Acyclic improper colourings of graphs with bounded maximum degree

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ABSTRACT

For graphs of bounded maximum degree, we consider acyclic t-improper colourings, that is, colourings in which each bipartite subgraph consisting of the edges between two colour classes is acyclic, and each colour class induces a graph with maximum degree at most t.

We consider the supremum, over all graphs of maximum degree at most d, of the acyclic t-improper chromatic number and provide t-improper analogues of results by Alon, McDiarmid and Reed [N. Alon, C.J.H. McDiarmid, B. Reed, Acyclic coloring of graphs, Random Structures Algorithms 2 (3) (1991) 277–288] and Fertin, Raspaud and Reed [G. Fertin, A. Raspaud, B. Reed, Star coloring of graphs, J. Graph Theory 47 (3) (2004) 163–182].

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1. Introduction

Given a graph $G = (V, E)$, a proper colouring $\mathcal{V} = (V_1, \ldots, V_k)$ of V is *acyclic* if for all $1 \leq i < j \leq k$, the subgraph of G induced by $V_i \cup V_j$, which we denote $G[V_i \cup V_j]$, contains no cycles (i.e., is a forest). The *acyclic chromatic number* $\chi_a(G)$ is the smallest value k for which there exists a proper acyclic k-colouring of G. It is easily seen that $\chi_a(G) \leq \Delta(G)^2 + 1$, as any proper colouring of the square G^2 of G is *de facto* a proper acyclic colouring of G, and G^2 has maximum degree at most $\Delta(G)^2$. (G^2 is the graph with vertex set V and edges between all pairs of vertices at distance at most 2 in G.) In 1976, Erdős (see (cf. [1])) conjectured that $\chi_a(G) = o(\Delta(G)^2)$; this conjecture was proved by Alon et al. [2], who showed the existence of a fixed constant $c < 50$ such that for all G, $\chi_a(G) \leq c\Delta(G)^{4/3}$. Alon et al. also showed that their bound was close to optimal by proving via probabilistic arguments that

$$\max\{\chi_a(G) : \Delta(G) \leq \Delta\} = \Omega\left(\frac{\Delta^{4/3}}{(\log \Delta)^{1/3}}\right).$$

When studying the asymptotics of $\chi_a(G)$ in terms of $\Delta(G)$, the restriction that the colouring be *proper* is not of great importance. Indeed, suppose we define the *laid-back acyclic chromatic number* $\chi_\ell(G)$ to be the smallest value k for which there exists a colouring $\mathcal{V} = (V_1, \ldots, V_k)$ of G such that, for all $1 \leq i < j \leq k$, $G[V_i \cup V_j]$ is a forest (placing no further restriction on edges within a given block $G[V_i]$). Clearly, $\chi_\ell(G) \leq \chi_a(G)$. On the other hand, given such a colouring, it follows in particular that for all $1 \leq i \leq k$, $G[V_i]$ is a forest, so $\chi(G[V_i]) \leq 2$. By splitting V_i into stable sets $V_i^{(1)}$ and $V_i^{(2)}$ (for each $1 \leq i \leq k$), we may then obtain an acyclic *proper* colouring of G with at most $2k$ colours. It follows that $\chi_a(G)$ and $\chi_\ell(G)$ are within a factor of two of each other.

In this paper we investigate another relaxation of the acyclic chromatic number; in order to define it we first note that we may reformulate the definition of $\chi_a(G)$, by observing that if V_i and V_j are distinct stable sets in G, then $G[V_i \cup V_j]$ is exactly

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the bipartite graph $G[V_i, V_j]$ containing all edges with one endpoint in V_i and one endpoint in V_j. We may then equivalently define $\chi_a(G)$ as the smallest value k for which there exists a proper colouring $\mathcal{V} = (V_1, \ldots, V_k)$ of V, such that for all $1 \leq i < j \leq k$, $G[V_i, V_j]$ is a forest (i.e. such that with this colouring, G contains no *alternating cycle*).

Starting from this definition, we may now relax the requirement that $\mathcal{V}$ be a proper colouring, while continuing to impose that G contain no alternating cycle. To wit: given an integer $t \geq 0$, we say that a colouring $\mathcal{V} = (V_1, \ldots, V_k)$ is *t-improper* if for all $1 \leq i \leq k$, $G[V_i]$ has maximum degree at most t (in this case we say that V_i is *t-dependent*, for each $1 \leq i \leq t$). The *t-improper acyclic chromatic number* $\chi_a^t(G)$ is the smallest k for which there exists a t-improper colouring $\mathcal{V} = (V_1, \ldots, V_k)$ such that with this colouring, G contains no alternating cycle.

For an integer $d \geq 0$, we let

$$\chi_a^t(d) = \max\{\chi_a^t(G) : \Delta(G) \leq d\}.$$

The object of this paper is to study how $\chi_a^t(d)$ varies as a function of t and of d. Clearly, for any d, $\chi_a^0(d) \geq \chi_a^1(d) \geq \cdots \geq \chi_a^d(d) = 1$.

It is easily seen that $\chi_a^t(d) = \Omega\left((d/t)^{4/3}/(\ln d)^{1/3}\right)$; given an acyclic t-improper colouring, by applying the first of the results from [2] mentioned above, we can acyclically colour each colour class with at most $ct^{4/3}$ new colours (where c is some fixed constant which is less than 50) to obtain an acyclic colouring of the entire graph. Our first result is to show that this straightforward lower bound on $\chi_a^t(d)$ can be much improved upon asymptotically, as long as $t \leq d - 10\sqrt{d \ln d}$. More fully,

Theorem 1. *If* $t \leq d - 10\sqrt{d \ln d}$, *then* $\chi_a^t(d) = \Omega\left((d - t)^{4/3}/(\ln d)^{1/3}\right)$.

In particular, if $t = (1 - \varepsilon)d$ for any fixed constant ε, $0 < \varepsilon \leq 1$, then we obtain the same asymptotic lower bound as Alon et al. Comparing this lower bound with the upper bound $\chi_a^t(d) = O(d^{4/3})$, we see the surprising fact that even allowing $t = \Omega(d)$ does not greatly reduce the number of colours needed for improper acyclic colourings of graphs with large maximum degree.

At some point, $\chi_a^t(d)$ must drop significantly as t increases, because $\chi_a^d(d) = 1$. Although we are unable to pin down the behaviour of $\chi_a^t(d)$ viewed as a function of t, we can improve upon the upper bound of Alon et al. when t is very close to d (more precisely, when $d - t = o(d^{1/3})$). We prove:

Theorem 2. $\chi_a^t(d) = O(d \ln d + (d - t)d)$.

As for lower bounds on $\chi_a^t(d)$ when $d - t = o(d)$, we first note that [3] showed $\chi_a^{d-2}(d) \geq 3$; we can straightforwardly generalise this result by showing that $\chi_a^t(d) \geq d - t + 1$. This is done as follows: if K_{d+1} is the complete graph on $d + 1$ vertices, then $\chi_a^t(K_{d+1}) \geq d - t + 1$, since, in any acyclic t-improper colouring of K_{d+1}, at most one colour class has more than one vertex and no colour class has more than $t + 1$ vertices. We can, however, improve upon this further and, in the final section, we exhibit a set of examples showing the following lower bound.

Theorem 3. $\chi_a^{d-1}(d) = \Omega(d^{2/3})$.

We would like to reduce the gaps between the lower and upper bounds on $\chi_a^t(d)$. For $t = d - 1$, the problem is particularly tantalising, and, in this case, the lower bound of Theorem 3 and the upper bound of Theorem 2 differ by a factor of $d^{1/3} \ln d$. For this choice of t, the problem also includes the conjecture from [3] that every subcubic graph is acyclically 2-improperly 2-colourable.

In the rest of the paper, we use the following notation. The degree of a given vertex v is denoted by $d(v)$. We denote by $N(v)$ the set of the neighbours of v. A k-cycle (resp. a $\geq k$-cycle) is a cycle containing k vertices (resp. at least k vertices). For a graph G and a vertex $v \in V(G)$, we denote by $G \setminus \{v\}$ the graph obtained from G by removing v and its incident edges; for an edge uv of $E(G)$, $G \setminus \{uv\}$ denotes the graph obtained from G by removing the edge uv. These notions are extended to sets of vertices and edges in an obvious way. Let G be a graph and f be a colouring of G. For a given vertex v of G, we denote by $\text{im}_f(v)$, or simply $\text{im}(v)$ when the colouring is clear from the context, the number of neighbours of v having the same colour as v and call this quantity the *impropriety* of the vertex v. For notation not defined here, we refer the reader to [9].

2. A probabilistic lower bound for $\chi_a^t(d)$

In this section, we prove Proposition 6, a more explicit version of Theorem 1. Our argument mirrors that of Alon et al. but uses upper bounds on the *t-dependence number* α^t, the size of a largest t-dependent set, in the random graph $G_{n,p}$. For more precise upper bounds on $\alpha^t(G_{n,p})$, consult [7].

Lemma 4. *Fix an integer* $n \geq 1$ *and* $p \in \mathbb{R}$ *with* $4(\ln n/n)^{1/4} \leq p \leq 1$. *Let* $m = \lfloor n - 128 \ln n/p^4 \rfloor$. *Then asymptotically almost surely and uniformly over* p *in the above range, any colouring of* $G_{n,p}$ *with* $k \leq (n - m)/4$ *colours and in which each colour class contains at most* m *vertices contains an alternating 4-cycle.*

Proof. As there are at most $k^n \leq n^n$ possible k-colourings of $G_{n,p}$, to prove the lemma it suffices to show that for any fixed k-colouring of the vertices of $G_{n,p}$ (which we denote $\{v_1, \ldots, v_n\}$) with colour classes $C_1, \ldots, C_k$ in which $|C_i| \leq m$ for all $1 \leq i \leq k$, the probability that $G_{n,p}$ does not contain an alternating 4-cycle is $o(n^{-n})$.

Fix a colouring as above, and let q be minimal such that $|C_1 \cup \cdots \cup C_q| \geq (n - m)/2$. Let $A = C_1 \cup \ldots \cup C_q$ and let $B = C_{q+1} \cup \ldots \cup C_k$. As no colour class has size greater than m, $|A| \leq (n + m)/2$ and so $|B| \geq (n - m)/2$. By symmetry, we may also assume that $|A| \geq n/2$.

Next, let $P = \{\{x_1, x_1'\}, \ldots, \{x_r, x_r'\}\}$ be a maximal collection of pairs of elements of A such that for $1 \leq i \leq r$, x_i and x_i' have the same colour, and for $1 \leq i < j \leq r$, $\{x_i, x_i'\}$ and $\{x_j, x_j'\}$ are disjoint. As we may place all but perhaps one vertex from each colour class C_i in some such pair (with one vertex left over precisely if $|C_i|$ is odd), it follows that

$$r \geq \frac{1}{2}(|A| - q) \geq \frac{1}{2}\left(\frac{n}{2} - k\right) \geq \frac{n}{8}.$$

Similarly, let $Q = \{\{y_1, y_1'\}, \ldots, \{y_s, y_s'\}\}$ be a maximal collection of pairs of elements of B satisfying identical conditions; by an identical argument to that above, it follows that $s \geq (n - m)/8$.

Let E be the event that for all $1 \leq i \leq r$, $1 \leq j \leq s$, $\{x_i, y_j, x_i', y_j'\}$ is not an alternating 4-cycle, and let E' be the event that $G_{n,p}$ contains no alternating 4-cycle; clearly $E' \subseteq E$. For *fixed* $1 \leq i \leq r$ and $1 \leq j \leq s$, the probability that $\{x_i, y_j, x_i', y_j'\}$ is not an alternating 4-cycle is $(1 - p^4)$ and this event is independent from all other such events. As $(n - m) \geq 128 \ln n/p^4$ it follows that

$$\mathbf{Pr}\left(E'\right) \leq \mathbf{Pr}(E) \leq (1 - p^4)^{rs} \leq e^{-p^4 rs}$$

$$\leq \exp\left\{-\frac{p^4 n(n - m)}{64}\right\} \leq e^{-2n \ln n} = o(n^{-n}),$$

as required. $\square$

Using this lemma, we next bound the acyclic t-improper chromatic number of $G_{n,p}$ for p in the range allowed in Lemma 4.

Lemma 5. *Fix an integer $n \geq 1$ and $p \in \mathbb{R}$ with $4(\ln n/n)^{1/4} \leq p \leq 1$. Let $m = \lfloor n - 128 \ln n/p^4 \rfloor$ and let $t(n, p) = p(m - 1) - 2\sqrt{np}$. Then asymptotically almost surely, for all integers $t \leq t(n, p)$, $\chi_a^t(G_{n,p}) \geq 32 \ln n/p^4$, uniformly over p and t in the above ranges.*

Proof. Fix n and p as above, and choose $t \leq t(n, p)$. We will show that asymptotically almost surely $G_{n,p}$ contains no t-dependent set of size greater than m, from which the claim follows immediately by applying Lemma 4 as $(n - m)/4 \geq 32 \ln n/p^4$. Let $G[m]$ represent the subgraph of $G_{n,p}$ induced by $\{v_1, \ldots, v_m\}$. By a union bound and symmetry, we have

$$\mathbf{Pr}\left(\alpha^t(G_{n,p}) \geq m\right) \leq \binom{n}{m} \mathbf{Pr}\left(\Delta(G[m]) \leq t\right) \leq 2^n \mathbf{Pr}(\Delta(G[m]) \leq t).$$

Since, if $\Delta(G[m]) \leq t$ then $G[m]$ has at most $tm/2$ edges, it follows that

$$\mathbf{Pr}\left(\alpha^t(G_{n,p}) \geq m\right) \leq 2^n \mathbf{Pr}\left(E(G[m]) \leq \frac{tm}{2}\right)$$

$$\leq 2^n \mathbf{Pr}\left(E(G[m]) - p\binom{m}{2} \leq \frac{tm}{2} - p\binom{m}{2}\right).$$

Finally, by a Chernoff bound and by the definition of $t(n, p)$, we conclude that

$$\mathbf{Pr}\left(\alpha^t(G_{n,p}) \geq m\right) \leq 2^n \exp\left\{-\left(\frac{tm}{2} - p\binom{m}{2}\right)^2 \cdot \left(2p\binom{m}{2}\right)^{-1}\right\}$$

$$\leq 2^n \exp\left\{-\frac{(t - p(m - 1))^2}{4p}\right\} \leq (2/e)^n = o(1),$$

as claimed. $\square$

Using Lemma 5, it is a straightforward calculation to bound $\chi_a^t(d)$ for d sufficiently large, and t sufficiently far from d.

Proposition 6. *For all sufficiently large integers d, and all non-negative integers $t \leq d - 10\sqrt{d \ln d}$,*

$$\chi_a^t(d) \geq \frac{(d - t)^{4/3}}{2^{15}(\ln d)^{1/3}}.$$

Proof. Choose n so that

$$2^{13} n^3 \ln n \leq d^3(d - t) \leq 2^{14} n^3 \ln n; \tag{1}$$

such a choice of n clearly exists as long as d is large enough. Let $p = (d - 4\sqrt{d \ln d})/n$; we first check that p and t satisfy the requirements of Lemma 5. Presuming d is large enough that $np \geq d/2$, by the lower bound in (1) and the fact that $d(d - t) \leq d^2$ we have

$$p \geq \frac{d}{2n} \geq \frac{(d^3(d-t))^{1/4}}{2n} \geq \frac{8n^{3/4}(\ln n)^{1/4}}{2n} = 4\left(\frac{\ln n}{n}\right)^{1/4}. \tag{2}$$

Furthermore, letting $m = \lfloor n - 128 \ln n/p^4 \rfloor$, we have

$$p(m - 1) - 2\sqrt{np} \geq np - \frac{128 \ln n}{p^3} - 2\sqrt{np} - 2 = d - 4\sqrt{d \ln d} - 2\sqrt{np} - 2 - \frac{128 \ln n}{p^3}$$

$$\geq d - 8\sqrt{d \ln d} - \frac{128 \ln n}{p^3}. \tag{3}$$

Since $p \geq d/2n$ and by the lower bound in (1),

$$\frac{128 \ln n}{p^3} \leq \frac{2^{10}n^3 \ln n}{d^3} \leq \frac{d - t}{8},$$

which combined with (3) yields

$$p(m - 1) - 2\sqrt{np} > d - 8\sqrt{d \ln d} - \frac{(d - t)}{8}$$

$$= t + \frac{7(d - t)}{8} - 8\sqrt{d \ln d} > t, \tag{4}$$

the last inequality holding since $t \leq d - 10\sqrt{d \ln d}$. As (2) and (4) hold, we may apply Lemma 5 to bound $\chi_a^t(G_{n,p})$ with this choice of t and p; as $n > d$, it follows that as long as d is sufficiently large,

$$\mathbf{Pr}\left(\chi_a^t(G_{n,p}) \geq \frac{32 \ln n}{p^4}\right) \geq \frac{3}{4}, \tag{5}$$

say. Furthermore, by a union bound and a Chernoff bound,

$$\mathbf{Pr}\left(\Delta(G_{n,p}) > d\right) \leq n\mathbf{Pr}\left(\mathrm{BIN}\left(n, \frac{d - 4\sqrt{d \ln d}}{n}\right) > d\right)$$

$$\leq ne^{-16 \ln d/3} \leq \frac{1}{n}, \tag{6}$$

the last inequality holding as $\ln d \geq \ln n/2$ (which is an easy consequence of (1)). Combining (5) and (6), we obtain that

$$\mathbf{Pr}\left(\chi_a^t(G_{n,p}) \geq \frac{32 \ln n}{p^4}, \Delta(G_{n,p}) \leq d\right) \geq \frac{3}{4} - \frac{1}{n} \geq \frac{1}{2}$$

as long as $n \geq 4$, so there is some graph G with maximum degree at most d and with $\chi_a^t(G) \geq 32 \ln n/p^4$. Since χ_a^t is monotonically increasing in d, it follows that

$$\chi_a^t(d) \geq \frac{32 \ln n}{p^4} > \frac{32n^4 \ln n}{d^4}. \tag{7}$$

An easy calculation using the upper bound in (1), and the fact that $\ln n < 2 \ln d$ gives the bound

$$d^4 < \frac{2^{20}n^4(\ln d)^{4/3}}{(d - t)^{4/3}},$$

so $32n^4 \ln n/d^4 > (d - t)^{4/3}/2^{15}(\ln d)^{1/3}$. By (7), it follows that

$$\chi_a^t(d) \geq \frac{(d - t)^{4/3}}{2^{15}(\ln d)^{1/3}},$$

as claimed. □

3. A probabilistic upper bound for $\chi_a^t(d)$

In this section, we study the situation when $d - t = o(d^{1/2})$. Theorem 2, which improves the upper bound of [2] when $d - t = o(d^{1/3})$, is a corollary of our main result here.

We analyse a different parameter from, but one that is closely related to, the acyclic t-improper chromatic number. A *star colouring* of G is a colouring, such that no path of length three (i.e. with four vertices) is alternating; in other words, each bipartite subgraph consisting of the edges between two colour classes is a disjoint union of stars. The *star chromatic number* $\chi_s(G)$ is the least number of colours needed in a proper star colouring of G. We analogously define the parameters $\chi_s^t(G)$ and $\chi_s^t(d)$ in the natural way. The star chromatic number was one of the main motivations for the original study of acyclic colourings [6]. Clearly, any star colouring is acyclic; thus, $\chi_a^t(d) \leq \chi_s^t(d)$. Fertin, Raspaud and Reed [5] showed that $\chi_s(d) = O(d^{3/2})$ and that $\chi_s(d) = \Omega\left(d^{3/2}/(\ln d)^{1/2}\right)$. We note that a natural adaptation to star colouring of the argument given in the last section gives the following:

Theorem 7. *There exists a fixed constant $C > 0$ such that, if $t \leq d - C\sqrt{d \ln d}$, then $\chi_s^t(d) = \Omega\left((d - t)^{3/2}/(\ln d)^{1/2}\right)$.*

Given a graph G of maximum degree d, the idea behind our method for improved upper bounds is to find a dominating set $\mathcal{D}$ and a function $g = g(d) = o(d^{3/2})$, such that $\left|(N(v) \cup N^2(v)) \cap \mathcal{D}\right| \leq g$ for all $v \in V(G)$. Given such a set $\mathcal{D}$ in G, we assign colours to the vertices in $\mathcal{D}$ by greedily colouring $\mathcal{D}$ in the square of G (i.e. vertices in $\mathcal{D}$ at distance at most two in G receive different colours) with at most $g + 1$ colours; then we give the vertices of $G \setminus \mathcal{D}$ the colour $g + 2$. It can be verified that this colouring prevents any alternating paths of length three (and so prevents alternating cycles), and ensures that every vertex has at least one neighbour of a different colour. Furthermore, we can generalise this idea by prescribing that our set $\mathcal{D}$ is k-*dominating* – each vertex outside of $\mathcal{D}$ has at least k neighbours in $\mathcal{D}$ – to give a bound on $\chi_s^{d-k}(d)$.

Theorem 8. $\chi_s^t(d) = O(d \ln d + (d - t)d)$.

This result provides an asymptotically better upper bound than $\chi_s^t(d) = O(d^{3/2})$ when $d - t = o(d^{1/2})$. It also provides a better bound than $\chi_a^t(d) = O(d^{4/3})$ when $d - t = o(d^{1/3})$. Theorem 8 is an easy consequence of the following lemma:

Lemma 9. *Given a d-regular graph G and an integer $k \geq 1$, let $\psi(G, k)$ be the least integer $k' \geq k$, such that there exists a k-dominating set $\mathcal{D}$ for which, for all $v \in V(G)$, $|N(v) \cap \mathcal{D}| \leq k'$. Let $\psi(d, k)$ be the maximum over all d-regular graphs G of $\psi(G, k)$. Then, for all d sufficiently large, $\psi(d, k) \leq \max\{3k, 31 \ln d\}$.*

We postpone the proof of this lemma, first using it to prove Theorem 8:

Proof of Theorem 8. We first remark that the function χ_s^t is monotonic with respect to graph inclusion in the following sense: if G and G' are graphs with $V(G) \subseteq V(G')$, and $E(G) \subset E(G')$, then $\chi_s^t(G) \leq \chi_s^t(G')$. As any graph G of maximum degree d is a subgraph of a d-regular graph (possibly with a greater number of vertices), to prove that $\chi_s^t(d) = O(d \ln d + (d - t)d)$ it therefore suffices to show that $\chi_s^t(G) = O(d \ln d + (d - t)d)$ for d-regular graphs G. We hereafter assume G is d-regular and d is large enough to apply Lemma 9. Let $k = d - t$. We will show that $\chi_s^t(G) \leq d\psi(d, k) + 2$, which proves the theorem.

By Lemma 9, there is a k-dominating set $\mathcal{D}$ such that $|N(v) \cap \mathcal{D}| \leq \psi(d, k)$ for all $v \in V(G)$. Fix such a dominating set $\mathcal{D}$ and form the auxiliary graph H as follows: let H have vertex set $\mathcal{D}$ and let uv be an edge of H precisely if u and v have graph distance at most two in G. As $|N(v) \cap \mathcal{D}| \leq \psi(d, k)$ for all $v \in V(G)$, H has maximum degree at most $d\psi(d, k)$.

To colour G, we first greedily colour H, using at most $d\psi(d, k) + 1$ colours, and assign each vertex v of $\mathcal{D}$ the colour it received in H. We next choose a new colour not used on the vertices of $\mathcal{D}$, and assign this colour to all vertices of $V(G) \setminus \mathcal{D}$. We remind the reader that $\mathrm{im}(v)$ denotes the number of neighbours of v of the same colour as v. If $v \in \mathcal{D}$ then $\mathrm{im}(v) = 0$, and if $v \in V \setminus \mathcal{D}$ then $\mathrm{im}(v) \leq d - |N(v) \cap \mathcal{D}| \leq d - k = t$, so the resulting colouring is t-improper.

Furthermore, given any path $P = v_1 v_2 v_3 v_4$ of length three in G, either two consecutive vertices v_i, v_{i+1} of P are not in $\mathcal{D}$ (in which case $c(v_i) = c(v_{i+1})$, and P is not alternating), or two vertices v_i, v_{i+2} are in $\mathcal{D}$ (in which case $c(v_i) \neq c(v_{i+2})$ and P is not alternating). Thus, the above colouring is a star colouring G of impropriety at most t and using at most $d\psi(d, k) + 2$ colours; as G was an arbitrary d-regular graph, it follows that $\chi_s^t(d) \leq d\psi(d, k) + 2$, as claimed. $\quad\square$

We next prove Lemma 9 with the aid of the following symmetric version of the Lovász Local Lemma:

Lemma 10 (*[4], cf. [8], Page 40*). *Let $\mathcal{A}$ be a set of bad events, such that for each $A \in \mathcal{A}$*

1. **Pr** $(A) \leq p < 1$, *and*
2. *A is mutually independent of a set of all but at most δ of the other events.*

If $4p\delta \leq 1$, then with positive probability, none of the events in $\mathcal{A}$ occurs.

Proof of Lemma 9. We may clearly assume that k is at least $(31/3) \ln d$, since, if the claim of the lemma holds for such k, then it also holds for smaller k. Let $p = 2k/d$ and let $\mathcal{D}$ be a random set obtained by independently choosing each vertex v with probability p. We claim that, with positive probability, $\mathcal{D}$ is a k-dominating set, such that $|N(v) \cap \mathcal{D}| \leq 3k$ for all $v \in V(G)$; we will prove our claim using the local lemma.

For $v \in V(G)$, let A_v be the event that either $|N(v) \cap \mathcal{D}| < k$ or $|N(v) \cap \mathcal{D}| > 3k$. By the mutual independence principle, cf. [8], page 41, A_v is mutually independent of all but at most d^2 events A_w (with $w \neq v$). Furthermore, since $|N(v) \cap \mathcal{D}|$ has a binomial distribution with parameters d and p, we have by a Chernoff bound that

$$\mathbf{Pr}\,(A_v) = \mathbf{Pr}\,(||N(v) \cap \mathcal{D}| - \mathbf{E}(|N(v) \cap \mathcal{D}|)| > k) \leq 2e^{-k/5} = o(d^{-2})$$

so $4\mathbf{Pr}\,(A_v)\,d^2 < 1$ for d large enough. By applying Lemma 10 with $\mathcal{A} = \{A_v \mid v \in V\}$, it follows that with positive probability none of the events A_v occur, i.e. $\mathcal{D}$ has the desired properties. $\quad\square$

4. A deterministic lower bound for $\chi_a^{d-1}(d)$

In this section, we concentrate on the case $t = d - 1$ and exhibit an example G_n, which gives the asymptotic lower bound of Theorem 3. Given a positive integer n, we construct the graph G_n as follows: G_n has vertex set $\{v_{ij} : i, j \in \{1, \ldots, n\}\} \cup \{w_{ij} : i, j \in \{1, \ldots, n\}\}$. For $i, j \in \{1, \ldots, n\}$ we let $\mathcal{V}_{ij} = \{v_{ij}, w_{ij}\}$. We can think of the set of vertices as an n-by-n matrix, each entry of which has been "doubled". Within each column $\mathcal{C}_i = \bigcup_{j=1}^n \mathcal{V}_{ij}$, and within each row $\mathcal{R}_j = \bigcup_{i=1}^n \mathcal{V}_{ij}$ we add all possible edges. The graph G_n has $2n^2$ vertices and is regular with degree $d = 4n - 3$. We will prove the following proposition, which directly implies Theorem 3:

Proposition 11. $\chi_a^{d-1}(G_n) \geq \frac{n}{n^{1/3}+1} + 1.$

Proof. Let $f : G_n \to \{1, \ldots, k\}$ be an acyclic $(d - 1)$-improper colouring of G_n; we will show that necessarily $k \geq \frac{n}{n^{1/3}+1}$. Since $n \geq 1$, it follows that $n/2 \geq \frac{n}{n^{1/3}+1}$ and thus we may assume that $k < n/2$. Clearly, some colour – say a_1 – appears on two vertices x, x' of $\mathcal{C}_1$. We call the colour a_1 "black" and refer to vertices receiving colour a_1 as black vertices. If $y, y' \in \mathcal{C}_1$ both receive colour $i \neq a_1$, then $xyx'y'$ forms an alternating cycle, so black is the only colour appearing twice in $\mathcal{C}_1$. It follows that at most $k - 1$ vertices in $\mathcal{C}_1$ are not black.

Applying the same logic to any column $\mathcal{C}_i$, we see that all but $k - 1$ vertices in $\mathcal{C}_i$ must receive the same colour, say a_i. Since $k < n/2$, it is easily seen, then, that there must be a row $\mathcal{R}_m$, such that v_{m1} and w_{m1} are both black, and v_{mi} and w_{mi} are both coloured black. This implies that a_i is the colour black, since otherwise $v_{m1}v_{mi}w_{m1}w_{mj}$ would be an alternating cycle. It follows that in all columns, at most $k - 1$ vertices receive a colour other than black. Symmetrically, there is a colour b such that in all rows, at most $k - 1$ vertices receive a colour other than b; clearly, it must the case that b is the colour black.

If there are $i, j \in \{1, \ldots, n\}$ such that both $\mathcal{R}_i$ and $\mathcal{C}_j$ are entirely coloured black, then all the neighbours of v_{ij}, w_{ij} are coloured black and the colouring is not $(d - 1)$-improper; therefore, it must be the case that either all rows, or all columns, contain a non-black vertex. Without loss of generality, we may assume that all rows contain a non-black vertex.

Let $x_1, \ldots, x_r$ be non-black vertices receiving the same colour, say red, and let $x_i \in \mathcal{V}_{\ell_i, m_i}$, for $1 \leq i \leq r$. As previously noted, no two of $x_1, \ldots, x_r$ may lie in the same row or column; i.e., for $i \neq j$, $\ell_i \neq \ell_j$ and $m_i \neq m_j$.

Claim 1. *At least* $3\binom{r}{2}$ *vertices of* $\bigcup_{1 \leq i \neq j \leq r} \mathcal{V}_{\ell_i, m_j}$ *receive a non-black colour other than red.*

Proof. No vertices in $\bigcup_{1 \leq i \neq j \leq r} \mathcal{V}_{\ell_i, m_j}$ are coloured red as each such vertex is in the same row as one of $x_1, \ldots, x_r$. On the other hand, for each pair i, j with $1 \leq i < j \leq r$, at least three of the vertices in $\mathcal{V}_{\ell_i, m_j} \cup \mathcal{V}_{\ell_j, m_i}$ must receive a colour other than black. For if $y, y' \in \mathcal{V}_{\ell_i, m_j} \cup \mathcal{V}_{\ell_j, m_i}$ are both coloured black, then $x_i y x_j y'$ forms an alternating cycle. The result follows, as there are $\binom{r}{2}$ pairs i, j with $1 \leq i < j \leq r$. $\quad\square$

Claim 2. *At least* r *distinct non-black colours appear on* $\bigcup_{1 \leq i < j \leq r} \mathcal{V}_{\ell_i, m_j}$.

Proof. By an argument just as above, each of $\mathcal{V}_{\ell_1, m_2}, \ldots, \mathcal{V}_{\ell_1, m_r}$ must contain a vertex receiving a colour other than black or red. These colours must all be distinct as $\mathcal{V}_{\ell_1, m_2}, \ldots, \mathcal{V}_{\ell_1, m_r}$ are all contained within $\mathcal{R}_{\ell_1}$. $\quad\square$

Let $\{a_2, a_3, \ldots, a_k\}$ be the set of non-black colours. Let $x_1^2, \ldots, x_{r_2}^2$ be the vertices receiving colour a_2, and for $i = 3, \ldots, k$ let $x_1^i, \ldots, x_{r_i}^i$ be the vertices receiving colour a_i which are in a different row from all vertices in $\bigcup_{j<i} \bigcup_{s \leq r_j} x_s^j$. As every row contains a non-black vertex, $\sum_{i=2}^k r_i = n$; it is possible that $r_i = 0$ for certain i, if there is a vertex coloured with one of $a_2, \ldots, a_{i-1}$ in every row.

For $i \in \{2, \ldots, k\}$ and $s \in \{1, \ldots, r_i\}$, say vertex $x_s^i \in \mathcal{V}_{\ell_s^i, m_s^i}$, and let

$$A_i = \bigcup_{1 \leq s < t \leq r_i} \mathcal{V}_{\ell_s^i, m_t^i} \cup \mathcal{V}_{\ell_t^i, m_s^i}.$$

By Claim 1, at least $3\binom{r_i}{2}$ vertices of A_i are non-black. Furthermore, if $i \neq i'$, then for any $s \in \{1, \ldots, r_i\}$, $s' \in \{1, \ldots, r_{i'}\}$, x_s^i and $x_{s'}^{i'}$ are in different rows — so A_i and $A_{i'}$ are disjoint. It follows that in $\bigcup_{i=2}^k A_i \cup \{x_1^i, \ldots, x_{r_i}^i\}$, at least

$$\sum_{i=2}^k \left(3\binom{r_i}{2} + r_i \right) \geq \sum_{i=2}^k r_i^2 \tag{8}$$

Table 1
Asymptotic bounds for $\chi_a^t(d)$ and $\chi_s^t(d)$

$d - t$	$\chi_a^t(d)$		$\chi_s^t(d)$	
	Lower	Upper	Lower	Upper
$\Theta(d)$	$\Omega\left(\frac{d^{4/3}}{(\ln d)^{1/3}}\right)$		$\Omega\left(\frac{d^{3/2}}{(\ln d)^{1/2}}\right)$	
$\omega(\sqrt{d \ln d})$	$\Omega\left(\frac{(d-t)^{4/3}}{(\ln d)^{1/3}}\right)$	$O(d^{4/3})$	$\Omega\left(\frac{(d-t)^{3/2}}{(\ln d)^{1/2}}\right)$	$O(d^{3/2})$
$O(d^{1/2})$	$\Omega\left(d^{2/3}\right)$		$\Omega\left(d^{2/3}\right)$	
$O(d^{1/3})$		$O((d-t)d)$		$O((d-t)d)$
$O(\ln d)$		$O(d \ln d)$		$O(d \ln d)$
0	1	1	1	1

vertices are non-black. As $\sum_{i=2}^{k} r_i = n$, it is easily seen that

$$\sum_{i=2}^{k} r_i^2 \geq (k-1)\left(\left\lfloor \frac{n}{k-1} \right\rfloor\right)^2.$$

As there are only $k - 1$ non-black colours, it follows that some non-black colour – say a_2 – appears at least $(\lfloor n/(k-1)\rfloor)^2$ times. If $(\lfloor n/(k-1)\rfloor)^2 \geq n^{2/3}$, then by Claim 2, at least $n^{2/3} + 1 > \frac{n}{n^{1/3}+1} + 1$ colours appear on G_n, so we may assume that $n^{2/3} > (\lfloor n/(k-1)\rfloor)^2 \geq (n/(k-1) - 1)^2$. But then $k > \frac{n}{n^{1/3}+1} + 1$, as claimed. $\square$

It is worth noting that the correct asymptotic order of $\chi_a^{d-1}(G_n)$ is unknown; it is even conceivable that $\chi_a^{d-1}(G_n) = \Theta(d)$.

5. Conclusion

In our view, the most surprising result of this paper is that the same asymptotic lower bound for ordinary acyclic chromatic number by Alon et al. also holds for the acyclic t-improper chromatic number for any $t = t(d)$, satisfying $d - t = \Theta(d)$. As $\chi_a(G) \geq \chi_a^t(G)$ for any $t \geq 0$, this means that, for $d - t = \Theta(d)$, Theorem 1 is asymptotically tight up to a factor of $(\ln d)^{1/3}$.

In the case that t is very close to d, Theorem 8 improves upon upper bounds for $\chi_a^t(d)$ and $\chi_s^t(d)$ implied by the results of Alon et al. and Fertin et al., respectively, giving for instance that $\chi_s^t(d) = O(d \ln d)$ for $d - t = O(\ln d)$. On the other hand, we showed that $\chi_a^{d-1}(d) = \Omega(d^{2/3})$ by a deterministic construction.

There is much remaining work in the case $d - t = o(d)$. Table 1 is a rough summary of the current bounds on $\chi_a^t(d)$, and $\chi_s^t(d)$ when d is large. A case of particular interest to the authors is when $d - t = 1$; in this case, it is unknown if $\chi_a^{d-1}(d)$ is $\Theta(d^{2/3})$, $\Theta(d \ln d)$ or lies somewhere strictly between these extremes.

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