$\quad 31^2 \implies 3 \cdot 331$	[3 is forbidden]
$\quad 31^e, e \geqslant 4$	$[5^2 \cdot 31^4 > 10^6]$
$5^4 \implies 11 \cdot 71$	$[5^4 \cdot 11^2 \cdot 71^2 > 10^6]$
$5^5 \implies 2 \cdot 3^2 \cdot 7 \cdot 13$	[3 is forbidden]
$5^6 \implies 19531$	$[5^6 \cdot 19531^2 > 10^6]$
$5^8 \implies 19 \cdot 31 \cdot 8291$	$[5^8 \cdot 19^2 \cdot 31^2 \cdot 8291^2 > 10^6]$
$5^e, e \geqslant 9$	$[5^9 > 10^6]$

Final argument

- Suppose N is an odd perfect number such that $GCD(N, 3 \cdot 5) = 1$.
- If $\omega(N) \leqslant 5$, then $\sigma_{-1}(N) < \sigma_{-1}(7^\infty \cdot 11^\infty \cdot 13^\infty \cdot 17^\infty \cdot 19^\infty) = 1.5592 \cdots < 2$.
- If $\omega(N) \geqslant 6$, then $N \geqslant 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 > 10^6$.

$N > 10^{2000}$

- We forbid $\{127, 19, 7, 11, 331, 31, 97, 61, 13, 398581, 1093, 3, 5, 307, 17, 23\}$
- Improved final argument.
- We circumvent roadblocks.

Example of roadblock :

$11^{18} \Longrightarrow 6115909044841454629 = P19$

$P19^{16} \Longrightarrow C301$

Circumventing roadblocks

Example of roadblock :
$11^{18} \Longrightarrow P19$
  $P19^{16} \Longrightarrow C301$

So $N = 11^{18} \cdot P19^{16} \cdot i$.
Let q be the smallest prime factor of i. Suppose $q \geqslant 947$.
If $\omega(i) \leqslant 564$, then
$\sigma_{-1}(N) < \sigma_{-1}(11^{18} \cdot P19^{16}) \cdot (1 + 1/946)^{564} < 2$.
If $\omega(i) \geqslant 565$, then
$N > 11^{18} \cdot P19^{16} \cdot 947^{565} > 10^{2000}$.

The contradiction gives $q < 947$.
To circumvent this roadblock, we branch on all the primes smaller than 946 to rule them out.

Circumventing roadblocks recursively

Example of roadblock :

$11^{18} \Longrightarrow P19$

$\quad P19^{16} \Longrightarrow C301$

We branch on the primes smaller than 946.
So we have to branch on the prime 3 and on the component 3^4, and we hit a new roadblock :

$11^{18} \Longrightarrow P19$

$\quad P19^{16} \Longrightarrow C301$

$\quad\quad 3^4 \Longrightarrow 11^2$

Then we branch on 5^1 and hit a new roadblock :

$11^{18} \Longrightarrow P19$

$\quad P19^{16} \Longrightarrow C301$

$\quad\quad 3^4 \Longrightarrow 11^2$

$\quad\quad\quad 5^1 \Longrightarrow 2 \cdot 3$

$\Omega(N) \geqslant (18\omega(N) - 31)/7$ - variables

- p_2 : number of distinct prime factors with exponent 2, distinct from 3 and the special prime
- $p_{2,1}$: number of distinct prime factors with exponent 2, congruent to 1 mod 3
- p_4 : number of distinct prime factors with exponent at least 4, distinct from 3 and the special prime
- f_4 : total number of prime factors with exponent at least 4, distinct from 3 and the special prime
- e : exponent of the special prime
- f_3 : exponent of the prime 3

$\Omega(N) \geqslant (18\omega(N) - 31)/7$ - inequalities

(1) $1 \leqslant e$

(2) $e + f_3 + 2p_2 + f_4 = \Omega$

(3) $4p_4 \leqslant f_4$

(4) $p_{2,1} \leqslant f_3$

(5) $\omega \leqslant f_3/2 + 1 + p_2 + p_4$

(6) $\omega \leqslant 2 + p_2 + p_4$

(7) $7\Omega \leqslant 18\omega - 32$

(8) $2p_2 \leqslant 1 + e + 3p_{2,1} + p_4 + f_4$

The combination
$5\times(1)+7\times(2)+5\times(3)+6\times(4)+2\times(5)+16\times(6)+(7)+2\times(8)$
gives $1 \leqslant 0$, a contradiction.
So (7) is false, thus $\Omega(N) \geqslant 18\omega(N) - 31)/7$.

Large factorizations

- $\sigma(2801^{78}) = C269 = P85 \cdot P184$
  27/10/2010, Tom Womack, MersenneForum (SNFS)
- $\sigma(3^{606}) = C290 = P85 \cdot P96 \cdot P110$
  01/11/2010, NFS@Home, Boinc (SNFS)
- $\sigma(2801^{82}) = C283 = P93 \cdot P193$
  15/03/2013, Ryan Propper (SNFS)
- $\sigma(547^{106}) = C291 = P60 \cdot P232$
  22/07/2013, Ryan Propper (SNFS)
- $\sigma(13^{269}) = C300 = P105 \cdot P195$
  24/02/2014, Ryan Propper (SNFS)
- $\sigma(11^{448}) = C468 = P68 \cdot P400$
  11/06/2014, Ryan Propper (ECM)

Merci