Graphs with mad< 3 and $\Delta \geq 17$ are list 2-distance $(\Delta + 2)$-colorable

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Extended Abstract

All the graphs considered here are simple and finite. A *2-distance k-coloring* of a graph G is a coloring of the vertices of G with k colors such that two vertices that are adjacent or have a common neighbor receive distinct colors. We define $\chi^2(G)$ as the smallest k such that G admits a 2-distance k-coloring. A generalization of the 2-distance k-coloring is the *list 2-distance k-coloring*, where instead of having the same list of k colors for the whole graph, every vertex is assigned some set of k colors and has to be colored from it. We define $\chi^2_\ell(G)$ as the smallest k such that G admits a list 2-distance k-coloring of G for any list assignment. Obviously, 2-distance coloring is a sub-case of list 2-distance coloring (where the same color list is assigned to every vertex), so for any graph G, $\chi^2_\ell(G) \geq \chi^2(G)$.

The study of $\chi^2(G)$ on planar graphs was initiated by Wegner in 1977 [7], and has been actively studied because of his conjecture, stated below. The *maximum degree* of a graph G is denoted $\Delta(G)$.

Conjecture 1 (Wegner [7]) *If G is a planar graph, then:*

- $\chi^2(G) \leq 7$ *if* $\Delta(G) = 3$
- $\chi^2(G) \leq \Delta(G) + 5$ *if* $4 \leq \Delta(G) \leq 7$
- $\chi^2(G) \leq \lfloor \frac{3\Delta(G)}{2} \rfloor + 1$ *if* $\Delta(G) \geq 8$

Note that any graph G satisfies $\chi^2(G) \geq \Delta(G) + 1$. Indeed, if we consider a vertex of maximal degree and its neighbors, they form a set of $\Delta(G) + 1$ vertices, any two of which are adjacent or have a common neighbor. Hence at least $\Delta(G) + 1$ colors are needed for a 2-distance coloring of G. It is therefore natural to ask when this lower bound is reached. For that purpose, we can study, as suggested by Wang and Lih [6], what conditions on the sparseness of the graph can be sufficient to ensure the equality holds. A first measure of the sparseness of a planar graph is its girth. The *girth* of a graph G, denoted $g(G)$, is the length of a shortest cycle.

Conjecture 2 (Wang and Lih [6]) *For any integer $k \geq 5$, there exists an integer $D(k)$ such that for every planar graph G verifying $g(G) \geq k$ and $\Delta(G) \geq D(k)$, $\chi^2(G) = \Delta(G)+1$.*

Conjecture 2 was proved by Borodin, Ivanova and Noestroeva [3, 4] to be true for $k \geq 7$, even in the case of list-coloring, and false for $k \in \{5, 6\}$.

Dvořák, Král, Nejedlý and Škrekovski [5] proved that it is off by just one for $k = 6$, i.e. for a planar graph G with girth 6 and sufficiently large $\Delta(G)$, $\chi^2(G) \leq \Delta(G) + 2$. They also conjectured that the same holds for planar graphs with girth 5, but this remains open. Borodin and Ivanova [1, 2] improved the corresponding bound for graphs of girth 6, and extended it to list-coloring.

Theorem 3 (Borodin and Ivanova [1]) *Every planar graph G with $\Delta(G) \geq 18$ and $g(G) \geq 6$ admits a 2-distance $(\Delta(G) + 2)$-coloring.*

Theorem 4 (Borodin and Ivanova [2]) *Every planar graph G with $\Delta(G) \geq 24$ and $g(G) \geq 6$ admits a list 2-distance $(\Delta(G) + 2)$-coloring.*

We improve the previous two theorems as follows.

Theorem 5 *Every planar graph G with $\Delta(G) \geq 17$ and $g(G) \geq 6$ admits a list 2-distance $(\Delta(G) + 2)$-coloring.*

Another way to measure the sparseness of a graph is through its maximum average degree as defined below. The *average degree* of a graph G, denoted $\text{ad}(G)$, is $\frac{\sum_{v \in V} d(v)}{|V|} = \frac{2|E|}{|V|}$. The *maximum average degree* of a graph G, denoted $\text{mad}(G)$, is the maximum of $\text{ad}(H)$ over all subgraph H of G. Using this measure, we, in fact, prove a more general theorem than Theorem 5.

Theorem 6 *Every graph G with $\Delta(G) \geq 17$ and $\text{mad}(G) < 3$ admits a list 2-distance $(\Delta(G) + 2)$-coloring.*

Euler's formula links girth and maximum average degree in the case of planar graphs, as it easy to check that for any planar graph G, $(\text{mad}(G) - 2)(g(G) - 2) < 4$. Thus, planar graphs of girth at least 6 have a maximum average degree smaller than 3, and Theorem 5 is a corollary of Theorem 6.

To prove Theorem 6, we use a global discharging method, that is, a discharging method where some forbidden configurations have unbounded size and where the weight can travel arbitrarily far.

An *injective k-coloring* of G is a (not necessarily proper) coloring of the vertices of G with k colors such that no vertex has two neighbors with the same color, or, in other words, such that two vertices that have a common neighbor receive distinct colors. A 2-distance k-coloring is also an injective coloring, but the reverse is not true. The list version of this coloring is a *list injective k-coloring* of G.

Some results on 2-distance coloring have their counterpart on injective coloring with one less color, and it is the case of Theorems 3 and 4. It happens that the proof of Theorem 6 also works with close to no alteration for list injective coloring, thus yielding a proof that every graph G with $\Delta(G) \geq 17$ and $\text{mad}(G) < 3$ admits a list injective $(\Delta(G) + 1)$-coloring.

References

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